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HYBRID ASYMPTOTIC METHOD FOR THE EFFECT OF THE LOCAL THICKNESS DEFECTS ON THE BUCKLING OF CYLINDRICAL SHELLS UNDER COMBINED LOADING

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1. INTRODUCTION

It is well known that the initial imperfections of the shells may have a remarkable effect on the buckling load. Many authors have discussed the effect of imperfections on the buckling of cylindrical shells under axial compression as well as under external pressure. We refer to the works by Koiter [1,2], Kan [3], Volmir [4], Gristchak [5], Abdelmoula et al. [6], Koiter et al. [7].

In many cases it is impossible to obtain an exact analytical solution of the problem. For this reason the applied mechanics problems are often solved by approximate analytical methods such as perturbation and variation ones.

In Gristchak [5] an approximate calculation of the effect of more or less localised initial imperfections has been carried out. Here the components of the displacements are presented as asymptotic expansions on the small parameter $\frac{\mu^2}{m^2}$ of the imperfection location. In Koiter et al. [7] the buckling of the circular cylindrical shell with small periodical thickness defects is considered. Here stress function is expressed in terms of the thickness variation parameter ε. In both works asymptotic formulas in terms of $\frac{\mu^2}{m^2}$ and ε have been derived by perturbation methods.

In this paper in the spirit of papers [5,7] the hybrid perturbation method is proposed for the effect of the local thickness defects and an external pressure on the buckling of an axially compressed cylindrical shell.

In spite of the proposed asymptotic method has been used in this paper only for the stability problem of the compressed cylindrical shell, it also can be applied for investigation of the behaviour of the thinwalled structures under combined loading.

2. DESCRIPTION OF THE HYBRID PERTURBATION METHOD

We consider a circular cylindrical shell of radius R, length L that is made of an isotropic, elastic material with Young's modulus E and Poisson's ratio ν. It is subjected to an axial compressive load P_0 and to an external pressure P_1. The coordinate system is taken as shown in Figure 1.

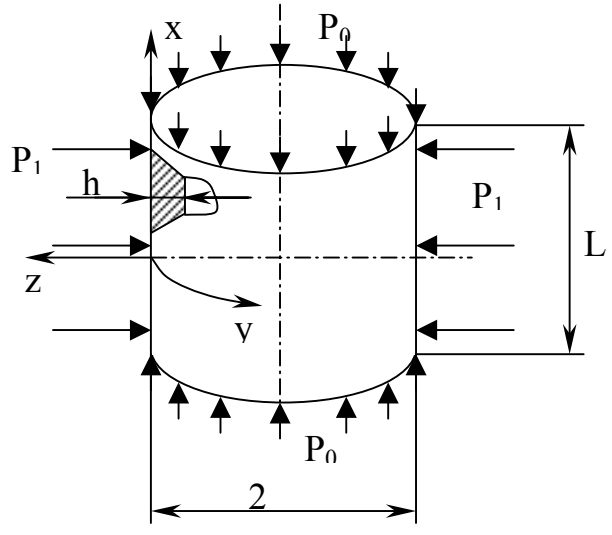


Figure 1. Cylindrical shell under axial compression and external pressure

The governing equations of the pressurised non-uniform cylindrical shell are as follows [7]

$$\frac{Eh^3(x)}{12\left(1-\nu^2\right)}\left(\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2\partial y^2}+\frac{\partial^4 W}{\partial y^4}\right)+\frac{3E}{12\left(1-\nu^2\right)}\left(2h(x)h'^2(x)+h^2(x)h''(x)\right)\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}\right)+$$

$$+\frac{1}{R}\frac{\partial^2 F}{\partial x^2}+\frac{6E}{12\left(1-\nu^2\right)}h^2(x)h'(x)\left(\frac{\partial^3 W}{\partial x^3}+\frac{\partial^3 W}{\partial x\partial y^2}\right)-\frac{3E}{12\left(1-\nu^2\right)}(1-\nu)\left(2h(x)h'^2(x)+h^2(x)h''(x)\right)\frac{\partial^2 W}{\partial y^2}+$$

$$+P_0\frac{\partial^2 W}{\partial x^2}+P_1R\frac{\partial^2 W}{\partial y^2}=0\,, \tag{1}$$

$$\frac{1}{Eh(x)}\left(\frac{\partial^4 F}{\partial x^4}+2\frac{\partial^4 F}{\partial x^2\partial y^2}+\frac{\partial^4 F}{\partial y^4}\right)-\frac{1}{E}\left(\frac{h''(x)h(x)-2h'^2(x)}{h^3(x)}\right)\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}\right)-\frac{1}{R}\frac{\partial^2 W}{\partial x^2}-$$

$$-\frac{2}{E}\frac{h'(x)}{h^2(x)}\left(\frac{\partial^3 F}{\partial x^3}+\frac{\partial^3 F}{\partial x\partial y^2}\right)+\frac{1+\nu}{E}\left(\frac{h''(x)h(x)-2h'^2(x)}{h^3(x)}\right)\frac{\partial^2 F}{\partial y^2}=0\,, \tag{2}$$

where $W(x,y)$ is the radial displacement (positive outward), $F(x,y)$ is the stress function, $h(x)$ is the shell thickness.

By introducing the following non-dimensional parameters

$$\xi=\frac{x}{L}\,,\ \eta=\frac{y}{L}\,,\ H(\xi)=\frac{h(x)}{h_0}\,,\ W^*(\xi,\eta)=\frac{W(x,y)}{h_0}\,,\ F^*(\xi,\eta)=\frac{F(x,y)}{D}\,,\ D=\frac{Eh_0{}^3}{12\left(1-\nu^2\right)}\,,$$

where h_0 is the nominal thickness of the shell, D is the bending stiffness,

the governing equations (1)-(2) can be rewritten into their non-dimensional form

$$H^3\left(\frac{\partial^4 W^*}{\partial\xi^4}+2\frac{\partial^4 W^*}{\partial\xi^2\partial\eta^2}+\frac{\partial^4 W^*}{\partial\eta^4}\right)+3\left(2HH'^2+H^2H''\right)\left(\frac{\partial^2 W^*}{\partial\xi^2}+\frac{\partial^2 W^*}{\partial\eta^2}\right)+\frac{L^2}{Rh_0}\frac{\partial^2 F^*}{\partial\xi^2}+$$

$$+6H^2H'\left(\frac{\partial^3 W^*}{\partial\xi^3}+\frac{\partial^3 W^*}{\partial\xi\,\partial\eta^2}\right)-3(1-\nu)\left(2HH'^2+H^2H''\right)\frac{\partial^2 W^*}{\partial\eta^2}+$$

$$\frac{P_0L^2}{D}\frac{\partial^2 W^*}{\partial\xi^2}+\frac{P_1RL^2}{D}\frac{\partial^2 W^*}{\partial\eta^2}=0\,, \tag{3}$$

$$H^2\left(\frac{\partial^4 F^*}{\partial\xi^4}+2\frac{\partial^4 F^*}{\partial\xi^2\partial\eta^2}+\frac{\partial^4 F^*}{\partial\eta^4}\right)+2H'^2\left(\frac{\partial^2 F^*}{\partial\xi^2}-\nu\frac{\partial^2 F^*}{\partial\eta^2}\right)-HH''\left(\frac{\partial^2 F^*}{\partial\xi^2}-\nu\frac{\partial^2 F^*}{\partial\eta^2}\right)-$$

$$-2HH'\left(\frac{\partial^3 F^*}{\partial\xi^3}+\frac{\partial^3 F^*}{\partial\xi\,\partial\eta^2}\right)-\frac{12\left(1-\nu^2\right)L^2}{Rh_0}H^3\frac{\partial^2 W^*}{\partial\xi^2}=0\,. \tag{4}$$

We seek the solution of the initial equations in the following form

$$\left.\begin{aligned}F^*(\xi,\eta)&=f(\xi)\cos\left(\frac{nL}{R}\eta\right)\\ W^*(\xi,\eta)&=w(\xi)\cos\left(\frac{nL}{R}\eta\right)\end{aligned}\right\}, \tag{5}$$

where n is the wave number in the circumferential direction.

We assume that an axial compressive load P_0 exceeds an external load P_1R in such a manner that the shape formation corresponds to an axial compression. Thus an external load P_1R can be expressed as

$$P_1R=\alpha P_0\,,$$

where $\alpha = \frac{P_1 R}{P_0}$ is the ratio ($0 \le \alpha \le 0{,}5$).

Substituting (5) into equations (3)-(4) and denoting

$$N = \frac{nL}{R}, z = \frac{L}{\sqrt{Rh_0}}, c = \sqrt{3\left(1-\nu^2\right)}, \lambda = \frac{P_0}{P_{cr}}, P_{cr} = \frac{Eh_0{}^2}{R\sqrt{3\left(1-\nu^2\right)}}, \tag{6}$$

where λ is the buckling load parameter, P_{cr} is the classical buckling load of the shell with constant thickness, we obtain the following ordinary differential equations

$$H^3 w^{(4)} + 6H^2 H' w''' + \left[-2N^2 H^3 + 6H H'^2 + 3H^2 H'' + 4\lambda c z^2\right] \cdot w'' -$$
$$- 6N^2 H^2 H' w' + \left[N^4 H^3 - 6\nu N^2 H H'^2 - 3\nu N^2 H^2 H'' - 4N^2 \alpha \lambda c z^2\right] \cdot w + z^2 f'' = 0, \tag{7}$$

$$H^2 f^{(4)} - 2HH' f''' + \left[-2N^2 H^2 + 2H'^2 - HH''\right] \cdot f'' + 2N^2 HH' f' +$$
$$+ \left[N^4 H^2 + 2\nu N^2 H'^2 - \nu N^2 HH''\right] \cdot f = 4c^2 z^2 H^3 w''. \tag{8}$$

We consider the local thickness defects in the form

$$H(\xi) = 1 - \varepsilon \exp\left(-\frac{\mu^2}{2}\xi^2\right), \tag{9}$$

where $\varepsilon << 1$ and μ^2 are parameters indicating the amplitude magnitude of the thickness defects and the character of the defect location respectively.

The buckling mode of a circular cylindrical shell can be written in the form

$$w(\xi) = \left[A\cos(m\xi) + B\cos(3m\xi)\right] \cdot \exp\left(-\frac{\mu^2}{2}\xi^2\right), \tag{10}$$

where $m = \frac{pL}{R}$, p is the number of half-waves along the shell length.

The first term in the brackets is the buckling pattern that satisfies the boundary conditions of the simple support. The second term and modulating factor $\exp\left(-\frac{\mu^2}{2}\xi^2\right)$ are introduced in (10) as the local thickness defects may initiate the local buckling mode.

In order to obtain an approximate analytical solution of the compatibility equation (8) we shall use the double asymptotic expansion method described above.

An unknown function $f(\xi)$ we seek as a two-terms expansion on the small defect parameter ε

$$f(\xi) = \tilde{\varphi}_0(\xi) \exp\left(-\frac{\mu^2}{2}\xi^2\right) + \varepsilon \tilde{\varphi}_1(\xi) \exp\left(-\mu^2 \xi^2\right) + \ldots. \tag{11}$$

The function $\tilde{\varphi}_0(\xi)$ can be represented as an asymptotic expansion on the small parameter $\beta^2 = \frac{\mu^2}{m^2}$ such as

$$\tilde{\varphi}_0(\xi) = \tilde{f}_0(\xi) + \beta^2 \tilde{f}_1(\xi) + \ldots. \tag{12}$$

Substituting expansions (11)-(12) into (8) and accounting (9)-(10) after collecting the coefficients with equal orders of parameters ε and β^2 we obtain a system with unknown functions $\tilde{f}_0(\xi)$, $\tilde{f}_1(\xi)$, $\tilde{\varphi}_1(\xi)$

$$\left(\beta^2\right)^0 : \tilde{f}_0{}^{(4)} - 2N^2 \tilde{f}_0'' + N^4 \tilde{f}_0 = 4c^2 z^2 \left[-A m^2 \cos(m\xi) - 9Bm^2 \cos(3m\xi)\right], \tag{13}$$

$$\left(\beta^2\right)^1 : \quad \tilde{f}_1^{(4)} - 2N^2 \tilde{f}_1'' + N^4 \tilde{f}_1 = 4c^2 z^2 \left[-A m^2 \cos(m\xi) - B m^2 \cos(3m\xi) + 2A m^3 \xi \sin(m\xi) +\right.$$

$$+6Bm^3\xi\sin(3m\xi)\Big]+4m^2\xi\,\widetilde{f}_0'''+6m^2\widetilde{f}_0''-4N^2m^2\xi\,\widetilde{f}_0'-2N^2m^2\widetilde{f}_0, \tag{14}$$

$$\varepsilon^1:\widetilde{\varphi}_1^{(4)}-2N^2\widetilde{\varphi}_1''+N^4\widetilde{\varphi}_1=12c^2z^2\Big[Am^2\cos(m\xi)+9Bm^2\cos(3m\xi)\Big]+2\widetilde{f}_0^{(4)}-4N^2\widetilde{f}_0''+2N^4\widetilde{f}_0. \tag{15}$$

We neglect the terms of higher order than ε and β^2. Omitting the details of calculations we obtain

$$\widetilde{f}_0(\xi)=a\cdot\cos(m\xi)+b\cdot\cos(3m\xi), \tag{16}$$

where
$$a=-\frac{4c^2z^2m^2}{\left(m^2+N^2\right)^2}A,\ b=-\frac{36c^2z^2m^2}{\left(9m^2+N^2\right)^2}B, \tag{17}$$

$$\widetilde{f}_1(\xi)=\left[\frac{8c^2z^2m^4\left(5N^2-m^2\right)}{\left(m^2+N^2\right)^4}-\frac{4c^2z^2m^2}{\left(m^2+N^2\right)^2}\right]\cdot A\cos(m\xi)+\left[\frac{8c^2z^2m^3\left(N^2-m^2\right)}{\left(m^2+N^2\right)^3}\right]\cdot A\xi\sin(m\xi)+$$

$$+\left[\frac{72c^2z^2m^4\left(5N^2-9m^2\right)}{\left(9m^2+N^2\right)^4}-\frac{4c^2z^2m^2}{\left(9m^2+N^2\right)^2}\right]\cdot B\cos(3m\xi)+\left[\frac{24c^2z^2m^3\left(N^2-9m^2\right)}{\left(9m^2+N^2\right)^3}\right]\cdot B\xi\sin(3m\xi), \tag{18}$$

$$\widetilde{\varphi}_1(\xi)=\frac{1}{\left(m^2+N^2\right)^2}\left[12c^2z^2m^2A+2m^4a+4N^2m^2a+2N^4a\right]\cos(m\xi)+$$

$$+\frac{1}{\left(9m^2+N^2\right)^2}\left[108c^2z^2m^2B+162m^4b+36N^2m^2b+2N^4b\right]\cos(3m\xi). \tag{19}$$

It should be noted that there are many coincident modes and the corresponding wave numbers n and p are located on the Koiter's circle. We consider that case when the buckling mode has the same wave numbers in both the axial and circumferential directions. Thus for this case we have relations

$$p=n=\frac{p_0}{2},\ p_0^2=2c\frac{R}{h_0},\ cz^2=2m^2. \tag{20}$$

We apply the Boobnov-Galerkin procedure to the equilibrium equation (7). In this connection it should be pointed out that in order to obtain analytical forms of all integrals in the solution we replace finite limits of integrals by infinite ones. It is possible if the process of deformation has the local character.

All integrals from (7) are the follows

$$\int_{-\infty}^{+\infty}\exp\left(-\beta^2\widetilde{\xi}^2\right)\cdot d\widetilde{\xi}=\frac{\sqrt{\pi}}{\beta},$$

$$\int_{-\infty}^{+\infty}\exp\left(-\beta^2\widetilde{\xi}^2\right)\cos\left(k\widetilde{\xi}\right)\cdot d\widetilde{\xi}=\frac{\sqrt{\pi}}{\beta}\exp\left(-\frac{k^2}{4\beta^2}\right), \tag{21}$$

$$\int_{-\infty}^{+\infty}\widetilde{\xi}\exp\left(-\beta^2\widetilde{\xi}^2\right)\sin\left(k\widetilde{\xi}\right)\cdot d\widetilde{\xi}=\frac{k\sqrt{\pi}}{2\beta^3}\exp\left(-\frac{k^2}{4\beta^2}\right).$$

The Boobnov-Galerkin procedure yields the eigenvalue problem

$$\left.\begin{aligned}a_{11}A+a_{12}B=0\\ a_{21}A+a_{22}B=0\end{aligned}\right\}, \tag{22}$$

where
$$a_{11}=4\sqrt{2}\left[\left(3-\beta^2\right)K^2-\left(1+\beta^2\right)-\alpha\left(1+K^2\right)\right]\lambda+4\sqrt{2}+4\sqrt{2}\beta^2+$$

$$+4\sqrt{2}\left(\beta^2-3\right)K^2-8\varepsilon(1+K),$$

$$a_{12}=4\sqrt{2}\left[\left(3-\beta^2\right)K^2+\left(15-\beta^2\right)K^8-\alpha\left(K^2+K^8\right)\right]\lambda+$$

$$+\left(27{,}6544\sqrt{2}\beta^2-65{,}248\sqrt{2}\right)K^2+\left(27{,}6544\sqrt{2}\beta^2-186{,}976\sqrt{2}\right)K^8-156{,}48\varepsilon\left(K^4+K\right), \tag{23}$$

$$a_{21}=4\sqrt{2}\left[\left(7-\beta^2\right)K^8-\left(5+\beta^2\right)K^2-\alpha\left(K^2+K^8\right)\right]\lambda+\left(4\sqrt{2}\beta^2+20\sqrt{2}\right)K^2+$$

$$+\left(4\sqrt{2}\beta^2-28\sqrt{2}\right)K^8-8\varepsilon\left(K+K^4\right),$$

$$a_{22}=4\sqrt{2}\left[\left(27-\beta^2\right)K^{18}-\left(9+\beta^2\right)-\alpha\left(1+K^{18}\right)\right]\lambda+$$

$$+\left(56{,}48\sqrt{2}+27{,}6544\sqrt{2}\beta^2\right)+\left(27{,}6544\sqrt{2}\beta^2-308{,}704\sqrt{2}\right)K^{18}-156{,}48\,\varepsilon\left(1+K^9\right),$$

$$K=exp\left(-\frac{1}{\beta^2}\right).$$

Using the requirement of nontrivial solution of the system (22) after simplification we obtain an asymptotic formula for the critical buckling load parameter λ

$$\lambda=1-\left[\frac{\sqrt{2}\varepsilon\left(1+K\right)+\alpha}{1+\beta^2+\alpha}\right]. \qquad (24)$$

As formula (24) shows the external pressure and the local thickness defects in the form (9) cause the reduction of critical compressive load.

3. ANALYSIS OF THE CRITICAL BUCKLING LOAD PARAMETER VALUES

Values of the critical buckling load parameter λ for different values of the external pressure and the thickness parameter ε are given in Figure 2.

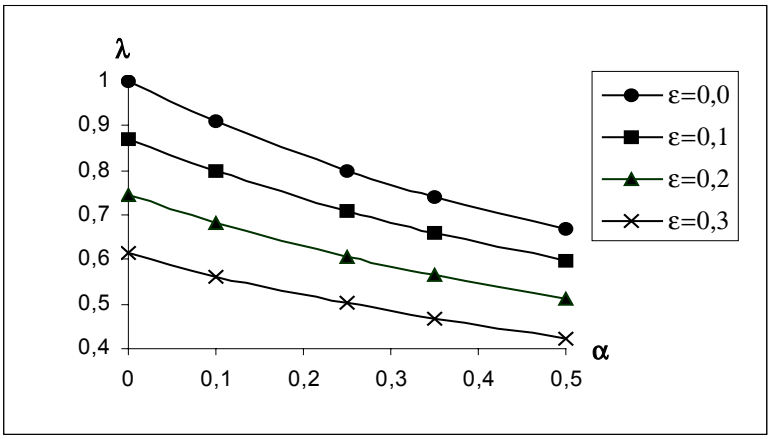


Figure 2. Reduction of the critical load parameter in the presence of the local thickness defects and the external pressure

The above results show that in situation when the defects of the shell thickness are absent ($\varepsilon=0$) the external pressure causes the reduction λ by 26% ($P_1\approx 35\%\,P_0$). In its turn, the local defects of thickness in the absence of an external pressure even if the amplitude of the thickness defects is 0,15 reduce the buckling load by 19% compared to the shell with constant thickness. So, the presence in the shell both the local thickness defects and the external pressure has more influence on the classical buckling load, reducing it by 53% ($\varepsilon=0{,}3$, $P_1\approx 35\%\,P_0$).

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