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INVESTIGATION OF LONGITUDINAL VIBRATIONS OF AN ELASTIC ROD WITH TRANSVERSE CRACK BY MEANS OF CLOSE SUMMATION OF TRIGONOMETRIC SERIES

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The paper deals with the problem of vibrations of a rod with one fixed end, which vibrates under the force of gravity. Close by the fixing there is a transverse crack. The solution is obtained by means of close summation of trigonometric series.

Amplitude and period of vibrations of unfixed end of the rod as functions of the crack depth are obtained.

STATEMENT OF THE PROBLEM

Let us concern a problem [1]: «Research vertical longitudinal vibrations of rod of length l, which is hung up by the end $x = 0$, granting the vibrations influenced by gravity. Take zero initial conditions».

The vibrations of the rod are given by the equation

$$u_{tt} = a^2 u_{xx} + g \tag{1}$$

($u = u(x,t)$ is a removal of the section of the rod with abscissa x at instant t, g is acceleration of gravity, $a = \sqrt{\frac{E}{\rho}}$, where E is the rod material modulus of elongation, ρ is the rod material density) under the boundary conditions

$$u(0,t) = 0, \quad u_x(l,t) = 0 \tag{2}$$

and the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 0\,. \tag{3}$$

Let S be a rod cross-section area. Consider now the case where in section with abscissa $x_0 = 0$ there is a transverse crack of area S_1. When in a small neighborhood of x_0 the rod is compressed (removals of sections with small x-coordinates are negative), the crack is closed, and it is open, when in a small neighborhood of x_0 the rod is stretched (removals of sections with small x-coordinates are positive). Then at t, for which the crack is closed, we have Eq. (1), and at t, for which the crack is open, the vibrations are specified by the equation

$$u_{tt} = b^2 u_{xx} + g\,, \tag{4}$$

where $b = a\sqrt{\frac{S - S_1}{S}}$ [2] under corresponding initial and boundary conditions

On imposed conditions the response-function for small x is positive for every t, thus the crack is constantly open.

For this reason, the following problem is of interest. A vertical rod of length l and weight $g\rho Sl$, the upper length of which is rigidly fixed, is initially compressed up to the length $l_1 < l$ and released without initial speed. Close by the fixing there is a transverse crack of area S_1 (Fig.1). Research longitudinal vibrations of such a rod.

Initially the crack is closed. The vibrations of the rod are given by the Eq.(1) on the initial and boundary conditions

$$u(x,0) = -rx, \quad u_t(x,0) = 0, \tag{5}$$

$$u(0,t) = 0, \quad u_x(l,t) = 0\,, \tag{6}$$

where $r = \dfrac{l - l_1}{l}$.

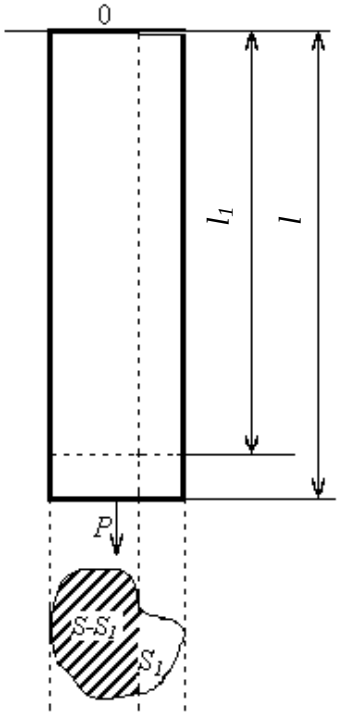


Fig 1. A rod with a transverse crack close by fixing

APPLICATION OF CLOSE SUMMATION OF TRIGONOMERTIC SERIES

The problem of vibrations of the rod with the transverse crack has as a solution the function

$$u(x,t) = \sum\nolimits_1 + \sum\nolimits_2\,, \tag{7}$$

where $\sum_1$ describes the motion of the rod of weight $g\rho Sl$ under zero conditions, and $\sum_2$ describes the motion of the rod, which was initially compressed up to the length l_1 and released without initial speed. Furthermore, $\sum_1$ satisfies Eq.1, while $\sum_2$ satisfies corresponding homogeneous equation. The solution, obtained by means of separation of variables method, is expressed as follows

$$\sum\nolimits_1 = \frac{16gl^2}{\pi^3 a^2}\sum_{n=0}^{\infty}\frac{1-\cos(2n+1)\dfrac{\pi at}{2l}}{(2n+1)^3}\sin(2n+1)\frac{\pi x}{2l}\ , \text{ [1],} \tag{8}$$

$$\sum\nolimits_2 = -\frac{8lr}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\dfrac{\pi at}{2l}}{(2n+1)^2}\sin(2n+1)\frac{\pi x}{2l}\ , \text{ [2].} \tag{9}$$

Now we transform the first series (this operation is permitted since the series converges absolutely):

$$\sum\nolimits_1 = \frac{16gl^2}{\pi^3 a^2}\left(\sum_{n=0}^{\infty}\frac{\sin(2n+1)\frac{\pi x}{2l}}{(2n+1)^3} - \sum_{n=0}^{\infty}\frac{\cos(2n+1)\frac{\pi at}{2l}}{(2n+1)^3}\sin(2n+1)\frac{\pi x}{2l}\right). \qquad (10)$$

The sum of the first series in this expression is

$$2\frac{gl^2}{a^2}\left(\frac{x}{2l} - \frac{x^2}{4l^2}\right) \qquad (11)$$

and is due to influence of gravity on the vibrations of the rod of length x. Given component is invariable as it is independent of t.

By means of [3] it is possible to obtain a response-function as the linear combination of the sums of the series given above. By the trigonometric functions product formulae we obtain:

$$\sum_{n=0}^{\infty}\frac{\cos(2n+1)\frac{\pi at}{2l}}{(2n+1)^3}\sin(2n+1)\frac{\pi x}{2l} = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{\sin(2n+1)\frac{\pi}{2l}(x-at)}{(2n+1)^3} + \sum_{n=0}^{\infty}\frac{\sin(2n+1)\frac{\pi}{2l}(x+at)}{(2n+1)^3}\right) \qquad (12)$$

It is known that

$$\sum_{n=0}^{\infty}\frac{\sin(2n+1)y}{(2n+1)^3} = \frac{\pi y}{8}(\pi - y) \qquad (13)$$

when $y \in [0,\pi)$. If the argument belongs to another interval, for instance, $[\pi, 2\pi)$, the series can be transformed as follows:

$$\sum_{n=0}^{\infty}\frac{\sin(2n+1)y}{(2n+1)^3} = -\sum_{n=0}^{\infty}\frac{\sin(2n+1)(y-\pi)}{(2n+1)^3}. \qquad (14)$$

Now the tabulated summation formula [3] is applicable.

When the crack is closed, the vibrations of the rod for $x \in \left[0, \frac{l}{2}\right]$, $t \in \left[0, \frac{3l+x}{a}\right)$ are given by the function:

$$u(x,t)=\begin{cases}\dfrac{gt^2}{2}-rx, & t\in\left[0,\dfrac{x}{a}\right)\\ \dfrac{gx}{2a^2}(2at-x)-rx, & t\in\left[\dfrac{x}{a},\dfrac{l-x}{a}\right)\\ \dfrac{gx}{2a^2}(2at-x)-r(l-at), & t\in\left[\dfrac{l-x}{a},\dfrac{l+x}{a}\right)\\ \dfrac{gx}{2a^2}(2at-x)+rx, & t\in\left[\dfrac{l+x}{a},\dfrac{2l-x}{a}\right)\\ -\dfrac{1}{2}\dfrac{g}{a^2}\left(2x^2-4xl+(at-2l)^2\right)+rx, & t\in\left[\dfrac{2l-x}{a},\dfrac{2l+x}{a}\right)\\ -\dfrac{1}{2}\dfrac{gx}{a^2}(-8l+x+2at)+rx, & t\in\left[\dfrac{2l+x}{a},\dfrac{3l-x}{a}\right)\\ -\dfrac{1}{2}\dfrac{gx}{a^2}(-8l+x+2at)-r(at-3l), & t\in\left[\dfrac{3l-x}{a},\dfrac{3l+x}{a}\right)\end{cases} \tag{15}$$

Explicit form of the function enables us to determine the instant of the crack opening t_0:

$$\text{for } l\le\frac{a^2r}{g}\quad t_0=\frac{l}{a},\quad \text{for } l>\frac{a^2r}{g}\quad t_0=\frac{ar}{g}.$$

Let t_1 be an instant when the crack is closed again. Then for $t\in[t_0,t_1)$ the vibrations of the rod are given by the function $v(x,t)$, which obeys:

$$\begin{aligned}&v_{tt}=b^2v_{xx}+g,\\ &v(x,t_0)=u(x,t_0),\quad v_t(x,t_0)=u_t(x,t_0),\\ &v(0,t)=0,\quad v_x(l,t)=0.\end{aligned} \tag{16}$$

By dint of separation of variables method we get:

$$v(x,t)=\frac{16gl^2}{\pi^3a^2}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\dfrac{\pi x}{2l}}{(2n+1)^3}-\frac{4gl^2}{\pi^3a^2}\left(1+\frac{a}{b}\right)\sum_{n=0}^{\infty}\frac{\cos(2n+1)\dfrac{\pi(bt-(b-a)t_0)}{2l}}{(2n+1)^3}\sin(2n+1)\frac{\pi x}{2l}-$$

$$-\frac{4gl^2}{\pi^3a^2}\left(1-\frac{a}{b}\right)\sum_{n=0}^{\infty}\frac{\cos(2n+1)\dfrac{\pi(bt-(b+a)t_0)}{2l}}{(2n+1)^3}-$$

$$-\frac{2lr}{\pi^2}\left(1+\frac{a}{b}\right)\sum_{n=o}^{\infty}\frac{(-1)^n\cos(2n+1)\dfrac{\pi(bt-(a-b)t_0)}{2l}}{(2n+1)^2}\sin(2n+1)\frac{\pi x}{2l}-$$

$$-\frac{2lr}{\pi^2}\left(1-\frac{a}{b}\right)\sum_{n=o}^{\infty}\frac{(-1)^n\cos(2n+1)\dfrac{\pi(bt-(a+b)t_0)}{2l}}{(2n+1)^2}\sin(2n+1)\frac{\pi x}{2l}. \tag{17}$$

As-summing for $x \in [0,(a-b)t_0)$ and $t \in \left[t_0, \dfrac{l-x+(b-a)t_0}{b}\right)$:

$$v(x,t)=\begin{cases} \dfrac{g}{a}xt-\dfrac{gx^2}{2a^2}-rx, & t\in\left[t_0,\dfrac{-x+(b+a)t_0}{b}\right) \\ \dfrac{1}{2}\dfrac{g}{a^2}bxt+\dfrac{1}{2}\dfrac{g}{a}xt+\dfrac{1}{2}gx\dfrac{t_0}{b}-\dfrac{1}{2}\dfrac{g}{a^2}xt_0b+\dfrac{1}{4}\dfrac{g}{a^2}b^2t^2-\dfrac{1}{4}\dfrac{g}{a}bt^2-\dfrac{1}{2}\dfrac{g}{a^2}b^2tt_0+ \\ +\dfrac{1}{2}gtt_0+\dfrac{1}{4}\dfrac{g}{a^2}b^2t_0^2+\dfrac{1}{4}\dfrac{g}{a}bt_0^2-\dfrac{1}{4}gt_0^2-\dfrac{1}{4}\dfrac{g}{a^2}x^2-\dfrac{1}{4}\dfrac{g}{ab}x^2-\dfrac{1}{4}\dfrac{g}{b}at_0^2-rx, \\ \qquad\qquad t\in\left[\dfrac{-x+(b+a)t_0}{b},\dfrac{x+(b+a)t_0}{b}\right) \\ \dfrac{g}{a^2}xbt+gx\dfrac{t_0}{b}-\dfrac{g}{a^2}xt_0b-\dfrac{1}{2}\dfrac{g}{a^2}x^2-rx, \qquad t\in\left[\dfrac{x+(b+a)t_0}{b},\dfrac{l-x+(b-a)t_0}{b}\right) \end{cases} \tag{18}$$

Let us now determine the instant t_1:

$$t_1=\frac{4l}{b}+t_0\left(1-\frac{a}{b}-\frac{a^2}{b^2}\right). \tag{19}$$

Starting with this time, that is for $[t_1,T)$, where T is the vibration period, the rod sections removals are given by the function $w(x,t)$, which obeys:

$$\begin{gathered} w_{tt}=a^2w_{xx}+g\,, \\ w(x,t_1)=v(x,t_1),\quad w_t(x,t_1)=v_t(x,t_1)\,, \\ w(0,t)=0,\quad w_x(l,t)=0\,. \end{gathered} \tag{20}$$

OBTAINED RESULTS

Let us designate as $U(x,t)$ the follow function:

$$U(x,t)=\begin{cases} u(x,t) & t\in[0,t_0), \\ v(x,t) & t\in[t_0,t_1), \\ w(x,t) & t\in[t_1,T). \end{cases} \tag{21}$$

The maximum A^+ of the one-variable function $U(l,t)$ represents the greatest positive removal of the rod end. The minimum of this function, according to the conditions of the problem, is attained for the instant T.

If A_0^+ is the greatest positive removal of an undamaged rod end, then

$$A_0^+=U\left(l,\frac{2l}{a}\right)=\frac{gl^2}{a^2}+rl\,, \tag{22}$$

$$A^+=U\left(l,\frac{2l}{b}+t_0\left(1-\frac{a^2}{b^2}\right)\right)=\frac{gl^2}{a^2}+rl+gt_0^2\left(\frac{a^2}{b^2}-1\right). \tag{23}$$

The greatest negative removals of the end both of damaged end undamaged rods coincide and are equal to $A_0^- = -rl$. How the difference $A - A_0$ of vibration amplitudes in the cases of damaged and undamaged rod depends on the crack depth, is shown on the Fig. 2.

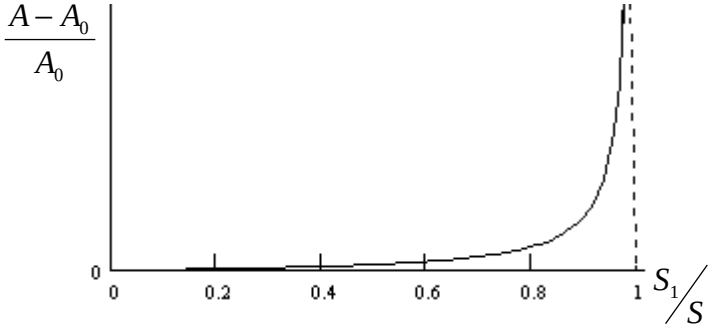


Fig.. 2.

The vibration period in case of the undamaged rod is equal to $T_0 = \frac{4l}{a}$, and for the damaged rod we have $T = \frac{4l}{b} + t_0\left(1 - \frac{a^2}{b^2}\right)$. How the difference $T - T_0$ depends on the crack depth, is shown on Fig. 3.

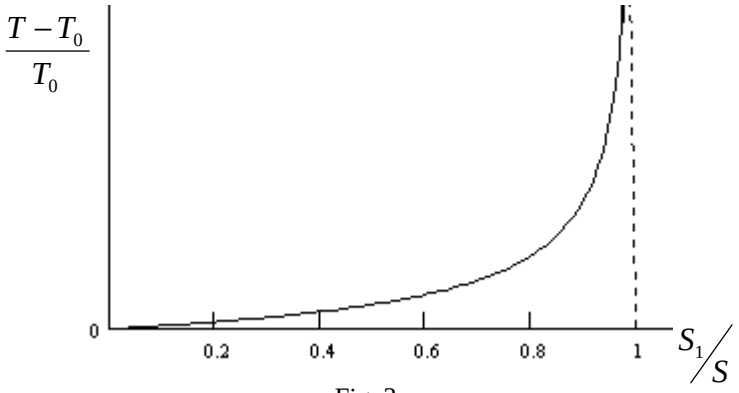


Fig..3

The graphs of movement of the rod end for different crack depths are presented on Figures 4,5,6. The following designations are used: 1 - $\frac{t_0}{T_0}$; 2 - $\frac{t_1}{T_0}$; 3 - 1; 4 - $\frac{T_1}{T_0}$.

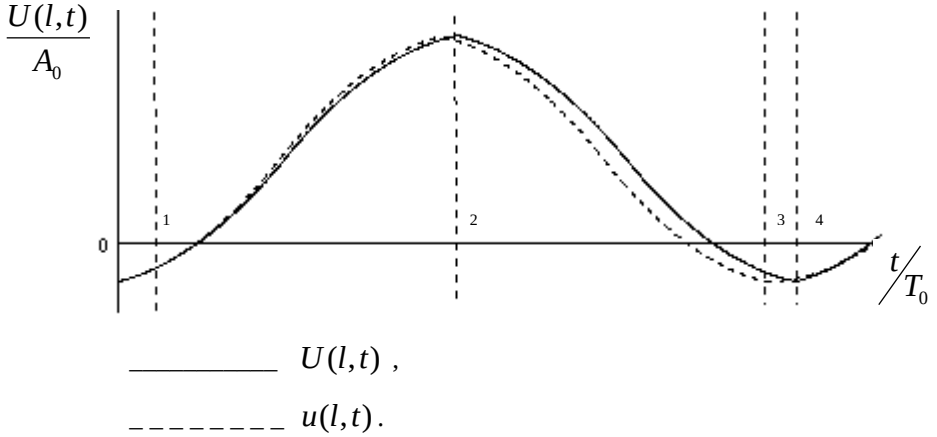


Fig. 4. The crack makes up 10% of the rod cross-section .

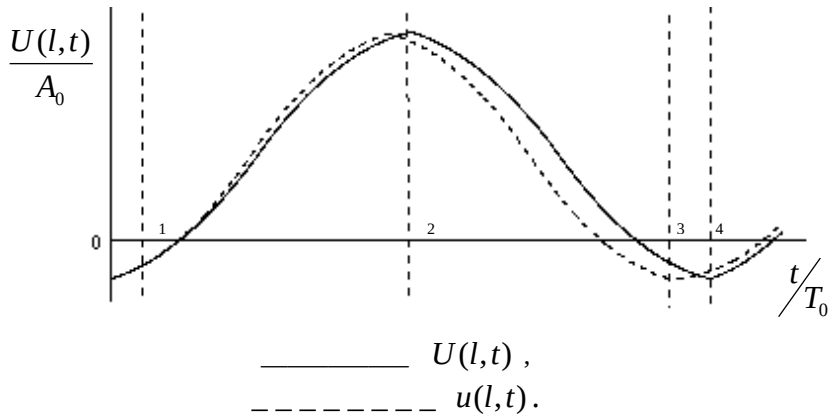


Fig. 5. The crack makes up 15% of the rod cross-section

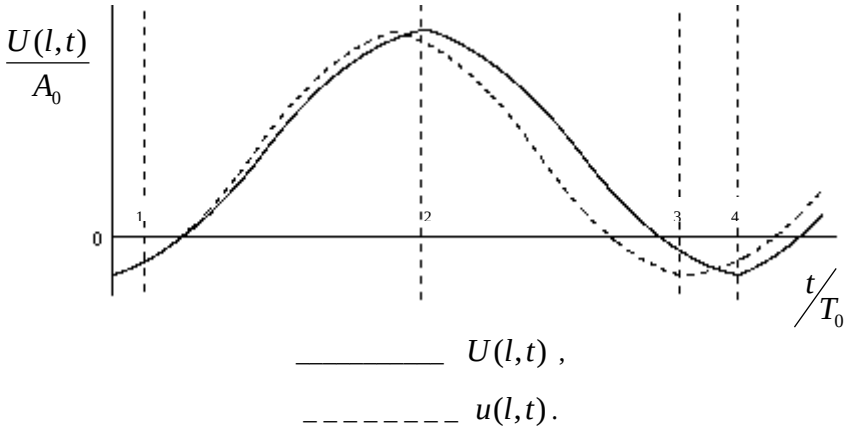


Fig. 6. The crack makes up 20% of the rod cross-section.

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