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A HYBRID PERTURBATION-GALERKIN TECHNIQUE WITH APPLICATION TO MIXED BOUNDARY VALUE PROBLEM

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1. INTRODUCTION

The problem of forced vibrations of thin-walled structures (plates and shells) could be reduced to the problem of searching of natural frequencies, i.e. eigenvalue problem. And further, knowing the eigenvalues makes it possible to obtain the forced solution in terms of these eigenvalues – the *spectral representation* or *modal expansion approach* [1]. Meanwhile, compared to large number of possible structure configurations (for instance, inhomogeneity of form or boundary conditions), very few exact solutions of plate and shell eigenvalue problem are possible, so that is why they are usually solved numerically using BEM, FEM etc. However, in many cases the asymptotic [2,3] and hybrid [4,5] approaches, provided they are accurate enough, enable to obtain the approximate analytical solutions with a rather high level of precision. That will be of great practical usefulness and especially effective in the case when machine computations are faced with difficulties.

Present paper deals with a two-step hybrid perturbation-Galerkin technique for the computation of the eigenfunctions and eigenfrequencies of natural oscillations of a round plate with inhomogeneous (mixed) boundary conditions. The basic idea of the method presented may be described as follows. In the first step, parameter ε is introduced into the boundary conditions in such a way that $\varepsilon = 0$ corresponds to the simple boundary problem and case $\varepsilon = 1$ corresponds to the problem under consideration. Then the ε-expansion of the solution is obtained. As a rule, just at point $\varepsilon = 1$ the ε-expansion of the solution is divergent. To remove this divergence, in step two we use a subset of the perturbation co-ordinate functions, determined in step one, in Galerkin type approximation. It is shown that the eigenfunctions and eigenfrequencies obtained by the hybrid technique using only a few terms from the perturbation solutions compare well with numerical results.

2. BASIC EQUATIONS

The differential equation that governs the natural non-axisymmetric vibrations of the circular plate with radius a subjected to mixed boundary conditions of the "simple support – clamping" kind along the edge of the plate may be written as follows

$$\nabla^2\nabla^2 u(x,\theta) - \lambda u(x,\theta) = 0\,, \qquad (1)$$

where $\lambda = \rho h\omega^2 a^4 D^{-1}$; ω is the natural frequency; $D = Eh^3/12(1-\nu^2)$; $x = r/a$; $u(x,\theta) = W(r,\theta)/a$; $\nabla^2 = \partial^2/\partial x^2 + x^{-1}\,\partial/\partial x + x^{-2}\,\partial^2/\partial\theta^2$ is the Laplace operator in polar co-ordinates; r – initial radial co-ordinate; $\theta \in [-\pi,\pi]$ – circumferential co-ordinate; $W(r,\theta)$ – initial mode shape.

Let us separate variables by writing $u(x,\theta) = \overline{u}(x)\cos(n\theta)$ where $n = 0,1,2,\ldots$.

In the first stage, the mixed boundary conditions of the type "simple support – clamping" with the help of so-called "combined" Heaviside function may be formalized as [2]

$$\begin{cases} \overline{u} = 0, \\ \left(1-\overline{H}(\theta,\mu)\right)\cdot M_x + \overline{H}(\theta,\mu)\cdot\overline{u}' = 0, \end{cases} \quad \text{when } x = 1\,, \qquad (2),\ (3)$$

where $M_x = \overline{u}'' + \nu\overline{u}'$ is a bending moment; $\overline{H}(\theta,\mu) = \sum_{i=1}^{p} H(\theta-\psi_i)\cdot H(\theta_{i+1}-\theta)$ with $H(\theta-\psi_i)$ and $H(\theta_{i+1}-\theta)$ as simple Heaviside functions; $\theta_i = -\pi + \frac{2\pi}{p}i$ and $\psi_i = -\pi + \frac{2\pi}{p}(i-1+\mu)$ with $i = 1,\ldots p$;

p – the number of the plate's boundary regions on which the mixed boundary conditions of the "simple support – clamping" kind take place; $\mu \in [0,1]$ – the distribution coefficient of two types boundary conditions at the i -th region.

Thus, according to (3), when $\mu = 0$ we have the plate clamped along the boundary; when $\mu = 1$ the plate is simply supported. The intermediate values of μ are related to mixed boundary conditions under consideration.

Eliminating the non-uniformity of function $\overline{H}(\theta,\mu)$ (we use for this purpose a routing asymptotic procedure) we obtain for this one the expression

$$\overline{H}(\theta,\mu) \cong f(\theta,\mu) = 1-\mu+\sum_{l=1}^{s}\left[(-1)^l \frac{2}{\pi l}\sin\frac{\pi l}{p}(1-\mu)\sum_{i=1}^{p}\cos\left(l\theta-\frac{\pi l}{p}(2i-1+\mu)\right)\right],$$

and further we'll use it instead of $\overline{H}(\theta,\mu)$.

In the second step, we introduce the parameter ε into the boundary conditions (3)

$$\overline{u}'' + \nu\overline{u}' = \varepsilon f(\theta,\mu)\{\overline{u}'' + (\nu-1)\overline{u}'\}, \qquad \text{when} \qquad x = 1. \tag{4}$$

The case $\varepsilon = 0$ brings us the plate simply supported along the boundary; the case $\varepsilon = 1$ corresponds to the problem under consideration (1) – (3). The intermediate values of ε are related to mixed boundary conditions too, but of the "simple support – elastic clamping" kind with elastic support coefficient $c = \varepsilon/(1-\varepsilon)$ at those regions of the plate's boundary, where earlier we assumed the clear clamping. It is easy to see if to rewrite (4) as

$$\overline{u}'' + \nu\overline{u}' = -\frac{\varepsilon f(\theta,\mu)}{1-\varepsilon f(\theta,\mu)}\overline{u}', \quad \text{when} \quad x = 1. \tag{5}$$

So we consider that the parameter μ relates to the distribution of the aforesaid mixed boundary conditions, and the parameter ε relates to theirs "quality".

3. A HYBRID PERTURBATION-GALERKIN TECHNIQUE

In step one of the method, eigenvalue λ and eigenfunction $\overline{u}(x)$ are presented by ε-based expansion

$$\lambda = \sum_{j=0}^{\infty}\lambda_j\varepsilon^j \quad \text{and} \quad \overline{u}(x) = \sum_{j=0}^{\infty}\overline{u}_j\varepsilon^j, \tag{6}$$

which are formally valid about "small" values of the parameter ε, meanwhile generally ε can be "large".

Substituting series (6) into the governing boundary problem (1), (2), (4) and splitting it with respect to powers of ε, yields the recurrent sequence of boundary problems.

For ε^0: $$\nabla_1^2\nabla_1^2\overline{u}_0 - \lambda_0\overline{u}_0 = 0, \tag{7}$$

$$\overline{u}_0 = 0 \quad \text{and} \quad \overline{u}_0'' + \nu\overline{u}_0' = 0, \quad \text{when} \quad x = 1. \tag{8}$$

For ε^j: $$\nabla_1^2\nabla_1^2\overline{u}_j - \lambda_0\overline{u}_j = \sum_{v=1}^{j}\lambda_v\overline{u}_{j-v}, \tag{9}$$

$$\overline{u}_j = 0 \quad \text{and} \quad \overline{u}_j'' + \nu\overline{u}_j' = -f(\theta,\mu)\sum_{v=0}^{j-1}\overline{u}_v', \quad \text{when} \quad x = 1, \tag{10}$$

$$j = 1,2,3,\ldots,$$

where $$\nabla_1^2 = \partial^2/\partial x^2 + x^{-1}\partial/\partial x - n^2x^{-2}\partial^2/\partial\theta^2.$$

The solution of the uniform differential equation (7) may be written as

$$\overline{u}_0(x) = A_0J_n(k_{nm}x) + B_0I_n(k_{nm}x), \tag{11}$$

where J_n, I_n – Bessel functions of the first kind of order n, $k_{nm} = \sqrt[4]{\lambda_0}$ (further defined as k) – roots of the characteristic equation

$$2kJ_n(k)I_n(k) + (\nu-1)\left(J_n(k)I_{n+1}(k) + J_{n+1}(k)I_n(k)\right) = 0.$$

Some first roots k (the Poisson ratio is chosen to be $\nu = 1/3$) are tabulated in Table 1, where n is the number of nodal diametral lines, m is the number of nodal concentric circles.

Table 1. Eigenvalues k

k	(n,m)	k	(n,m)
2.23245	0,0	6.96534	1,1
3.73359	1,0	7.54181	4,0
5.06484	2,0	8.37577	2,1
5.45511	0,1	8.61349	0,2
6.32419	3,0	8.73154	5,0

Constants A_0, B_0 in (11) may be taken in the form

$$A_0 = I_n(k)/\left(J_n^2(k) + I_n^2(k)\right) \text{ and } \quad B_0 = -J_n(k)/\left(J_n^2(k) + I_n^2(k)\right).$$

When we set $j = 1$ in (9), we can obtain the solution of the corresponding non-uniform differential equation as

$$\overline{u}_1(x) = \frac{\lambda_1}{2k^3}\int_0^x \left[J_n(kx)\frac{Y_n(k\xi)}{\delta_{1n}(\xi)} - Y_n(kx)\frac{J_n(k\xi)}{\delta_{1n}(\xi)} + I_n(kx)\frac{K_n(k\xi)}{\delta_{2n}(\xi)} - K_n(kx)\frac{I_n(k\xi)}{\delta_{2n}(\xi)}\right]\overline{u}_0(\xi)d\xi +$$

$$+ A_1 J_n(kx) + C_1 I_n(kx), \qquad (12)$$

where $\delta_{1n}(x) = J_{n+1}(kx)Y_n(kx) - J_n(kx)Y_{n+1}(kx), \quad \delta_{2n}(x) = I_{n+1}(kx)K_n(kx) + I_n(kx)K_{n+1}(kx)$,

Y_n, K_n – Bessel functions of the first and second kind of order n and A_1, C_1 – certain constants.

We notice that eigenfunction $\overline{u}_1(x)$ contains three unknowns λ_1 and A_1, C_1, meanwhile there are only two available boundary conditions (10) occur. Later we will eliminate such a discrepancy.

And now eigenfunction $\overline{u}(x)$ in form of a truncated perturbation expansion (6) (two terms of expansion) can be obtained as

$$\overline{u}(x) = \overline{u}_0(x) + \varepsilon\overline{u}_1(x). \qquad (13)$$

In step two of the hybrid method, we seek new approximate solution, using the coordinate functions $\overline{u}_j(x)$ determined in step one of the method, as

$$\widetilde{u}(x,\theta) = u_0(x,\theta) + \sum_{j=1}^{N-1} a_j u_j(x,\theta) = \left\{\overline{u}_0(x) + \sum_{j=1}^{N-1} a_j \overline{u}_j(x)\right\}\cos(n\theta), \qquad (14)$$

where a_j represents new unknown "amplitudes" of $\overline{u}_j(x)$.

Requiring that the residual is orthogonal to each $u_j(x,\theta)$, $0 \le j \le N-1$, according to the Galerkin technique, we may determine the amplitudes a_j, as well as $\widetilde{\lambda}$, from the condition

$$\iint_\Omega \left[\nabla^2\nabla^2\widetilde{u}(x,\theta) - \widetilde{\lambda}\widetilde{u}(x,\theta)\right]u_j(x,\theta)x d\Omega = 0, \text{ where } \Omega = [0,1]\times[-\pi,\pi]. \qquad (15)$$

Thus if we set $N = 2$ in (14) we may find new approximate solution for $\widetilde{\lambda}$ in the form

$$\widetilde{\lambda} = \lambda_0 + \frac{\iint_\Omega \overline{u}_0^2\cos^2(n\theta)x d\Omega \iint_\Omega \lambda_1\overline{u}_0\overline{u}_1\cos^2(n\theta)x d\Omega - \iint_\Omega \lambda_1\overline{u}_0^2\cos^2(n\theta)x d\Omega \iint_\Omega \overline{u}_0\overline{u}_1\cos^2(n\theta)x d\Omega}{\iint_\Omega \overline{u}_0^2\cos^2(n\theta)x d\Omega \iint_\Omega \overline{u}_1^2\cos^2(n\theta)x d\Omega - \left(\iint_\Omega \overline{u}_0\overline{u}_1\cos^2(n\theta)x d\Omega\right)^2}, \qquad (16)$$

where $\overline{u}_1(x) = \frac{\lambda_1}{2k^3}\left[S(x) - \frac{S(1)}{I_n(k)}I_n(kx)\right] + A_1\left[J_n(kx) - \frac{J_n(k)}{I_n(k)}I_n(kx)\right]$ could be obtained from the expression (12) after its subjecting to the first boundary condition (10),

and where $S(x)=\int_0^x\left[J_n(kx)\frac{Y_n(k\xi)}{\delta_{1n}(\xi)}-Y_n(kx)\frac{J_n(k\xi)}{\delta_{1n}(\xi)}+I_n(kx)\frac{K_n(k\xi)}{\delta_{2n}(\xi)}-K_n(kx)\frac{I_n(k\xi)}{\delta_{2n}(\xi)}\right]\overline{u}_0(\xi)d\xi$,

A_1 – unknown coefficient, which could be determined from the conditions $\begin{cases}\widetilde{\lambda}(\mu=0)=\lambda_{clamping}\\ \widetilde{\lambda}(\mu=1)=\lambda_0\end{cases}$,

$$\lambda_1=2k^3 f(\theta,\mu)\mathrm{P},$$

$$\mathrm{P}=\frac{\overline{u_0'}(1)J_n(k)}{S(1)\left[J_n''(k)+\nu J_n'(k)\right]-J_n(k)\left[S''(1)+\nu S'(1)\right]},$$

$\mu\in[0,1]$ – the distribution coefficient of the "simple support – clamping" boundary conditions at the i-th region.

The results of calculations are displayed in figure 1. Data obtained for $n=m=0$, $k=2.23245$ and $p=2$ by formula (16) are shown as continuous lines, and circles represent the numerical results obtained by the method of finite differences[6].

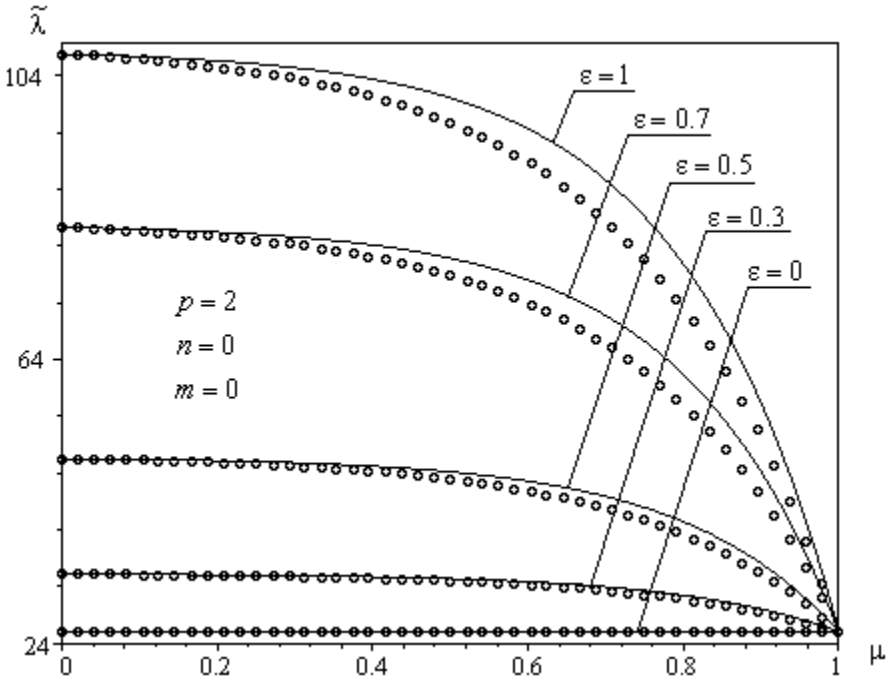


Figure 1.

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