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OPTIMIZATION METHODS

Міністерство освіти і науки України
Чернівецький національний університет
імені Юрія Федьковича

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OPTIMIZATION METHODS

Study guide

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МЕТОДИ ОПТИМІЗАЦІЇ

Навчальний посібник

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The textbook outlines the basic concepts of the course "Optimization Methods", which the author teaches to students of the Faculty of Mathematics and Informatics. In addition to the theoretical material, the textbook contains practical problems for each chapter.

The textbook will be useful for students and anyone who wants to get acquainted with the basic concepts of probability theory.

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INTRODUCTION

The concept of optimality, as well as the optimization process, is a central point not only in economics, engineering, management and business. It is also used in social, biological and many other sciences.

The term "optimal" is most often interpreted as favorable, maximum (minimum), most effective, etc. Each person every day (not always realizing it) solves the problem of obtaining the greatest effect with limited resources. Solving an optimization problem means finding the best solution among possible options.

If the optimization is related to the calculation of optimal parameter values for a given structure of the object, then it is called parametric. The task of choosing the optimal structure is structural optimization.

The solution of each optimization problem is based on the construction of a mathematical model of the object under study and conducting a computational experiment (not with the object itself, but with the model), which allows you to investigate its properties in arbitrary situations.

The theory of optimization is a set of fundamental mathematical results and numerical methods, which allows avoiding a complete search of all solutions.

Optimization methods are methods of building algorithms for finding the optimal value of some function.

The term "optimum" was introduced in the 18th century by Gottfried Wilhelm Leibniz. In the same century, there were a number of works by Daniel Bernoulli, Leonhard Euler and Joseph-Louis Lagrange, which were devoted to calculus of variations. Later, in the 19th century, Karl Weierstrass and Carl Jacobi were engaged in such tasks.

The first extremum search problems studied in detail were linear programming problems. Back in 1820, Jean Baptiste Fourier, and then Leonid Kantorovych in 1939, George Danzig in 1947 formulated the problem of linear programming and proposed a method for its solution. Later, Richard Bellman developed a dynamic programming method that allows solving problems for systems whose characteristics depend on time.

Lev Pontryagin made a significant improvement to mathematical programming and optimal control by developing the calculus of variations section.

Thus, by the 70s of the 20th century, the section of applied mathematics – theory and methods of optimization – was basically formed.

General formulation of the problem

Modeling of many practical operations, in which it is appropriate to use the apparatus of functions of many variables, is carried out in two stages.

1. The image of the pursued goal in the form of some dependence on the sought values (income from the sale of manufactured products, costs for the performance of certain work, etc.). The resulting expression is called the objective function.

2. Formation of conditions for the sought quantities. They arise from available resources, from the need to satisfy certain needs, from the conditions of technology, and other economic and technical factors. As a rule, these conditions are equalities or inequalities. They form a system of restrictions.

If the objective function expresses a positive factor, then it should be maximized, otherwise it should be minimized.

In general, the problem can be written as follows:

$$f\left(x_1, x_2, \ldots, x_n\right) \rightarrow extr; \qquad (1)$$

$$\begin{cases} \varphi_i\left(x_1, x_2, \ldots, x_n\right) \le b_i, \quad i = 1, \ldots, m_1, & (2) \\ \varphi_i\left(x_1, x_2, \ldots, x_n\right) = b_i, \quad i = m_1 + 1, \ldots, m, & (3) \end{cases}$$

where f and φ_i, $i = 1, \ldots, m$, – are given functions, b_i, $i = 1, \ldots, m$, – are given numbers.

Solving the problem (1)-(3) means finding values of, which would satisfy the system of constraints (2)-(3), at which the objective function from (1) would reach an extreme value.

The set D of such values $X=\left(x_1, x_2,..., x_n\right)$ for which (2)-(3) is fulfilled is called an admissible set, and the points X of this set are admissible solutions of the problem.

$$D=\left\{X=\left(x_1, x_2,..., x_n\right) \mid \varphi_i\left(X\right) \le b_i,\ i=1,...,m_1,\ \varphi_i\left(X\right)=b_i,\ i=m_1+1,...,m\right\}.$$

The point $X^*=\left(x_1^*, x_2^*,..., x_n^*\right)\in D$, at which the function f from (1) reaches an extreme value is called the optimal solution of problem (1)-(3), and the value of the function at this point $f\left(X^*\right)$ is called an extreme value.

To solve the problem (1)-(3) means to find an optimal solution X^*, or to make sure that f is unbounded ($f\to\pm\infty$), or to show that $D=\varnothing$.

CHAPTER 1

LINEAR PROGRAMMING

§1. Examples of linear programming problems

L.V. Kantorovych in 1939, engaged in planning the work of units of a plywood factory, solved several such problems: about the best loading of equipment, about cutting materials with the lowest costs, etc. With this, he formed a new class of conditionally extremal problems – linear programming problems.

Most problems of operational long-term planning of an enterprise or industry, problems of optimal concentrations, transportation, and many others can be reduced to linear programming problems.

Item 1.1. The task of optimal production planning

Consider the activity of some production facility (factory, workshop, etc.). Let the facility have *m* types of resources $S_1, S_2, ..., S_m$ at its disposal, from which the facility produces *n* types of products $P_1, P_2, ..., P_n$. Let's assume that:

1) a_{ij}, $i = 1,...,m$, $j = 1,...,n$ – is the number of units of resource S_i, $i = 1,...,m$, which is required for the production of one unit of product P_j, $j = 1,...,n$.

2) b_i, $i = 1,...,m$ – is the number of units of the available S_i, $i = 1,...,m$ resource.

3) c_j, $j = 1,...,n$ – is the price of one unit of production P_j, $j = 1,...,n$.

To build a mathematical model of the problem of determining such a production plan, which would maximize the total cost of manufactured products under the given conditions.

◄ Let's make a mathematical model of the problem. Let x_j, $j = 1,...,n$, be the number of production units P_j in the plan. It is obvious that $x_j \geq 0$. With such a production plan, the cost f of products manufactured at the facility is equal to

$$f = c_1x_1 + c_2x_2 + ... + c_nx_n = \sum_{j=1}^{n} c_jx_j .$$

The costs of raw material S_i for the production of products are equal to

$$\sum_{j=1}^{n} a_{ij}x_j , \quad i = 1,...,m .$$

Taking into account the limited resources and the purpose of production, we will get the following task:

$$f = \sum_{j=1}^{n} c_jx_j \rightarrow \max ;$$

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i , \quad i = 1,...,m ;$$

$$x_j \geq 0 , \quad j = 1,...,n .$$

This is a linear programming problem. It contains *n* unknowns and *m* constraints. ►

Item 1.2. Diet problem

This is the task of planning the most economical diet that meets medical requirements. Such a problem arises, as a rule, when it is necessary to feed a large number of people (army, sanatorium, etc.).

The food ration consists of n types of products $P_1, P_2, ..., P_n$. It is known that a unit of product P_j contains a_{ij} units of nutrient R_i, $i = 1,...,m$, $j = 1,...,n$. Let c_j, $j = 1,...,n$ be the price of one unit of product P_j, $j = 1,...,n$, b_i, $i = 1,...,m$ be the minimum consumption rate of nutrient R_i, $i = 1,...,m$ in the diet. With the specified restrictions, build a mathematical model of the problem of determining the minimum cost ration.

◀ Let x_j, $j = 1,...,n$ be the number of units of product P_j in the diet. The cost f of such a ration is $f = \sum_{j=1}^{n} c_j x_j$. The substance R_i in such a ration is $\sum_{j=1}^{n} a_{ij} x_j$ units. Taking into account the goal and medical requirements, we get the following linear programming problem:

$$f = \sum_{j=1}^{n} c_j x_j \to \min;$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = 1,...,m;$$

$$x_j \geq 0, \quad j = 1,...,n. \quad \blacktriangleright$$

Item 1.3. Transport problem

The process of production and consumption of homogeneous products is considered. There are m points $P_1, P_2, ..., P_m$ that produce homogeneous products in quantities a_i, $i = 1, ..., m$, respectively. The produced products are consumed in n points $Q_1, Q_2, ..., Q_n$, respectively, in quantities b_j, $j = 1, ..., n$.

Let c_{ij}, $i = 1, ..., m$, $j = 1, ..., n$ be the cost of transportation of a product unit from the point of production P_i to the point of consumption Q_j. Build a model for finding such a product transportation plan that would satisfy consumer needs while minimizing transportation costs.

◄ Let x_{ij} be the number of cargo units transported from the point of production P_i to the point of consumption Q_j $i = 1, ..., m$, $j = 1, ..., n$. At the same time, the total costs f for cargo transportation are equal to:

$$f = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij} .$$

$\sum_{j=1}^{n} x_{ij}$ units of cargo will be exported from production point P_i. $\sum_{i=1}^{m} x_{ij}$ units of cargo will be delivered to the point of consumption Q_j. It is clear that

$$\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j .$$

We received the following task:

$$f = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \rightarrow \min;$$

$$\begin{cases} \sum_{j=1}^{n} x_{ij} \le a_i, \ i = 1,\dots,m, \\ \sum_{i=1}^{m} x_{ij} \ge b_j, \ j = 1,\dots,n; \end{cases}$$

$$x_{ij} \ge 0, \ i = 1,\dots,m, \ j = 1,\dots,n. \ ▶$$

§2. Different forms of recording linear programming problems, their equivalence

In the most general case, a linear programming problem has the following form

$$f = \sum_{j=1}^{n} c_j x_j \ \rightarrow \text{extr}; \quad (4)$$

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1,\dots,m_1, & (5_1) \\ \sum_{j=1}^{n} a_{ij} x_j \ge b_i, \ i = m_1 + 1,\dots,m_2, & (5_2) \\ \sum_{j=1}^{n} a_{ij} x_j = b_i, \ i = m_2 + 1,\dots,m, & (5_3) \end{cases}$$

additional conditions on variables:

$$\begin{aligned} x_j \ge 0, \quad j = 1,\dots,n_1, \\ x_j \le 0, \quad j = n_1 + 1,\dots,n_2, \end{aligned} \quad (6)$$

here x_j for $j = n_2 + 1,\dots,n$ can be arbitrary.

In this problem, the constraints have the form of equalities or inequalities, no additional conditions are imposed on all variables.

To solve the problem (4)-(6) means to find such a plan $X^* = \left(x_1^*, x_2^*,\dots, x_n^*\right)$ that satisfies the constraint (5) under the conditions (6) in which the objective function from (4) reaches its optimal value or to prove its non-existence, that is, that the system of constraints (5) is not compatible $(D = \varnothing)$ or that the objective function is unbounded $(f \rightarrow \pm\infty)$.

If the linear programming problem is written in the form

$$f = \sum_{j=1}^{n} c_j x_j \to \max; \quad (7)$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1,\dots,m, \quad (8)$$

$$x_j \ge 0, \quad j = 1,\dots,n, \quad (9)$$

then it is said that it is written in a **symmetrical form**.

At the same time, the number of constraints *m* and the number of unknowns *n* are not related in any way.

If the linear programming problem has the form

$$f = \sum_{j=1}^{n} c_j x_j \to \max; \quad (10)$$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1,\dots,m; \quad (11)$$

$$x_j \ge 0, \quad j = 1,\dots,n, \quad (12)$$

then they say that it is **canonical (standard)**.

In problem (10)-(12), the number of equations *m* is less than the number of unknowns *n*. In the future, without limiting the generality, we will assume that the rank of the matrix of coefficients of system (11) is equal to *m*.

It is obvious that the symmetric problem (7)-(9) and the canonical problem (10)-(12) are partial cases of the general problem (4)-(6).

Consider matrices

$$C=(c_1,c_2,...,c_n),\ X=\begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix},\ A=\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & . & . & . \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix},\ A_0=\begin{pmatrix} b_1 \\ b_2 \\ ... \\ b_m \end{pmatrix}.$$

The canonical problem in matrix form has the form

$$\begin{aligned} & f=C\cdot X\to \max; \\ & AX=A_0, \\ & X\geq 0. \end{aligned} \tag{13}$$

In the future, we will assume that $rangA=rang\bar{A}=m<n$.

An arbitrary linear programming problem can be reduced to one of the above three forms using the following equivalent transformations:

1) transition from minimization of the objective function f to maximization of the function $-f$;
2) the transition from the inequality with the sign $\geq$ to the inequality with the sign $\leq$ is carried out by multiplying the first by –1 and vice versa:

$$\text{from } \sum_{j=1}^{n} a_{rj}x_j\geq b_r \quad \text{to } -\sum_{j=1}^{n} a_{rj}x_j\leq -b_r;$$

3) the transition from inequality with the sign $\geq$ to equality is carried out by subtracting the auxiliary variable $x_{n+1}\geq 0$:

$$\text{from } \sum_{j=1}^{n} a_{rj}x_j\geq b_r \quad \text{to } \sum_{j=1}^{n} a_{rj}x_j-x_{n+1}=b_r;$$

4) the transition from inequality with sign $\leq$ to equality is carried out by adding the auxiliary varible $x_{n+1}\geq 0$:

$$\text{from } \sum_{j=1}^{n} a_{rj} x_j \le b_r \quad \text{to} \quad \sum_{j=1}^{n} a_{rj} x_j + x_{n+1} = b_r;$$

5) transition from the equality $\sum_{j=1}^{n} a_{rj} x_j = b_r$ to the system of inequalities:

$$\begin{cases} \sum_{j=1}^{n} a_{rj} x_j \ge b_r, \\ \sum_{j=1}^{n} a_{rj} x_j \le b_r; \end{cases}$$

6) the transition from an additional restriction of the type $x_j \le 0$ to a restriction of the type $x'_j \ge 0$ is carried out by replacing $x'_j = -x_j$;

7) the transition from the variable x_j, the sign of which is not subject to conditions, to the non-negative variables $x'_j \ge 0$, $x''_j \ge 0$ is carried out by replacing $x_j = x'_j - x''_j$;

8) the transition from the variable $x_j \ge d_j$ to $x'_j \ge 0$ is carried out by replacing $x'_j = x_j - d_j$.

Remark. *The problem in the canonical form can be reduced to a symmetric form by reducing the number of variables.*

Example 1. Reduce the problem to a symmetrical form

$$f = 27 + x_1 - 3x_2 + 2x_4 \to \max;$$

$$\begin{cases} 2x_1 + x_2 + x_3 - 3x_4 = 7, \\ 3x_1 + x_2 + 2x_3 + x_4 = 5; \end{cases}$$

$$x_j \ge 0, \quad j = 1, \dots, 4.$$

◄ *The first way*. Let's move from each equality to a pair of inequalities according to 5), and then, using 2), we will get the problem in a symmetrical form:

$$f = 27 + x_1 - 3x_2 + 2x_4 \to \max;$$

$$\begin{cases} 2x_1 + x_2 + x_3 - 3x_4 \le 7, \\ -2x_1 - x_2 - x_3 + 3x_4 \le -7, \\ 3x_1 + x_2 + 2x_3 + x_4 \le 5, \\ -3x_1 - x_2 - 2x_3 - x_4 \le -5; \end{cases}$$

$$x_j \ge 0, \quad j = 1,\dots,4,$$

which contains four (more than the initial) constraints.

The second way. The problem under consideration has a canonical form. Let's reduce it to symmetric by reducing the number of variables. Let's make the transformation

	A_1	A_2	A_3	A_4	A_0
	2	[1]	1	–3	7
	3	1	2	1	5
x_2	2	1	1	–3	7
	1	0	[1]	4	–2
x_2	1	1	0	–7	9
x_3	1	0	1	4	–2

We received the following task:

$$f = 27 + x_1 - 3x_2 + 2x_4 \to \max;$$

$$\begin{cases} x_1 + x_2 \qquad - 7x_4 = 9, \\ x_1 \qquad + x_3 + 4x_4 = -2; \end{cases}$$

$$x_j \ge 0, \quad j = 1,\dots,4.$$

From the equations of the obtained system, we find

$$\begin{cases} x_2 = 9 - x_1 + 7x_4, \\ x_3 = -2 - x_1 - 4x_4. \end{cases}$$

Then, given that $x_2 \geq 0$, we have $9 - x_1 + 7x_4 \geq 0$. Hence $x_1 - 7x_4 \leq 9$. Similarly, given that $x_3 \geq 0$, we have $-2 - x_1 - 4x_4 \geq 0$ and $x_1 + 4x_4 \leq -2$. In the constraints, x_1 and x_4 remained unknown. Express x_2 and x_3 in terms of x_1 and x_4 in the objective function *f*:

$$f = 27 + x_1 - 3(9 - x_1 + 7x_4) + 2x_4 = 4x_1 - 19x_4.$$

We received the following task:

$$f = 4x_1 - 19x_4 \to \max;$$

$$\begin{cases} x_1 - 7x_4 \leq 9, \\ x_1 + 4x_4 \leq -2; \end{cases}$$

$$x_1 \geq 0, \quad x_4 \geq 0.$$

This problem has a symmetrical form and contains two unknowns.

§3. Properties of linear programming problems

Item 3.1. Properties of the admissible set

Let $X^i = \left(x_1^i, x_2^i, ..., x_n^i\right) \in \mathbb{R}^n$, $i = 1,...,r$, $r \geq 2$.

Definition. *A set of points of the form* $X = \sum_{i=1}^{r} \lambda_i X^i$, *where* $\lambda_i \geq 0$, $i = 1,...,r$ *and* $\sum_{i=1}^{r} \lambda_i = 1$, *is called* ***a convex linear envelope*** *of points* $X^1, X^2, ..., X^r$.

A point of a convex linear envelope is called **a convex linear combination** of points $X^1, X^2, ..., X^r$.

At $r = 2$, the convex linear envelope is the segment connecting the points X^1 and X^2. Then the points of the segment

$$X = \lambda X^1 + \left(1 - \lambda\right) X^2, \quad \lambda \in [0,1]. \tag{14}$$

Definition. *A* ***set*** *is called* ***convex*** *if, together with its arbitrary two points* X^1 *and* X^2, *it contains all points of the form (14), that is, the entire line segment.*

We recall that the set of points $X \in \mathbb{R}^n$ for which (5_1) or (5_2) are fulfilled is called a half-space in $\mathbb{R}^n$, and in the case of fulfilling (5_3) – a hyperplane in $\mathbb{R}^n$.

It is easy to show that:

1) the intersection of convex sets is a convex set;
2) the half-space is a convex set;
3) the hyperplane is a convex set.

Definition. *The intersection of a finite number of half-spaces is called a* ***polyhedral set****.*

A bounded polyhedral set is called a polyhedron.

It is clear that the admissible set D of the linear programming problem, if it is not empty, is a polyhedral set.

Theorem 1 follows from the above.

Theorem 1. *The admissible set D of the linear programming problem is a convex polyhedral set.*

A point X^* of a convex set Ω is called **a corner (extreme) point** if it cannot be represented in the form

$$X^* = \lambda X^1 + (1-\lambda) X^2, \;\; \lambda \in (0,1), \;\; X^1 \in \Omega, \;\; X^2 \in \Omega, \;\; X^1 \neq X^2 .$$

For geometric reasons, a point is an extreme point of a set if it cannot be located inside the segment whose endpoints belong to this set.

The corner points of a convex polyhedral set are called its **vertices**.

Item 3.2. Properties of linear programming problem solutions

Let us consider some theorems reflecting the fundamental properties of linear programming problems. For simplicity, consider the case when *D* is a polyhedron.

Theorem 2. *The objective function of the linear programming problem reaches an optimal value at the vertex of the solution polyhedron. If this value is reached at two or more points, then it reaches the same value at any point that is a convex linear combination of them.*

Therefore, the solutions of the linear programming problem should be sought among the vertices of its admissible set. However, in the general case, this is difficult to do due to the large number of vertices and the difficulty of finding them.

Consider the canonical problem in the vector form

$$f = (C, X) \to \max;$$
$$A_1 x_1 + \ldots + A_n x_n = A_0;$$
$$X \geq 0,$$

where the matrices $C = (c_1 \ c_2 \ldots c_n)$, $X = (x_1 \quad x_2 \quad \ldots \quad x_n)^T$,
$A_j = (a_{1j} \quad a_{2j} \quad \ldots \quad a_{mj})^T$, $j = 1, \ldots, n$, $A_0 = (b_1 \quad b_2 \quad \ldots \quad b_m)^T$,
(C, X) – is the scalar product of vectors *C* and *X*.

Definition*. A nonzero admissible* **solution** *X of a linear programming problem is called* **basic** *if the system of vectors* A_j *corresponding to the positive components* x_j *of this solution is linearly independent. We will always consider the zero admissible solution as the base solution.*

Since the rank of the matrix of coefficients of the system of constraints is equal to the rank of the expanded matrix and is equal to *m*, $m < n$, then the maximum number of positive components of the solution of the linear programming problem is equal to *m*.

Definition. *A basic solution is called* ***nondegenerate*** *if it contains exactly m positive components, and* ***degenerate*** *if the number of positive components is less than m.*

For example, for the system

$$\begin{cases} x_1 \quad + 2x_3 + x_4 = 7, \\ \quad x_2 + x_3 - 3x_4 = 2, \end{cases}$$

the solution $(7;\ 2;\ 0;\ 0)$ is a nondegenerate basic solution. Appropriate vectors $A_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form the basis. Variables x_1, x_2 – are basic, x_3, x_4 – are non-basic.

Theorem 3. *An admissible solution X of a linear programming problem is a vertex of its admissible set D if and only if X is a basic solution.*

This theorem makes it possible to formalize (and thereby simplify) the process of transition from one vertex to another.

§4. Geometric interpretation of linear programming problems

Consider the problem of linear programming with two variables:

$$f = c_1 x_1 + c_2 x_2 \to \text{extr} \tag{15}$$

$$a_{i1} x_1 + a_{i2} x_2 \le b_i, \quad i = 1, \ldots, m, \tag{16}$$

$$x_1 \ge 0, \quad x_2 \ge 0. \tag{17}$$

This problem has a simple geometric interpretation. Each of the inequalities (16) in $\mathbb{R}^2$ defines a half-plane. The domain of admissible values given by (16) and (17) is a convex polygon called the solution polygon. The solution of problem (15)-(17), if it exists, is among the vertices of this polygon.

Definition*. The* ***level line*** *of a function is a set of points from its definition area in which the function reaches the same fixed value*.

For the linear function (15) of two variables, the level line is determined by the equality

$$c_1 x_1 + c_2 x_2 = h, \quad h = \text{const}. \tag{18}$$

Level lines (18) are a family of parallel straight lines with normal vector $\vec{n} = (c_1, c_2)$.

Recall that the gradient of the function f is the vector $\text{grad}\, f = \left(\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2} \right)$, which indicates the direction of the fastest growth of the function and is oriented perpendicular to the level line. Therefore, the value of h from (18) increases if these lines move in the direction of $\vec{n}$. At the same time, either a moment will come when the line will become a reference (area D is located on one side of it and has at least one point in common with D) or it will become clear that no line can be a reference. If the reference line exists, then the optimal solutions will be those points from D that lie on the reference line.

Remark. *When solving linear programming problems, the following cases are possible:*

1) One point $X^* \in D$ belongs to the reference line. This means that the problem has a unique solution (Fig. 1).

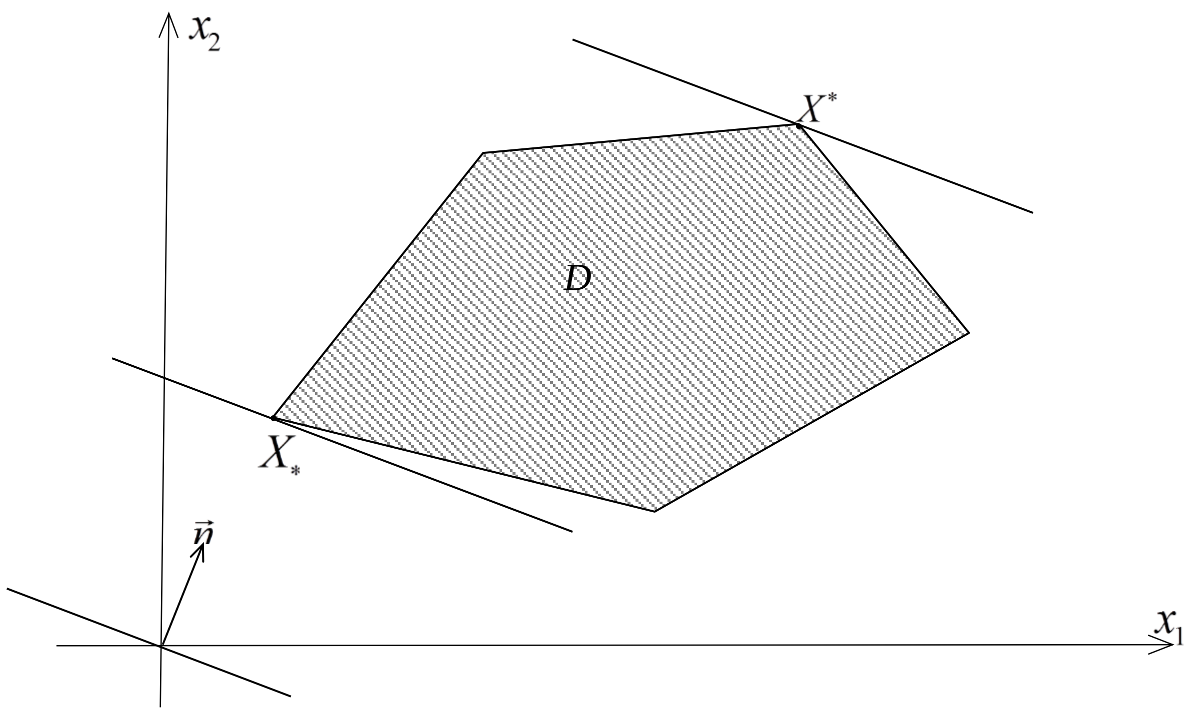


Fig. 1. A situation when the range of permissible values is limited and the problem has only one maximum and minimum points

If the function is examined for the maximum, then the line of the highest level is the reference straight line that passes through the point X^*, then $f_{\max} = f\left(X^*\right)$.

If the function is examined for a minimum, then the line of lowest level is the reference line that passes through the point X_*, then $f_{\min} = f\left(X_*\right)$.

2) The reference line includes an infinite set of points $X \in D$, which coincides with one of the sides of the set *D* (Fig. 2, Fig. 3).

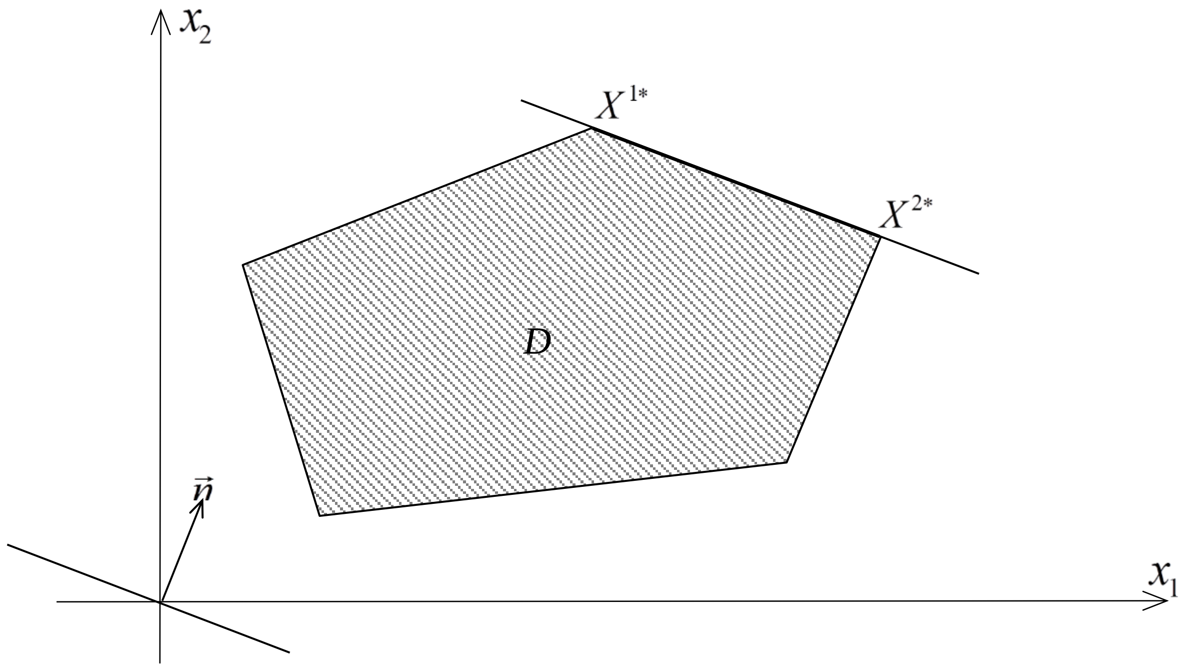


Fig. 2. The case of a limited range of admissible values, and the maximum is reached on the segment

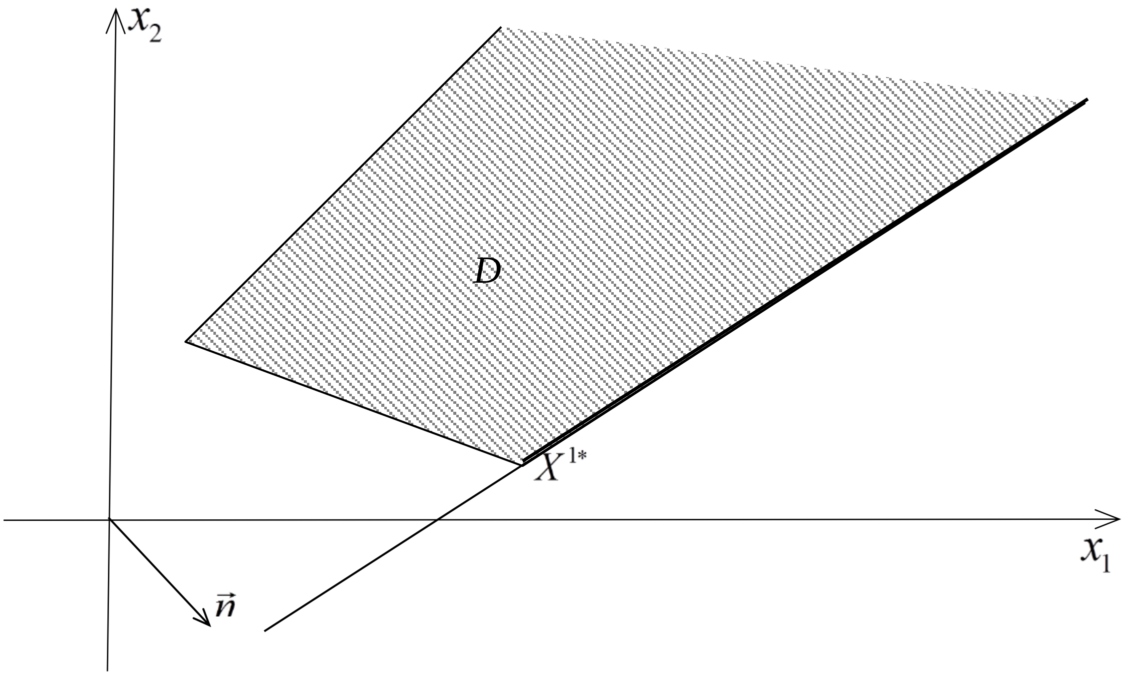


Fig. 3. The case of an unbounded domain, moreover the maximum value is reached on the beam

3) There is no reference line. In this case, the level line moving in the direction of $\vec{n}$ (for the problem at the maximum) or in the direction opposite to $\vec{n}$ (for the problem at the minimum) constantly crosses the polygon of solutions (Fig. 4).

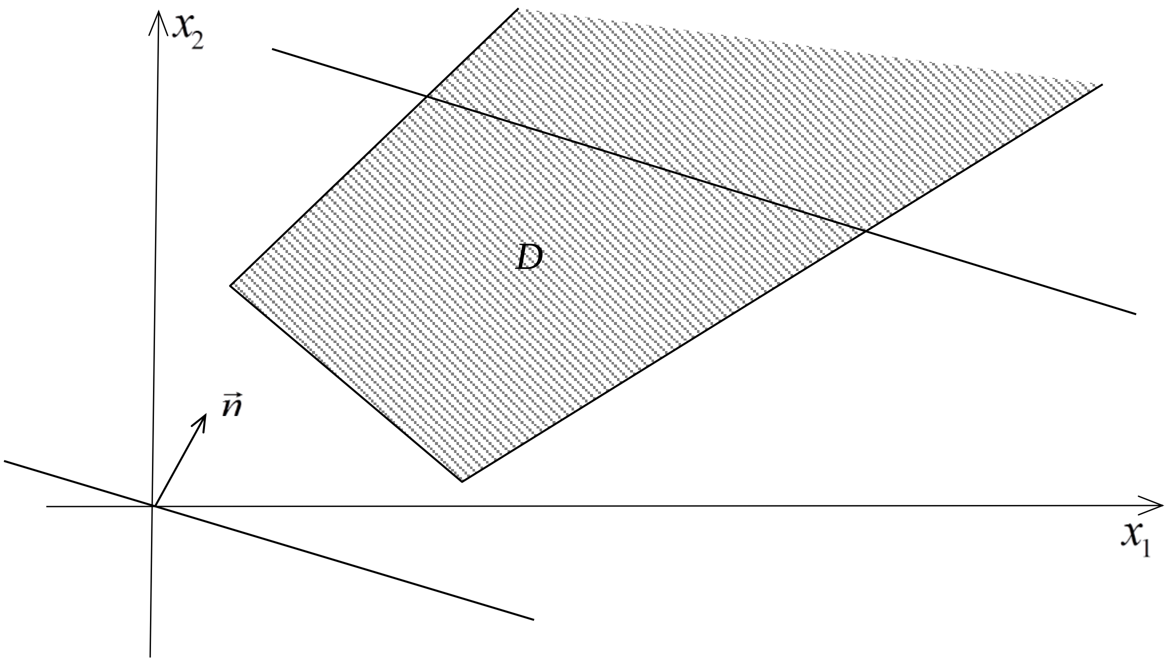


Fig. 4. The case of an unlimited range of admissible values, on which the function is unbounded from above

In this case, the objective function is said to be unbounded from above (for a maximum problem) $f_{\max} \to +\infty$ or unbounded from below (for a minimum problem) $f_{\min} \to -\infty$.

4) $D = \varnothing$. Then the problem has no solutions.

Example 2. Solve the problem of linear programming

$$f = x_1 + 2x_2 \to \text{extr};$$

$$\begin{cases} -5x_1 + 2x_2 \le 10, \\ x_1 + 2x_2 \ge 6, \\ 2x_1 + 9x_2 \ge 18; \end{cases}$$

$$x_1 \ge 0, \ \ x_2 \ge 0.$$

◀ The problem contains two variables. Therefore, let's break it down graphically (Fig. 5).

1. In $\mathbb{R}^2$, we construct a polygon of solutions (region D of admissible values).

2. We build lines of level $x_1 + 2x_2 = h$, $h = \text{const}$, normal line vector of level $\vec{n} = (1, 2)$. We draw a reference line. The line of the lowest level is the reference line that passes through the segment $\left[X_1^*, X_2^*\right]$. Let's find the coordinates of points X_1^* and X_2^*.

X_1^* – the point of intersection of the line II with the axis Oy – $X_1^*(0, 3)$.

X_2^* – is the point of intersection of lines II and III, its coordinates are the solution of the system of equations

$$\begin{cases} x_1 + 2x_2 = 6, \\ 2x_1 + 9x_2 = 18, \end{cases} \quad \text{whence} \quad \begin{cases} x_1 = \dfrac{18}{5}, \\ x_2 = \dfrac{6}{5}. \end{cases}$$

Therefore $X_2^* = \left(\dfrac{18}{5}, \dfrac{6}{5}\right)$.

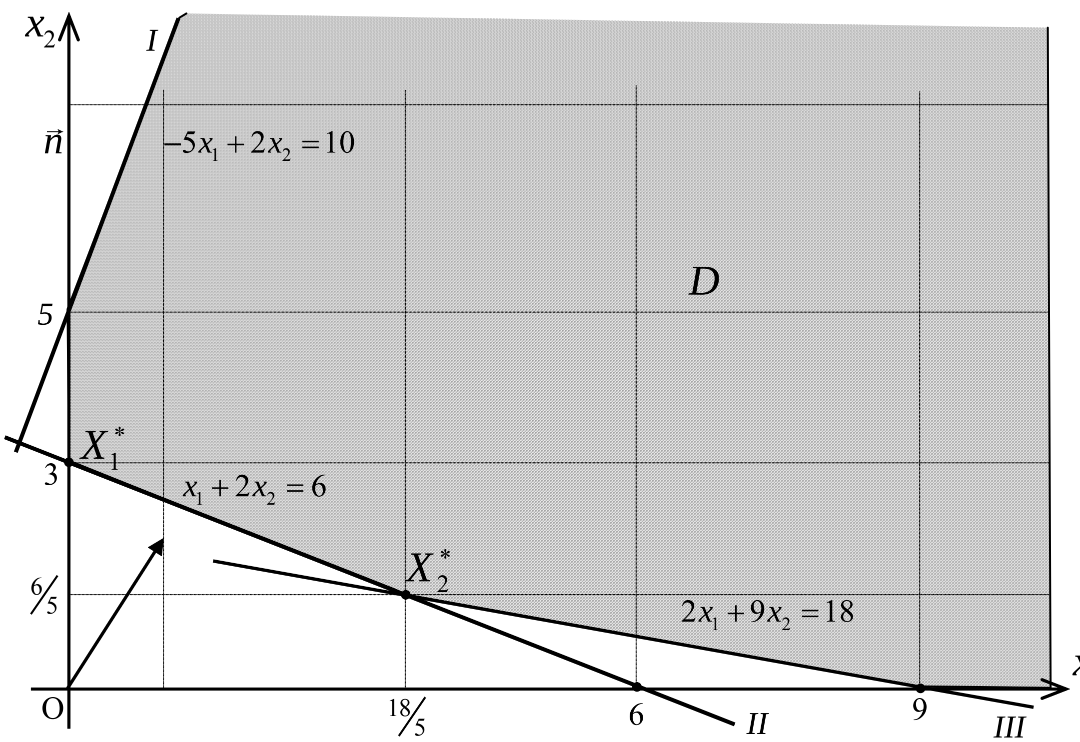


Fig. 5

The segment points can be written in the form

$$X_* = \lambda X_1^* + (1-\lambda) X_2^*, \quad \lambda \in [0,1],$$

where

$$X_* = \left(0 + (1-\lambda)\frac{18}{5}; 3\lambda + (1-\lambda)\frac{6}{5} \right),$$

$$X_* = \left(\frac{18}{5} - \frac{18}{5}\lambda; \frac{6}{5} + \frac{9}{5}\lambda \right), \ \lambda \in [0,1].$$

The value of the function $f_{\min} = f\left(X_*\right) = f\left(X_1^*\right) = 6$.

When moving the level line in the direction $\vec{n}$, no line will be a reference in the problem to the maximum. That is, there will be no line of the highest level. This means that the objective function is unbounded from above, i.e. $f \to +\infty$. ►

Example 3. Solve the linear programming problem from example 1:

$$f = 27 + x_1 - 3x_2 + 2x_4 \to \max;$$

$$\begin{cases} 2x_1 + x_2 + x_3 - 3x_4 = 7, \\ 3x_1 + x_2 + 2x_3 + x_4 = 5, \end{cases}$$

$$x_j \ge 0, \quad j = 1,\dots,4.$$

◄ Having a canonical problem with $n = 4$, $m = 2$, we reduce it to a problem with two variables, discarding the basic variables,

$$f = 4x_1 - 19x_4 \to \max;$$

$$\begin{cases} x_1 - 7x_4 \le 9, \\ x_1 + 4x_4 \le -2, \end{cases}$$

$$x_1 \ge 0, \ x_4 \ge 0.$$

Let's solve it graphically (Fig. 6).

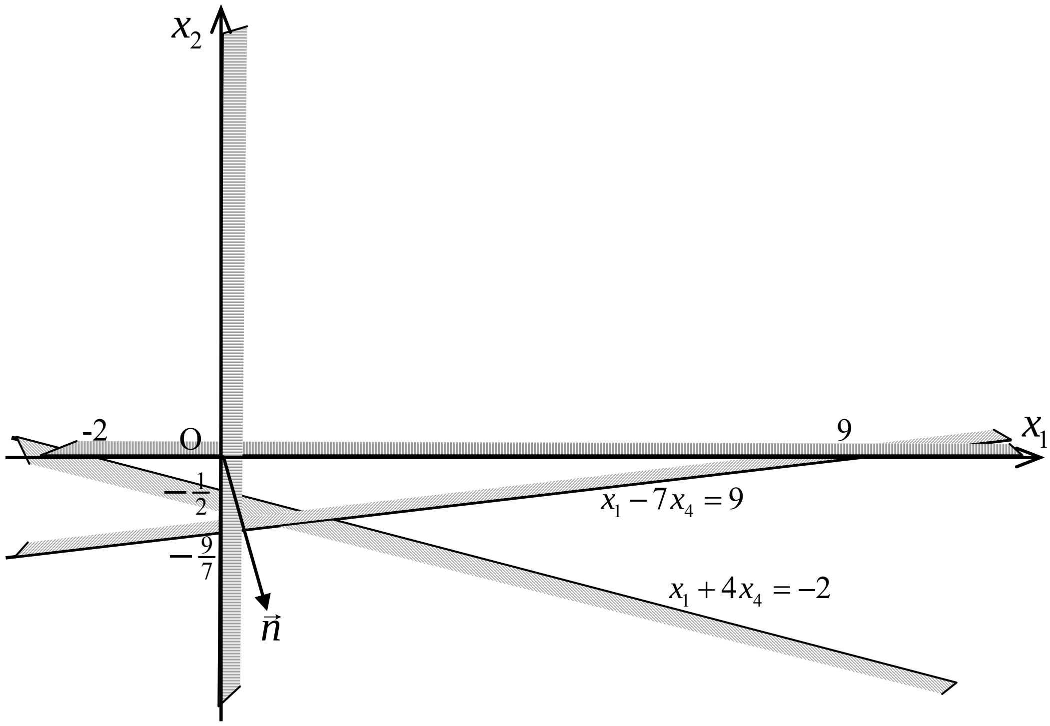


Fig. 6.

The range of valid values is empty. Therefore, the problem with two variables has no solutions. And therefore, the original problem has no solutions. ►

§5. The simplex method for solving linear programming problems

The search for solutions to linear programming problems, based on the properties of such problems, is reduced, at the principle level, to a sequential selection of extreme points of a set of admissible plans, or, what is the same, a selection of admissible basic plans. Such a selection for real multidimensional problems is only theoretically possible.

The means of solving such problems have become applied optimization methods, which are based on a consistent purposeful search of the basic plans of the problems.

The simplex method, developed in 1947 by the American mathematician George Danzig, became the classical method of solving the problem of linear programming.

Item 5.1. Basics of the simplex method

Consider the canonical problem of linear programming

$$f = \sum_{j=1}^{n} c_j x_j \to \max; \tag{19}$$

$$x_i + \sum_{j=m+1}^{n} a_{ij} x_j = b_i, \quad i = 1,...,m, \tag{20}$$

$$x_j \ge 0, \quad j = 1,...,n, \tag{21}$$

where $b_1 \ge 0,\ b_2 \ge 0, \ldots,\ b_m \ge 0$. Here, the variables $x_1,\ x_2, \ldots,\ x_m$ are basic, $x_{m+1}, \ldots,\ x_n$ are non-basic. Let's write (20) in expanded form

$$\begin{cases} x_1 \qquad + a_{1m+1}x_{m+1} + a_{1m+2}x_{m+2} + ... + a_{1s}x_s + ... + a_{1n}x_n = b_1, \\ \quad x_2 \qquad + a_{2m+1}x_{m+1} + a_{2m+2}x_{m+2} + ... + a_{2s}x_s + ... + a_{2n}x_n = b_2, \\ \ldots \\ \qquad x_r \quad + a_{rm+1}x_{m+1} + a_{rm+2}x_{m+2} + ... + a_{rs}x_s + ... + a_{rn}x_n = b_r, \\ \ldots \\ \qquad\quad x_m + a_{mm+1}x_{m+1} + a_{mm+2}x_{m+2} + ... + a_{ms}x_s + ... + a_{mn}x_n = b_m. \end{cases} \tag{20}$$

Problem (19)-(21) has an admissible basic (reference) solution:

$$X = \left(b_1, b_2, ..., b_m, 0, ..., 0\right). \tag{22}$$

Let's move from the basic solution (22) to the new one

$$X' = \left(b_1', b_2', ..., b_{r-1}', 0, b_{r+1}', ..., b_m', 0, ..., 0, b_s', 0, ..., 0\right),\ s \in \{m+1, ..., n\}. \tag{23}$$

To do this, we will perform the following transformations: divide the *r*-th equation by the coefficient $a_{rs} \ne 0$, and subtract the *r*-th multiplied by $\dfrac{a_{is}}{a_{rs}}$ from the other *i*-th equations. Let's get it

$$
\begin{cases}
x_1 + \left(-\dfrac{a_{1s}}{a_{rs}}\right)\cdot x_r + \left(a_{1m+1} - \dfrac{a_{rm+1}\cdot a_{1s}}{a_{rs}}\right)x_{m+1} + \ldots + 0\cdot x_s + \ldots + \\
\qquad + \left(a_{1n} - \dfrac{a_{rn}\cdot a_{1s}}{a_{rs}}\right)x_n = b_1 - \dfrac{b_r\cdot a_{1s}}{a_{rs}}, \\
x_2 + \left(-\dfrac{a_{2s}}{a_{rs}}\right)\cdot x_r + \left(a_{2m+1} - \dfrac{a_{rm+1}\cdot a_{2s}}{a_{rs}}\right)x_{m+1} + \ldots + 0\cdot x_s + \ldots + \\
\qquad + \left(a_{2n} - \dfrac{a_{rn}\cdot a_{2s}}{a_{rs}}\right)x_n = b_2 - \dfrac{b_r\cdot a_{2s}}{a_{rs}}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\dfrac{1}{a_{rs}}x_r + \left(\dfrac{a_{rm+1}}{a_{rs}}\right)x_{m+1} + \ldots + 1\cdot x_s + \ldots + \\
\qquad + \left(\dfrac{a_{rn}}{a_{rs}}\right)x_n = \dfrac{b_r}{a_{rs}}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\left(-\dfrac{a_{ms}}{a_{rs}}\right)\cdot x_r + x_m + \left(a_{mm+1} - \dfrac{a_{rm+1}\cdot a_{ms}}{a_{rs}}\right)x_{m+1} + \ldots + 0\cdot x_s + \ldots + \\
\qquad + \left(a_{mn} - \dfrac{a_{rn}\cdot a_{ms}}{a_{rs}}\right)x_n = b_m - \dfrac{b_r\cdot a_{ms}}{a_{rs}},
\end{cases}
$$

or

$$
\begin{cases}
x_s + \dfrac{1}{a_{rs}}x_r + \sum\limits_{\substack{j\in\{m+1,\ldots,n\},\\ j\neq s}} \left(\dfrac{a_{rj}}{a_{rs}}\right)x_j = \dfrac{b_r}{a_{rs}}, \\
x_i + \left(-\dfrac{a_{is}}{a_{rs}}\right)\cdot x_r + \sum\limits_{\substack{j\in\{m+1,\ldots,n\},\\ j\neq s}} \left(a_{ij} - \dfrac{a_{rj}\cdot a_{is}}{a_{rs}}\right)x_j = b_i - \dfrac{b_r\cdot a_{is}}{a_{rs}}, \\
\qquad i = 1,\ldots,m,\ i\neq r.
\end{cases}
\tag{24}
$$

In system (24), the basic variables are $x_1,\ x_2,\ldots,\ x_{r-1},\ x_{r+1},\ldots,\ x_m,\ x_s$. Let's introduce the following notation

$$a'_{ij} = a_{ij} - \frac{a_{rj} \cdot a_{is}}{a_{rs}},\ \ b'_i = b_i - \frac{b_r \cdot a_{is}}{a_{rs}} \ \text{ at } \ i = 1,\ldots,m,\ \ i \neq r,$$

$$b'_r = \frac{b_r}{a_{rs}},\ \ a'_{rj} = \frac{a_{rj}}{a_{rs}} \ \text{ at } i = r, \qquad (25)$$

$$j = 1,\ldots,n.$$

Note that $a'_{rj} = 0$ for all $j \in \{1,\ldots,r-1,r+1,\ldots,m\}$ and $a'_{rs} = 1$ for $j = s$.

Note that the transformation (25) is easily realized using the rectangle rule.

The solution (23) will be basic when

$$\frac{b_r}{a_{rs}} \geq 0,\ \ b_i - \frac{b_r \cdot a_{is}}{a_{rs}} \geq 0,\ \ i = 1,\ldots,m,\ \ i \neq r.$$

It is clear that the first inequality is satisfied if $a_{rs} > 0$. If $a_{is} \leq 0$, then the second inequality is always true. If $a_{is} > 0$, then to satisfy the second inequality, the following inequality must hold:

$$\frac{b_r}{a_{rs}} \leq \frac{b_i}{a_{is}}.$$

Let us take *r* such that

$$\theta_s := \frac{b_r}{a_{rs}} = \min_{i:\, a_{is}>0} \left\{ \frac{b_i}{a_{is}} \right\}. \qquad (26)$$

Then, according to (23) and (25),

$$X' = \big(b_1 - \theta_s \cdot a_{1s}, b_2 - \theta_s \cdot a_{2s}, \ldots, b_{r-1} - \theta_s \cdot a_{r-1s},\ 0,$$
$$b_{r+1} - \theta_s \cdot a_{r+1s}, \ldots, b_m - \theta_s \cdot a_{ms}, 0, \ldots, 0, \theta_s, 0, \ldots, 0\big).$$

Note that the coordinate with the number r in X' can be written as follows: $b_r - \theta_s \cdot a_{rs}$ and this coordinate is zero. Using this, we find

$$f(X') = \sum_{i=1}^{m} c_i (b_i - \theta_s \cdot a_{is}) + c_s \cdot \theta_s =$$

$$= \sum_{i=1}^{m} c_i b_i - \theta_s \left(\sum_{i=1}^{m} c_i a_{is} - c_s \right) = f(X) - \theta_s \cdot \Delta_s,$$

where

$$\Delta_s := \sum_{i=1}^{m} c_i a_{is} - c_s .$$

Note that here $s = m+1, ..., n$.

Let us define a similar expression for the base variables (22), i.e., in the case of $s = 1, ..., m$, in the form

$$\Delta_s = \sum_{i=1}^{m} c_i a_{is} - c_s = c_s - c_s .$$

Therefore, we can consider

$$\Delta_j := \sum_{i=1}^{m} c_i a_{ij} - c_j, \quad j = 1, ..., n. \qquad (27)$$

Expression (27) is called **simplex differences**.

Since $\theta_s > 0$, the index of the variable to be included in the number of basis variables must be chosen so that $\Delta_s < 0$, since $f(X') > f(X)$. If $\Delta_s = 0$, then $f(X') = f(X)$ and this means that a degenerate basis solution is obtained.

Theorem 4. (Optimality criterion for the basic solution of a linear programming problem).

If for some basic solution X the inequalities*

$$\Delta_j \geq 0, \quad j = 1,...,n$$

are satisfied, then X is an optimal solution of the linear programming problem.*

Theorem 5. (Criterion for the unboundedness of the objective function).

If for some base solution X there exists at least one $j \in \{1,...,n\}$ *such that* $\Delta_j < 0$ *and* $a_{ij} \leq 0$, $i = 1,...,m$, *then the objective function is unbounded on the admissible set.*

Item 5.2. Algorithm of the simplex method

Let us consider the linear programming problem (19)-(21), which has an initial basic (reference) plan (22). Let us formulate the algorithm of the simplex method:

1. We calculate the simplex differences Δ_j, $j = 1,...,n$, by formula (27). If $\Delta_j \geq 0$, $j = 1,...,n$, then the reference plan is optimal, otherwise, proceed to the next step.

2. If there exists at least one index $j \in \{1,...,n\}$ such that $\Delta_j < 0$ and $a_{ij} \leq 0$, $i = 1,...,m$, then the problem is unsolvable $(f \to +\infty)$. Otherwise, proceed to the next step.

3. Find the indices s from the condition:

$$\Delta_s = \min_{j:\,\Delta_j<0} \{\Delta_j\} \tag{28}$$

and r by rule (26).

4. Using the transformations (25), we move to the new reference plan (23) by introducing x_s instead of x_r into the basis.

5. Go to step 1.

Note that the element a_{rs} is called the **leading element**, the r-th row is called the **leading row**, and the s-th column is called the **leading column**. The sequence of actions 2.-4. is called an **iteration**.

If you want to maximize the objective function in one iteration, then s should be chosen not according to (28), but according to the formula

$$\theta_s \Delta_s = \min_{j:\Delta_j<0} \{\theta_j \Delta_j\}.$$

Lemma 1. *If the objective function of a canonical linear programming problem excludes the base variables, then the coefficients of the non-base variables will be the corresponding simplex differences with the „−" sign, and the free term is equal to the value of the function at the base point.*

From the lemma, it follows that the simplex differences of variables in a canonical linear programming problem can be easily obtained if, at the initial iteration, we add the equation $f(X)=0$ to the original indirect constraints of the linear programming problem, from which the basis variables are excluded. In subsequent iterations, the simplex transformation is applied to this new equation as well.

From the point of view of ensuring the rationality and visibility of calculations, it is convenient to draw up the implementation of the simplex method in the form of the following simplex tables:

i	B	C_b	A_0	c_1	$\dots$	c_m	c_{m+1}	$\dots$	c_n
				A_1	$\dots$	A_m	A_{m+1}	$\dots$	A_n
1	A_1	c_1	b_1	1		0	a_{1m+1}		a_{1n}
2	A_2	c_2	b_2	0		0	a_{2m+1}		a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
m	A_m	c_m	b_m	0		1	a_{mm+1}		a_{mn}
$m+1$			f_0	$\Delta_1=0$	$\dots$	$\Delta_m=0$	Δ_{m+1}	$\dots$	Δ_n

B – column of basis vectors,
C_b – column of coefficients in the objective function at the basis variables,
A_0 – free terms, $m+1$ – evaluation string.

Remarks.

1. The solution is unique if among the estimates of Δ_j, $j=1,\dots,n$, only those corresponding to the underlying variables are zero (and the rest are positive).

2. If among the estimates Δ_j, $j = 1,...,n$ of the optimal solution, not only those corresponding to the basis variables are zero, then the solution is not unique. To find all solutions, we introduce variables into the basis that correspond to those indices $j \in \{1,...,n\}$ for which $\Delta_j = 0$ and make a convex linear combination of them.

3. In the minimum problem, the optimality criterion is

$$\Delta_j \le 0, \ j = 1,...,n,$$

and the leading column is the one that corresponds $\max\limits_{j:\,\Delta_j > 0} \{\Delta_j\}$.

Example 4: The shop produces 3 types of products: P1, P2, P3, while having four types of raw materials A, B, C, D in quantities of 18, 16, 8, 6 units, respectively. The consumption rates of each type of raw material per unit of product P1 are 1, 2, 1, 0; P2 – 2, 1, 1, 1; P3 – 1, 1, 0, 1. The profit from the sale of a unit of product type P1 is 3 units, P2 – 4 units, P3 – 2 units. Draw up a plan for the production of three types of products to maximize profit.

◄ Let's build a mathematical model of the problem. Let x_j be the number of units of product P$_j$, $j = 1,...,3$, in the production plan. Then we have the following mathematical model of the problem:

$$f = 3x_1 + 4x_2 + 2x_3 \to \max;$$

$$\begin{cases} x_1 + 2x_2 + x_3 \le 18, \\ 2x_1 + x_2 + x_3 \le 16, \\ x_1 + x_2 \le 8, \\ x_2 + x_3 \le 6; \end{cases}$$

$$x_1 \ge 0, \ x_2 \ge 0, \ x_3 \ge 0.$$

Let's reduce the problem to its canonical form by introducing additional variables. We have

$$f = 3x_1 + 4x_2 + 2x_3 \rightarrow \max;$$

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 18, \\ 2x_1 + x_2 + x_3 + x_5 = 16, \\ x_1 + x_2 + x_6 = 8, \\ x_2 + x_3 + x_7 = 6; \end{cases}$$

$$x_j \geq 0, \quad j = 1,...,7.$$

The canonical form contains a unit base (the variables x_4, x_5, x_6, x_7 are the base variables) and nonnegative right-hand sides. Therefore, $X^0 = (0,0,0,18,16,8,6)$ is the reference plan. Fill in the first simplex table

i	B	C_b	A_0	3	4	2	0	0	0	0
				A_1	A_2	A_3	A_4	A_5	A_6	A_7
1	A_4	0	18	1	2	1	1	0	0	0
2	A_5	0	16	2	1	1	0	1	0	0
3	A_6	0	8	1	1	0	0	0	1	0
4	A_7	0	6	0	[1]	1	0	0	0	1
$m+1$			0	-3	-4	-2	0	0	0	0

$X^0 = (0,0,0,18,16,8,6)$ – is not optimal, s is determined by the condition $\min\{-3,-4,-2\} = -4$, so $s = 2$, $\theta_2 = \min\left\{\frac{18}{2}; \frac{16}{1}; \frac{8}{1}; \frac{6}{1}\right\} = 6$, then $r = 4$. We create a new simplex table

i	B	C_b	A_0	3	4	2	0	0	0	0
				A_1	A_2	A_3	A_4	A_5	A_6	A_7
1	A_4	0	6	1	0	-1	1	0	0	-2
2	A_5	0	10	2	0	0	0	1	0	-1
3	A_6	0	2	[1]	0	-1	0	0	1	-1
4	A_2	4	6	0	1	1	0	0	0	1
$m+1$			24	-3	0	2	0	0	0	4

$X^1 = (0,6,0,6,10,2,0)$ is not optimal, $s = 1$, $\theta_1 = \min\left\{\frac{6}{1}; \frac{10}{2}; \frac{2}{1}\right\} = 2$, then $r = 3$. We create a new simplex table

i	B	C_b	A_0	3	4	2	0	0	0	0
				A_1	A_2	A_3	A_4	A_5	A_6	A_7
1	A_4	0	4	0	0	0	1	0	-1	-1
2	A_5	0	6	0	0	$\boxed{2}$	0	1	-2	1
3	A_1	3	2	1	0	-1	0	0	1	-1
4	A_2	4	6	0	1	1	0	0	0	1
$m+1$			30	0	0	-1	0	0	3	1

$X^2 = (2,6,0,4,6,0,0)$ is not optimal, $s = 3$, $\theta_3 = \min\left\{\frac{6}{2}; \frac{6}{1}\right\} = 3$, then $r = 2$. We create a new simplex table

i	B	C_b	A_0	3	4	2	0	0	0	0
				A_1	A_2	A_3	A_4	A_5	A_6	A_7
1	A_4	0	4	0	0	0	1	0	-1	-1
2	A_3	2	3	0	0	1	0	$1/2$	-1	$1/2$
3	A_1	3	5	1	0	0	0	$1/2$	0	$-1/2$
4	A_2	4	3	0	1	0	0	$-1/2$	1	$1/2$
$m+1$			33	0	0	0	0	$1/2$	2	$3/2$

The plan $X^3 = (5,3,3,4,0,0,0)$ is optimal and unique. Therefore, $X^* = (5,3,3)$, $f_{\max} = 33$.

Note that raw materials B, C, and D are scarce, since the second, third, and fourth constraints are satisfied as equals. Raw material A is non-scarce. ►

Example 5. Solve a linear programming problem

$$f = x_1 - x_2 + x_3 + x_4 + x_5 - x_6 \to \min$$

$$\begin{cases} x_1 + x_4 + x_6 = 9, \\ 3x_1 + x_2 - 4x_3 + 2x_6 = 2, \\ x_1 - 2x_3 + x_5 + 2x_6 = 6, \end{cases} \qquad (*)$$

$$x_j \ge 0, \quad j = 1,...,6.$$

◄ The problem has a canonical form (minimizing the objective function f is suitable for the simplex method). There is a unit basis (x_4, x_2, x_5) and the right-hand sides of the constraints are nonnegative. Therefore, there is a reference plan and the problem can be solved by the simplex method. Fill in the first simplex table

i	B	C_b	A_0	1	-1	1	1	1	-1
				A_1	A_2	A_3	A_4	A_5	A_6
1	A_4	1	9	1	0	0	1	0	1
2	A_2	-1	2	3	1	-4	0	0	$\boxed{2}$
3	A_5	1	6	1	0	-2	0	1	2
$m+1$			13	-2	0	1	0	0	2

$X^0 = (0,2,0,9,6,0)$ is not optimal, s is determined from the condition $\max\{1,2\} = 2$, so $s = 6$, $\theta_6 = \min\left\{\frac{9}{1}; \frac{2}{2}; \frac{6}{2}\right\} = \frac{2}{2} = 1$, then $r = 2$.

We create a new simplex table

i	B	C_b	A_0	1	-1	1	1	1	-1
				A_1	A_2	A_3	A_4	A_5	A_6
1	A_4	1	8	$-1/2$	$-1/2$	2	1	0	0
2	A_6	-1	1	$3/2$	$1/2$	-2	0	0	1
3	A_5	1	4	-2	-1	$\boxed{2}$	0	1	0
$m+1$			11	-5	-1	5	0	0	0

The resulting plan $X^1 = (0,0,0,8,4,1)$ is not optimal, $s = 3$, $\theta_3 = \min\left\{\frac{8}{2}; \frac{4}{2}\right\} = \frac{4}{2} = 2$, then $r = 3$. Create a new simplex table

i	B	C_b	A_0	1	-1	1	1	1	-1
				A_1	A_2	A_3	A_4	A_5	A_6
1	A_4	1	4	$3/2$	$\boxed{1/2}$	0	1	-1	0
2	A_6	-1	5	$-1/2$	$-1/2$	0	0	1	1
3	A_3	1	2	-1	$-1/2$	1	0	$1/2$	0
$m+1$			1	0	$3/2$	0	0	$-5/2$	0

$X^2 = (0,0,2,4,0,5)$ is not optimal, $s = 2$, $r = 1$. Create a new simplex table

i	B	C_b	A_0	1	-1	1	1	1	-1
				A_1	A_2	A_3	A_4	A_5	A_6
1	A_2	-1	8	3	1	0	2	-2	0
2	A_6	-1	9	1	0	0	1	0	1
3	A_3	1	6	$1/2$	0	1	1	$-1/2$	0
$m+1$			-11	$-9/2$	0	0	-3	$1/2$	0

X^3 is not optimal, $s = 5$ There are no positive elements in the resulting leading column. Therefore, the objective function is unbounded from below ($f \to -\infty$). ►

The question of the number of iterations of the simplex method required to find an optimal solution to a linear programming problem is quite complex. It should be noted that practice shows that on average this number is approximately equal to the number of m indirect constraints of a canonical linear programming problem. It should be noted that artificial examples of linear programming problems have been constructed, in which the simplex algorithm "searches" all the vertices of the admissible domain. Thus, the simplex algorithm is an algorithm of exponential complexity.

§6. Duality in linear programming

Item 6.1. Basic concepts

Consider the canonical linear programming problem

$$\begin{aligned} & f = C \cdot X \to \max; \\ & AX = A_0; \\ & X \geq 0. \end{aligned} \qquad (29)$$

If the objective function *f* reaches a maximum value on *D*, then it is quite reasonable to construct an upper bound for it on *D*.

Since $A_0 - AX = 0$, the equality $Y\left(A_0 - AX\right) = 0$ holds for any vector $Y = \left(y_1 \ y_2 ... y_m\right)$ and therefore

$$CX = CX + Y\left(A_0 - AX\right) = \left(C - YA\right)X + YA_0.$$

Let's set the requirement that $C - YA \leq 0$, or, equivalently, $YA \geq C$. Then the last equality implies that for $X \geq 0$

$$CX \leq YA_0. \qquad (30)$$

It is also natural to hope that $\max\limits_{X \in D} CX = \min\limits_{YA \geq C} YA_0$. Seeking to obtain the best estimate of (30), we come to a new optimization problem called a dual problem.

The dual problem to problem (29) is the following:

$$\begin{aligned} & F = YA_0 \to \min; \\ & YA \geq C. \end{aligned} \qquad (31)$$

In this regard, we call the canonical linear programming problem (29) a direct (original) linear programming problem. The variables y_i, $i = 1,...,m$ are called **dual variables** (Lagrange multipliers, simplex multipliers, shadow estimates, shadow prices).

The above definition of duality for the canonical problem (29) can be extended to the general case.

The dual problem to the general linear programming problem

$$f=\sum_{j=1}^{n} c_j x_j \ \rightarrow \ \max;$$

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i=1,...,m_1, \\ \sum_{j=1}^{n} a_{ij} x_j \ge b_i, \ i=m_1+1,...,m_2, \\ \sum_{j=1}^{n} a_{ij} x_j = b_i, \ i=m_2+1,...,m; \end{cases}$$

$$x_j \ge 0, \quad j=1,...,n_1,$$
$$x_j \le 0, \quad j=n_1+1,...,n_2$$

is the following problem

$$F=\sum_{i=1}^{m} b_i y_i \ \rightarrow \ \min;$$

$$\begin{cases} \sum_{i=1}^{m} a_{ij} y_i \ge c_j, \quad j=1,...,n_1, \\ \sum_{i=1}^{m} a_{ij} y_i \le c_j, \ j=n_1+1,...,n_2, \\ \sum_{i=1}^{m} a_{ij} y_i = c_j, \ j=n_2+1,...,n; \end{cases}$$

$$y_i \ge 0, \quad i=1,...,m_1,$$

$$y_i \le 0, \quad i=m_1+1,...,m_2,$$

$$y_i - \text{arbitrary}, \ \ i=m_2+1,...,m.$$

The above definition implies an important property: the symmetry of the duality relation. That is, a dual to dual problem coincides with a direct problem. In this regard, it is more logical to talk not about a straightforward and a dual linear programming problem, but about a pair of dual linear programming problems.

The following pair of dual linear programming problems is called symmetric dual linear programming problems:

$$\begin{array}{ll} f = C \cdot X \to \max; & F = YA_0 \to \min; \\ AX \le A_0; & YA \ge C; \\ X \ge 0, & Y \ge 0. \end{array}$$

Example 6. Build a dual problem for the task

$$f = 2x_1 + x_2 - x_3 + 5x_4 \to \min;$$

$$\begin{cases} x_1 - 3x_2 \qquad + x_4 \le 5, \\ 2x_1 \qquad + x_3 \qquad \ge 7, \\ 4x_1 - 2x_2 + 9x_3 + x_4 = 5; \end{cases}$$

$$x_1 \ge 0, \qquad x_3 \le 0.$$

◄ The dual problem to this one is the following

$$F = 5y_1 + 7y_2 + 5y_3 \to \max;$$

$$\begin{cases} y_1 + 2y_2 + 4y_3 \le 2, \\ -3y_1 \qquad - 2y_3 = 1, \\ \qquad y_2 + 9y_3 \ge -1, \\ y_1 \qquad + y_3 = 5; \end{cases}$$

$y_1 \le 0,\ y_2 \ge 0,\ y_3$ – any value. ►

Example 7. Build a dual problem for the task

$$f = x_1 - 2x_2 + x_3 - x_4 + x_5 \to \min;$$

$$\begin{cases} x_1 - 2x_2 + x_3 + 3x_4 - 2x_5 = 6, \\ 2x_1 + 3x_2 - 2x_3 - x_4 + x_5 \le 4, \\ x_1 + 3x_3 - 4x_5 \ge 8; \end{cases}$$

$$x_1 \ge 0, \qquad x_3 \ge 0, \qquad x_5 \ge 0.$$

◄ The dual problem to this one is the following

$$F = 6y_1 + 4y_2 + 8y_3 \to \max;$$

$$\begin{cases} y_1 + 2y_2 + y_3 \le 1, \\ -2y_1 + 3y_2 = -2, \\ y_1 - 2y_2 + 3y_3 \le 1, \\ 3y_1 - y_2 = -1, \\ -2y_1 + y_2 - 4y_3 \le 1; \end{cases}$$

$$y_2 \le 0, \; y_3 \ge 0.$$

►

Item 6.2. Duality theorems and their applications

Since each problem can be reduced to a canonical form, problems (29) and (31) can be considered the main ones when studying the properties of a pair of mutual dual problems and their joint solution.

Theorem 6. *If X and Y are admissible plans of problems*

$$\begin{aligned} &f = C \cdot X \to \max; \\ &AX = A_0, \\ &X \ge 0 \end{aligned} \qquad (29)$$

and

$$\begin{aligned} &F = YA_0 \to \min; \\ &YA \ge C \end{aligned} \qquad (31)$$

respectively, then

$$f(X) \le F(Y).$$

Theorem 7. *If for some admissible plans* X^* *and* Y^* *of the pair of dual problems (29) and (31) the equality* $f(X^*) = F(Y^*)$ *holds, then* X^* *and* Y^* *are optimal solutions to problems (29) and (31), respectively.*

Theorem 8. (The first duality theorem).

If one of the problems of a dual pair has a solution, then the other problem also has a solution. In this case, for optimal solutions X^* *and* Y^* *the equality*

$$f(X^*) = F(Y^*).$$

holds.

Theorem 9. *If the objective function of the f-problem (29) is unbounded from above, then the dual problem (31) has no admissible plans, i.e.* $D_d = \varnothing$.

Theorem 10 (Second duality theorem).

For the admissible solutions $X^* = \left(x_1^*, x_2^*, ..., x_n^*\right)$ *and* $Y^* = \left(y_1^*, y_2^*, ..., y_m^*\right)$ *of problems (29) and (31) to be optimal, respectively, it is necessary and sufficient that the equations*

$$\left(\sum_{j=1}^{n} a_{ij} x_j^* - b_i\right) y_i^* = 0, \qquad i = 1,...,m,$$
$$\left(\sum_{i=1}^{m} a_{ij} y_i^* - c_j\right) x_j^* = 0, \qquad j = 1,...,n. \qquad (32)$$

are true.

Example 12: Solve the dual problem to the problem from Example 2.

◄ I. Let's consider the problem

$$f = x_1 + 2x_2 \to \min;$$

$$\begin{cases} -5x_1 + 2x_2 \le 10, \\ x_1 + 2x_2 \ge 6, \\ 2x_1 + 9x_2 \ge 18; \end{cases} \qquad (33)$$

$$x_1 \ge 0, \;\; x_2 \ge 0.$$

The solution to this problem was found using the graphical method in example 2

$$X^* = \left(\frac{18}{5} - \frac{18}{5}\lambda; \frac{6}{5} + \frac{9}{5}\lambda\right), \; \lambda \in [0,1], \; f_{\min} = 6.$$

The dual problem to (*) is the problem

$$F = 10y_1 + 6y_2 + 18y_3 \to \max;$$

$$\begin{cases} -5y_1 + y_2 + 2y_3 \le 1, \\ 2y_1 + 2y_2 + 9y_3 \le 2; \end{cases} \qquad (34)$$

$$y_1 \le 0, \quad y_2 \ge 0, \quad y_3 \ge 0.$$

According to the first duality theorem $F_{\max} = f_{\min} = 6$.

Let us find $Y^* = \left(y_1^*, y_2^*, y_3^*\right)$.

1) Write out the conditions (32) of the second duality theorem for problem (33), which has already been solved. We have

$$\begin{cases} \left(-5x_1^* + 2x_2^* - 10\right)y_1^* = 0, \\ \left(x_1^* + 2x_2^* - 6\right)y_2^* = 0, \\ \left(2x_1^* + 9x_2^* - 18\right)y_3^* = 0. \end{cases}$$

$y_1^* = 0$, since $\left(-5x_1^* + 2x_2^* - 10\right) \ne 0$. $\left(x_1^* + 2x_2^* - 6\right) = 0$ for $\lambda \in [0,1]$. Therefore, y_2^* must be found. $\left(2x_1^* + 9x_2^* - 18\right) = 0$ for $\lambda = 0$ and $\left(2x_1^* + 9x_2^* - 18\right) \ne 0$ for $\lambda \in (0,1]$, so $y_3^* = 0$.

2) Write out the conditions of the second duality theorem for the problem we are solving:

$$\begin{cases} \left(-5y_1^* + y_2^* + 2y_3^* - 1\right)x_1^* = 0, \\ \left(2y_1^* + 2y_2^* + 9y_3^* - 2\right)x_2^* = 0. \end{cases}$$

Since $x_1 = 0$ for $\lambda = 1$ and $x_1 \ne 0$ for $\lambda \in [0,1)$, we have

$$\begin{cases} -5y_1^* + y_2^* + 2y_3^* = 1, \\ 2y_1^* + 2y_2^* + 9y_3^* = 2. \end{cases}$$

So, $y_2^* = 1$. $x_1 = 0$ for $\lambda = 1$ i $x_1 \neq 0$ for $\lambda \in [0,1)$, we have

$$\begin{cases} -5y_1^* + y_2^* + 2y_3^* = 1, \\ 2y_1^* + 2y_2^* + 9y_3^* = 2. \end{cases}$$

So, $y_2^* = 1$.

II. Let's consider the problem

$$f = x_1 + 2x_2 \to \max;$$

$$\begin{cases} -5x_1 + 2x_2 \leq 10, \\ x_1 + 2x_2 \geq 6, \\ 2x_1 + 9x_2 \geq 18; \end{cases} \quad (35)$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

For this problem, $f_{\max} = +\infty$.

The dual problem to problem (35) is the problem

$$F = 10y_1 + 6y_2 + 18y_3 \to \min;$$

$$\begin{cases} -5y_1 + y_2 + 2y_3 \geq 1, \\ 2y_1 + 2y_2 + 9y_3 \geq 2; \end{cases} \quad (36)$$

$$y_1 \geq 0, \quad y_2 \leq 0, \quad y_3 \leq 0.$$

According to Theorem 9, the dual problem has no solution $D_d = \varnothing$. ▶

Formula

$$Y^* = C_{bas}^* \cdot B^{-1} \quad (37)$$

defines the optimal solution to the dual problem.

Example 13: Solve the dual problem to the problem from Example 4.

$$f = 3x_1 + 4x_2 + 2x_3 \to \max;$$

$$\begin{cases} x_1 + 2x_2 + x_3 \le 18, \\ 2x_1 + x_2 + x_3 \le 16, \\ x_1 + x_2 \qquad \le 8, \\ \qquad x_2 + x_3 \le 6; \end{cases} \qquad (38)$$

$$x_j \ge 0, \quad j = 1,...,3.$$

◀ This problem in Example 4 was solved by the simplex method and the following solution was found: $X^* = (5,3,3)$, $f_{\max} = 33$.

The solution to the dual problem can be found in the (m+1) row of the last simplex table of the direct problem. The reference plan is determined by the base variables A_4, A_3, A_1, A_2. In the original problem, they form the matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

The inverse of the matrix B^{-1} is the columns of the last table that were the basis in the first table

$$B^{-1} = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1/2 & -1 & 1/2 \\ 0 & 1/2 & 0 & -1/2 \\ 0 & -1/2 & 1 & 1/2 \end{pmatrix}.$$

Since $Y^* = C^*_{bas} \cdot B^{-1}$, this vector has actually already been calculated in the (m+1) row of the last simplex table:

$$Y^* = (0+0;\ 1/2+0;\ 2+0;\ 3/2+0). \ ►$$

§7. The dual simplex method for solving of linear programming problems

Consider the canonical linear programming problem

$$f = \sum_{j=1}^{n} c_j x_j \to \max; \tag{19}$$

$$x_i + \sum_{j=m+1}^{n} a_{ij} x_j = b_i, \quad i = 1,\dots,m, \tag{20}$$

$$x_j \geq 0, \quad j = 1,\dots,n. \tag{21}$$

Let this problem have a unit basis, but among the components $b_1,\dots,b_m$ there are one or more negative ones.

Such a problem is called a problem in dual basis form. A variant of the method for solving such a problem is called the dual simplex method. A solution $X_{ps} = (b_1,\dots,b_m, 0,\dots, 0)$ is called a **pseudo-solution** of problem (19)-(21) if the simplex differences

$$\Delta_j = \sum_{i=1}^{m} c_i a_{ij} - c_j \geq 0, \quad j = 1,\dots,n,$$

but its components $b_1,\dots,b_m$ have one or more negative ones.

Theorem 11. *If a pseudo-solution of problem (19)-(21) has at least one component $b_i < 0$ for which $a_{ij} \geq 0$, $j = 1,\dots,n$, then problem (19)-(21) have no solution $(D = \varnothing)$.*

Theorem 12. *Suppose that in problem (19)-(21),* $\Delta_j \geq 0$ *for all* $j \in \{1,...,n\}$, *and the pseudo-solution* X_{ps} *is such that for all* $i \in \{1,...,m\}$, *for which* $b_i < 0$, *at least one* $j \in \{1,...,n\}$ *for which* $a_{ij} < 0$. *Let r be such a number that* $b_r < 0$ *and*

$$b_r = \min_{i:\, b_i < 0} \{b_i\}, \qquad (39)$$

and let s be such that $a_{rs} < 0$ *and*

$$\frac{\Delta_s}{a_{rs}} = \max_{\substack{j: a_{rj} < 0, \\ \Delta_j \geq 0}} \left\{ \frac{\Delta_j}{a_{rj}} \right\}. \qquad (40)$$

Then the Jordaan transformation with leading element a_{rs} *leads to a new plan in which:*

1) all $\Delta_j \geq 0$, $j = 1,...,n$;

2) the value of the objective function does not increase.

Here is the algorithm of the dual simplex method.

1. If all $\Delta_j \geq 0$, $j = 1,...,n$ and $b_i \geq 0$, $i = 1,...,m$, then the optimal solution is found – $X^* = (b_1,...,b_m, 0,..., 0)$. If there is a pseudo-plan, then proceed to step 2.
2. If the pseudo-plan has at least one component $b_i < 0$ for which $a_{ij} \geq 0$, $j = 1,...,n$, then the problem has no solution $(D = \varnothing)$. Otherwise, go to step 3.
3. Find *r* from condition (39). The vector A_r is derived from the basis.
4. Find *s* from condition (40). The vector A_s is introduced into the basis.
5. Perform the Jordan transform with the leading element a_{rs}.
6. Go to step 1.

In the case of a minimum problem:

1) the leading line is selected in the same way as in the maximum problem;
2) if there is a pseudo-plan ($\Delta_j \le 0$), the leading element is selected using the formula

$$\frac{\Delta_s}{a_{rs}} = \min_{\substack{j: a_{rj} < 0, \\ \Delta_j \le 0}} \left\{ \frac{\Delta_j}{a_{rj}} \right\}.$$

Example 14. Solve the dual problem to the problem from Example 4 using the dual simplex method.

◀ The dual problem is the following:

$$F = 18y_1 + 16y_2 + 8y_3 + 6y_4 \to \min;$$

$$\begin{cases} y_1 + 2y_2 + y_3 \ge 3, \\ 2y_1 + y_2 + y_3 + y_4 \ge 4, \\ y_1 + y_2 + y_4 \ge 2; \end{cases} \tag{41}$$

$$y_i \ge 0, \ i = 1,...,4.$$

To solve this problem, let's reduce it to its canonical form:

$$F = 18y_1 + 16y_2 + 8y_3 + 6y_4 \to \min;$$

$$\begin{cases} y_1 + 2y_2 + y_3 - y_5 = 3, \\ 2y_1 + y_2 + y_3 + y_4 - y_6 = 4, \\ y_1 + y_2 + y_4 - y_7 = 2; \end{cases}$$

$$y_i \ge 0, \ i = 1,...,7.$$

To obtain the unit basis, we multiply each of the equations of the constraint system by (-1). Then we get the canonical form with a unit basis

$$F = 18y_1 + 16y_2 + 8y_3 + 6y_4 \to \min;$$

$$\begin{cases} -y_1 - 2y_2 - y_3 + y_5 = -3, \\ -2y_1 - y_2 - y_3 - y_4 + y_6 = -4, \\ -y_1 - y_2 - y_4 + y_7 = -2, \end{cases}$$

$$y_i \ge 0, \ i = 1,...,7.$$

However, the right-hand sides are negative. We solve this problem using the dual simplex method.

i	B	C_b	A_0	18	16	8	6	0	0	0
				A_1	A_2	A_3	A_4	A_5	A_6	A_7
1	A_5	0	-3	-1	-2	-1	0	1	0	0
2	A_6	0	-4	-2	-1	-1	-1	0	1	0
3	A_7	0	-2	-1	-1	0	-1	0	0	1
$m+1$			0	-18	-16	-8	-6	0	0	0

The first table contains a pseudo-plan. By (39), we define *r*: $\min\{-3; -4; -2\}$, $r = 2$. We define *s* from the condition: $\min\left\{\frac{-18}{-2}, \frac{-16}{1}, \frac{-8}{-1}, \frac{-6}{-1}\right\} = 6$, $s = 4$.

i	B	C_b	A_0	18	16	8	6	0	0	0
				A_1	A_2	A_3	A_4	A_5	A_6	A_7
1	A_5	0	-3	-1	-2	-1	0	1	0	0
2	A_4	6	4	2	1	1	1	0	-1	0
3	A_7	0	2	1	0	1	0	0	-1	1
$m+1$			24	-6	-10	-2	0	0	-6	0

The second table contains a pseudo-plan. Define *r*: $\min\{-3\} = -3$, $r = 1$. Define *s* from the condition: $\min\left\{\frac{-6}{-1}, \frac{-10}{-2}, \frac{-2}{-1}\right\} = 2$, $s = 3$.

i	B	C_b	A_0	18	16	8	6	0	0	0
				A_1	A_2	A_3	A_4	A_5	A_6	A_7
1	A_3	8	3	1	2	1	0	-1	0	0
2	A_4	6	1	1	-1	0	1	1	-1	0
3	A_7	0	-1	0	[-2]	0	0	1	-1	1
$m+1$			30	-4	-6	0	0	-2	-6	0

The third table has a pseudo-plan. Define *r*: $\min\{-1\}=-1$, $r=3$. Define *s* from the condition: $\min\left\{\frac{-6}{-2},\frac{-6}{-1}\right\}=3$, $s=2$.

i	B	C_b	A_0	18	16	8	6	0	0	0
				A_1	A_2	A_3	A_4	A_5	A_6	A_7
1	A_3	8	2	1	0	1	0	0	-1	1
2	A_4	6	$3/2$	1	0	0	1	$1/2$	$-1/2$	$-1/2$
3	A_2	16	$1/2$	0	1	0	0	$-1/2$	$1/2$	$-1/2$
$m+1$			33	-4	0	0	0	-5	-3	-3

The plan obtained in the fourth table is the optimal plan $Y^*=(0;\ 1/2;\ 2;\ 3/2)$, $F_{\min}=33$.

Find the solution of the dual problem to problem (41), i.e. problem (38). *B* is the matrix that gives the optimal solution Y^*, i.e., it is a matrix consisting of columns A_3, A_4, A_2:

$$B=\begin{pmatrix}1 & 0 & 2\\ 1 & 1 & 1\\ 0 & 1 & 1\end{pmatrix}.$$

And then

$$X^*=C^*_{bas}\cdot B^{-1}=(8\ 6\ 16)\cdot B^{-1}=(8\ 6\ 16)\cdot\begin{pmatrix}0 & 1 & -1\\ -1/2 & 1/2 & 1/2\\ 1/2 & -1/2 & 1/2\end{pmatrix},$$

$$X^*=(-(-5+0);-(-3+0);-(-3+0))=(5,3,3). \quad ►$$

CHAPTER 2

SPECIAL PROBLEMS OF LINEAR PROGRAMMING

§1. The transportation problem

Item 1.1. Properties of the problem

A substantive statement of the problem is made in Chapter 1.

The process of production and consumption of homogeneous products is considered. There are m points $A_1, A_2, ..., A_m$, which produce homogeneous products in quantities a_i, $i = 1, ..., m$, respectively. The produced goods are consumed in n points $B_1, B_2, ..., B_n$ in quantities b_j, $j = 1, ..., n$, respectively.

Let c_{ij}, $i = 1, ..., m$, $j = 1, ..., n$ be the cost of transportation of a unit of product from the point of production A_i to the point of consumption B_j. If we denote by x_{ij} the number of units of cargo transported from point of production A_i to point of consumption B_j, $i = 1, ..., m$, $j = 1, ..., n$, then the mathematical model for finding a transportation plan that would satisfy the needs of consumers while minimizing transportation costs is as follows:

$$f=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\ \rightarrow\ \min;$$

$$\begin{cases}\sum_{j=1}^{n}x_{ij}\le a_i,\ \ i=1,\dots,m,\\ \sum_{i=1}^{m}x_{ij}\ge b_j,\ \ j=1,\dots,n;\end{cases}$$

$$x_{ij}\ge 0,\ \ i=1,\dots,m,\ \ j=1,\dots,n.$$

Let us consider the ideal case where the sum of possible supplies equals the sum of needs:

$$\sum_{i=1}^{m}a_i=\sum_{j=1}^{n}b_j\,. \qquad (1)$$

In this case, the constraints of the problem are inequalities.

Condition (1) is called the **balance condition**, and the problem is called a **balanced (closed) problem**. In the following, a balanced problem will be called a **T-problem**.

Theorem 1. (Criterion for the existence of admissible solutions of the T-problem).

The balance condition (1) is a necessary and sufficient condition for the existence of feasible solutions to the T-problem.

Theorem 2. *The rank of the matrix of the constraint system of the T-problem is* $m+n-1$.

Theorem 3. *A balanced transportation problem always has an optimal solution.*

Item 1.2. Reference plans of the T-problem and their properties

The data of the linear programming transportation problem and the results of the calculations associated with its solution are entered into the transportation table.

Points of shipments	Destinations					Inventory
	B_1	…	B_j	…	B_n	
A_1	c_{11} x_{11}	…	c_{1j} x_{1j}	…	c_{1n} x_{1n}	a_1
…	…	…	…	…	…	…
A_i	c_{i1} x_{i1}	…	c_{ij} x_{ij}	…	c_{in} x_{in}	a_i
…	…	…	…	…	…	…
A_m	c_{m1} x_{m1}	…	c_{mj} x_{mj}	…	c_{mn} x_{mn}	a_m
Needs	b_1	…	b_j	…	b_n	

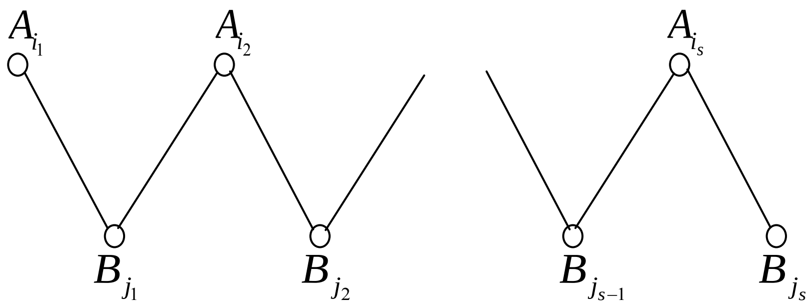
In this case, the transportation cost c_{ij} is recorded in the upper right corner of the corresponding cell of the transportation table. As a rule, the components x_{ij} of the transportation matrix are entered into the transportation table only when $x_{ij} > 0$. In this case, the corresponding cell of the transportation table is called filled, otherwise it is called free.

For a linear programming transportation problem, the concept of a basic solution plays an important role. According to Theorem 2, a nondegenerate basis solution of a linear programming transportation problem contains the $(m+n-1)$ positive component, and a degenerate basis solution contains a smaller number of positive components.

In a linear programming transportation problem, the concept of a basic solution can be given a clear geometric interpretation. To do this, we will introduce a number of definitions and formulate some statements.

Definition 1. *The* ***route*** *connecting points A and B is called a communication sequence*

$$\overline{A_{i_1}B_{j_1}},\ \overline{A_{i_2}B_{j_1}},\ \overline{A_{i_2}B_{j_2}},\ldots,\ \overline{A_{i_s}B_{j_{s-1}}},\ \overline{A_{i_s}B_{j_s}}.$$



We denote this route by *M*.

Definition 2*. A route M to which communication AB is added is called a closed route or* ***cycle****.*

Denoting the cycle by *C*, we have $C = M \cup \overline{A_{i_1}B_{j_s}}$.

If we replace the communications in the route and cycle definitions with the corresponding cells of the transport table of the T-task, we get the route and cycle definitions of the transport table.

Theorem 4. *The system of condition vectors* $R=\{P_{ij}\}$, $i=1,...,m$, $j=1,...,n$ *of a linear programming transportation problem is linearly independent if and only if it is impossible to form a cycle from the communications corresponding to these vectors.*

Definition 3*.* ***A communication*** $\overline{A_i B_j}$ *is called* ***basic*** *for a solution X if* $x_{ij} > 0$.

Definition 4*.* ***A plan*** *of a T-problem is called a* ***basic*** *plan if its basic communications cannot be used to form a cycle.*

Let us state a simple corollary of Theorem 4.

Theorem 5. *An admissible solution to a linear programming transportation problem is a basic solution if and only if its main communications cannot form a cycle.*

The above theorems can be transformed to apply to the transportation table. In particular, if it is impossible to form a cycle from the filled cells $(x_{ij} > 0)$ of some admissible solution to a linear programming transportation problem, then this solution is a basic solution, otherwise it is not.

Definition 5. The cells *of the transport table that correspond to the basic variables are called* **basic**, *the rest are called non-basic.*

Item 1.3. Methods for finding the initial reference plans for a T-problem

For a standard linear programming problem, finding an initial basis solution requires the use of an artificial basis method, or M-method. For a transportation linear programming problem, the search for an initial basis solution is much simpler due to its specifics. Let's consider some of the most popular methods.

1. Northwest corner method

The cell (1, 1) of the transportation table (its northwest corner) is selected and loaded with the maximum possible transportation. There are three possible cases:

1) $x_{11} = \min(a_1, b_1) = a_1$,
2) $x_{11} = \min(a_1, b_1) = b_1$,
3) $x_{11} = \min(a_1, b_1) = a_1 = b_1$.

In the first case, the first row of the transport table is excluded from further consideration, we set $b_1' = b_1 - x_{11}$, in the second case, the first column is excluded, we set $a_1' = a_1 - x_{11}$, in the third case, both the first column and the first row are excluded from further consideration. In the transport table thus reduced, its upper left cell (northwest corner) is located, loaded as much as possible, etc.

It is clear that the transportation plan constructed in this way is a valid solution to the linear programming transportation problem. In addition, this plan is a basic one.

Note that when constructing the initial base solution using the northwest corner method, transportation costs c_{ij} are not taken into account at all. Therefore, as a rule, this plan is far from optimal.

Example 1. Find a reference plan for a problem

Points of shipments	Destinations				Inventory
	B_1	B_2	B_3	B_4	
A_1	2 x_{11}	2 x_{12}	3 x_{13}	5 x_{14}	110
A_2	9 x_{21}	6 x_{22}	3 x_{23}	5 x_{24}	150
A_3	8 x_{31}	12 x_{32}	11 x_{33}	6 x_{34}	250
Needs	100	180	140	90	510

The problem is balanced because the total demand $\sum_{j=1}^{n} b_j = 510$ is equal to the total supply $\sum_{i=1}^{m} a_i = 510$.

Let's find a feasible solution to the transportation problem using the northwest corner method. Consistently fill in the cells, starting from the upper left (northwest corner), exhausting the reserves and satisfying the needs:

Points of shipments	Destinations				Inventory
	B_1	B_2	B_3	B_4	
A_1	2 **100**	2 **10**	3	5	110
A_2	9	6 **150**	3	5	150
A_3	8	12 **20**	11 **140**	6 **90**	250
Needs	100	180	140	90	510

We get the following valid solution

$$X_{\text{M1}} = \begin{pmatrix} 100 & 10 & 0 & 0 \\ 0 & 150 & 0 & 0 \\ 0 & 20 & 140 & 90 \end{pmatrix},$$

$$f_{\text{M1}} = 2\cdot 100 + 2\cdot 10 + 6\cdot 150 + 12\cdot 20 + 11\cdot 140 + 6\cdot 90 = 3440.$$

2. Minimum element method

The idea of this method is to maximize the transportation load of communications with the minimum transportation cost. In fact, this method differs from the Northwest corner method only in that at each step of building the initial basic solution, the cell with the minimum value of c_{ij} is selected for loading.

The minimum element method also results in a valid basic solution to the linear programming transportation problem.

Example 2. Find a feasible solution to the transportation problem from Example 1 using the minimum element method.

Fill in the cells of the table in ascending order of transportation costs, exhausting stocks and satisfying needs:

Points of shipments	Destinations				Inventory
	B_1	B_2	B_3	B_4	
A_1	2	2 **110**	3	5	110
A_2	9	6	3 **140**	5 **10**	150
A_3	8 **100**	12 **70**	11	6 **80**	250
Needs	100	180	140	90	510

We get the following feasible solution

$$X_{M2} = \begin{pmatrix} 0 & 110 & 0 & 0 \\ 0 & 0 & 140 & 10 \\ 100 & 70 & 0 & 80 \end{pmatrix},$$

$$f_{M2} = 2\cdot 110 + 3\cdot 140 + 5\cdot 10 + 8\cdot 100 + 12\cdot 70 + 6\cdot 80 = 2810.$$

§2. The method of potentials for solving a transportation problem

Item 2.1. Duality in the transportation problem

Let's write the T-problem in expanded form

$$f=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}\ \to\ \min;$$

$$\begin{cases} x_{11}+x_{12}+\ldots+x_{1n} = a_1, \\ x_{21}+x_{22}+\ldots+x_{2n} = a_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_{m1}+x_{m2}+\ldots+x_{mn}=a_m, \\ x_{11}+x_{21}+\ldots+x_{m1} = b_1, \\ x_{12}+x_{22}+\ldots+x_{m2} = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_{1n}+x_{2n}+\ldots+x_{mn}=b_n, \end{cases} \qquad (2)$$

$$x_{ij}\ge 0,\ \ i=1,\ldots,m,\ \ j=1,\ldots,n.$$

Let us construct a binary problem to the T-problem (2) by matching the first m constraints with the binary variables $u_1,\ldots,u_m$, and the next n constraints with the binary variables $v_1,\ldots,v_n$. If $Y=\left(u_1,\ldots,u_m,v_1,\ldots,v_n\right)$, then the binary problem to problem (2) is as follows:

$$F = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \to \max$$

$$u_i + v_j \le c_{ij}, \qquad (3)$$

$$i = 1,...,m, \quad j = 1,...,n.$$

Definition 6. *The variable* u_i *is called* ***the potential of the production point*** A_i***,*** *and the variable* v_j *is called* ***the potential of the consumption point*** B_j*,* $j = 1,...,n$.

Since there are no restrictions on the sign of the binary variables, the vector of potentials can be chosen in the form $Y = \left(-u_1,...,-u_m,v_1,...,v_n\right)$. Then we get the following dual problem to (2):

$$F = -\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \to \max$$

$$v_j - u_i \le c_{ij}, \qquad (3')$$

$$i = 1,...,m, \quad j = 1,...,n.$$

Item 2.2. The method of potentials

The method of potentials essentially uses a dual optimality criterion for a linear programming transportation problem. Let u_i, $i=1,...,m$, v_j, $j=1,...,n$ be the potentials of points A_i and B_j, respectively. The value $c_{ij}-\left(u_i+v_j\right)=\Delta_{ij}$ is called the **simplex difference** (relative value) of the variable x_{ij}.

Theorem 6. (Dual optimality criterion for the linear programming transportation problem).

A basic solution X is optimal if and only if there exist potentials u_i, $i=1,...,m$, v_j, $j=1,...,n$, *such that*

$$\Delta_{ij}=0 \quad \textit{for basic cells,} \tag{4}$$

$$\Delta_{ij}\geq 0 \quad \textit{for non-basic cells.} \tag{5}$$

The algorithm of the method of potentials consists of several steps.

1. Find the initial nondegenerate solution X of the linear programming transportation problem using one of the known methods.
2. Calculate the potentials u_i, $i=1,...,m$, v_j, $j=1,...,n$, so that in each basis cell the relation $\Delta_{ij}=0$ is fulfilled, or, equivalently,

$$u_i+v_j=c_{ij},\ (i,j)-\text{basic cells.} \tag{6}$$

Note that system (6) contains $m+n-1$ equations with $m+n$ unknowns. Therefore, one of the unknowns (u_1, for example) is assumed to be equal to an arbitrary constant, which is usually zero. The remaining unknowns are found from system (6).

3. Find Δ_{ij} for the non-basic cells. If all of them are nonnegative, that is, if the inequalities $u_i + v_j \le c_{ij}$ are satisfied for all nonbasic cells, then the basic solution X is optimal. Otherwise, it can be improved by including one of the cells where $\Delta_{ij} < 0$, i.e., a cell for which $u_i + v_j > c_{ij}$. As a rule, the cell $\left(i_0, j_0\right)$ for which $\Delta_{i_0 j_0} = \min\limits_{i,j} \Delta_{ij}$ is included in the number of base cells.

4. Redistribute the transportation: include cell $\left(i_0, j_0\right)$ (i.e., variable $x_{i_0 j_0}$) in the number of base cells, exclude cell $\left(k, l\right)$ for

$$x_{kl} = \theta = \min_{i,j:(i,j)\in C^-} x_{ij}$$

(i.e., variable x_{kl}) from the number of base cells. Go to step 2 of the algorithm.

Remarks. *If the vector of binary variables is chosen as* $Y = \left(-u_1, \ldots, -u_m, v_1, \ldots, v_n\right)$, *then the simplex differences* Δ_{ij} *are set equal to* $\Delta_{ij} = v_j - u_i - c_{ij}$. *Then, in step 2 of the algorithm, condition (6) is written as follows:*

$$v_j - u_i = c_{ij},\ (i, j) \text{ – are the basic cells,}$$

and in step 3 of the algorithm, the optimality condition is as follows:

$$v_j - u_i \le c_{ij}, \text{ that is } \Delta_{ij} \le 0 \text{ for non-basic cells.}$$

Example 3. Find the optimal transportation plan for the previous examples. Let's present the problem data in the form of a table

Points of shipments	Destinations				Inventory
	B_1	B_2	B_3	B_4	
A_1	2 x_{11}	2 x_{12}	3 x_{13}	5 x_{14}	110
A_2	9 x_{21}	6 x_{22}	3 x_{23}	5 x_{24}	150
A_3	8 x_{31}	12 x_{32}	11 x_{33}	6 x_{34}	250
Needs	100	180	140	90	510

◄ Let's use the feasible solution to the transportation problem found by the northwest angle method:

Points of shipments	Destinations				Inventory
	B_1	B_2	B_3	B_4	
A_1	2 **100**	2 **10**	3	5	110
A_2	9	6 **150**	3	5	150
A_3	8	12 **20**	11 **140**	6 **90**	250
Потреби	100	180	140	90	510

We got the following valid solution

$$X^1 = \begin{pmatrix} 100 & 10 & 0 & 0 \\ 0 & 150 & 0 & 0 \\ 0 & 20 & 140 & 90 \end{pmatrix},$$

$$f(X^1) = 2\cdot 100 + 2\cdot 10 + 6\cdot 150 + 12\cdot 20 + 11\cdot 140 + 6\cdot 90 = 3440.$$

Let us check this solution for optimality using the method of potentials. Let's assign the potentials u_i to the points of shipment A_i and the potentials v_j to the points of destination B_j. Let's construct the system of equations $u_i + v_j = c_{ij}$ for all the base cells:

$$\begin{cases} u_1 + v_1 = 2, \\ u_1 + v_2 = 2, \\ u_2 + v_2 = 6, \\ u_3 + v_2 = 12, \\ u_3 + v_3 = 11, \\ u_3 + v_4 = 6. \end{cases}$$

Points of shipments	Destinations				Inventory	u_i
	B_1	B_2	B_3	B_4		
A_1	2 **100**	2 **10**	3	5	110	0
A_2	9	6 **150**	3	5	150	4
A_3	8	12 **20**	11 **140**	6 **90**	250	10
Needs	100	180	140	90	510	
v_j	2	2	1	-4		

Setting $u_1 = 0$, we find the potentials u_i, v_{jj} by solving the system of equations. In order for an admissible plan to be optimal, it is necessary and sufficient that the relative estimates

$$\Delta_{ij} = c_{ij} - \left(u_i + v_j\right),$$

calculated for the free cells are nonnegative. We have

$$\Delta_{13} = 3 - 0 - 1 = 2 > 0,$$

$$\Delta_{14} = 5 - 0 - (-4) = 1 > 0,$$

$$\Delta_{21} = 9 - 2 - 4 = 3 > 0,$$

$$\Delta_{23} = 3 - 4 - 2 = -3 < 0,$$

$$\Delta_{24} = 5 - 4 - (-4) = 5 > 0,$$

$$\Delta_{31} = 8 - 10 - 2 = -4 < 0.$$

The optimality criterion is not met for cells (2, 3) and (3, 1).

a_i \ b_j	100	180	140	90	u_i
110	2 – **100**	2 + **10**	3	5	0
150	9	6 **150**	3	5	4
250	8 +	12 – **20**	11 **140**	6 **90**	10
v_j	2	2	1	-4	

Therefore, we enter the cell (3, 1) into the base and recalculate the table by setting $\theta = 20$. We get the following table:

a_i \ b_j	100	180	140	90	u_i
110	– 2 **80**	+ 2 **30**	3	5	0
150	9	– 6 **150**	+ 3	5	4
250	+ 8 **20**	12	– 11 **140**	6 **90**	6
v_j	2	2	5	0	

A new transportation plan

$$X^2 = \begin{pmatrix} 80 & 30 & 0 & 0 \\ 0 & 150 & 0 & 0 \\ 20 & 0 & 140 & 90 \end{pmatrix},$$

the cost of transportation for which is

$$f(X^2) = 2 \cdot 80 + 2 \cdot 30 + 6 \cdot 150 + 8 \cdot 20 + 11 \cdot 140 + 6 \cdot 90 = 3360.$$

Let's check this solution for optimality using the method of potentials. The optimality criterion is not met for cells (1, 3) and (2, 3). We enter cell (2, 3) into the basis and recalculate the table by setting $\theta = 80$. We get the following table:

a_i \ b_j	100	180	140	90	u_i
110	2	2 **110**	3	5	0
150	9	− 6 **70**	+ 3 **80**	5	4
250	8 **100**	+ 12	− 11 **60**	6 **90**	12
v_j	-4	2	-1	-6	

The transportation plan

$$X^3 = \begin{pmatrix} 0 & 110 & 0 & 0 \\ 0 & 70 & 80 & 0 \\ 100 & 0 & 60 & 90 \end{pmatrix},$$

has a transportation cost

$$f(X^3) = 2 \cdot 110 + 6 \cdot 70 + 3 \cdot 80 + 8 \cdot 100 + 11 \cdot 60 + 6 \cdot 90 = 2880.$$

Let's check this solution for optimality using the method of potentials. The optimality criterion is not met for cell (3, 2). Recalculate the table by setting $\theta = 60$. We get the following table:

a_i \ b_j	100	180	140	90	u_i
110	2	2 **110**	3	5	0
150	9	6 **10**	3 **140**	5	4
250	8 **100**	12 **60**	11	6 **90**	12
v_j	-4	2	-1	-6	

Since all relative scores calculated for the free cells are nonnegative, the resulting solution is optimal.

Thus, the optimal transportation plan

$$X^* = \begin{pmatrix} 0 & 110 & 0 & 0 \\ 0 & 10 & 140 & 0 \\ 100 & 60 & 0 & 90 \end{pmatrix},$$

and the minimum cost of transportation on this plan is $f_{\min} = 2760$ units. ►

§3. Unbalanced transportation problems

Above, we have considered a balanced model of a linear programming transportation problem characterized by the fulfillment of the condition

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \,. \qquad (1)$$

If condition (1) is violated, then we speak of an unbalanced (open) model of the linear programming transportation problem. The following cases are possible:

$$\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j \quad \text{or} \quad \sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j \,.$$

The mathematical formulation of the problem also changes. In the first case, the linear programming transportation problem takes the following form

$$f = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} x_{ij} \to \min$$

$$\sum_{j=1}^{n} x_{ij} = a_i, \; i = 1,\dots,m,$$

$$\sum_{i=1}^{m} x_{ij} \le b_j, \; j = 1,\dots,n,$$

$$x_{ij} \ge 0, \; i = 1,\dots,m, \; j = 1,\dots,n.$$

The mathematical model of the linear programming transportation problem for the second case is written in a similar way.

It is clear that in the first case, not all consumption points B_j, $j \in \{1,...,n\}$, will satisfy the demand for products. The total amount of unmet demand is

$$\sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i .$$

In the second case, at some points of production A_i, $i \in \{1,...,m\}$, there are still unsold products of the total volume

$$\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j .$$

However, in each of these cases, a transportation plan that meets the minimum transportation costs can be constructed. To do this, in the first case, a fictitious production point A_{m+1} is introduced with a production volume of

$$a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$$

units, and transportation costs from this point are zero, i.e., $c_{m+1,j} = 0$, $j = 1,...,n$. In the second case, a fictitious consumer B_{n+1} is added with a demand volume

$$b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$$

units, and transportation costs to which are zero, i.e., $c_{i,n+1} = 0$, $i = 1,...,m$.

It is easy to see that the proposed extensions of the original unbalanced problems are balanced linear programming transportation problems that can be solved, for example, by the method of potentials. Let $\bar{X}^*$ be their optimal solution. Then the optimal solution X^* of the original problem in the first case is obtained by discarding the last line of the solution $\bar{X}^*$, which corresponds to a fictitious production point whose positive elements determine the amount of shortages to the corresponding consumers. Similarly, for the second case, the optimal solution X^* is obtained by discarding the last column of the solution $\bar{X}^*$, whose positive elements determine the amount of unsold products from the respective producers.

Remarks. When finding the initial basic solution of an extended linear programming transportation problem using the minimum element method (or another method that takes into account transportation costs), you should first select the cells for loading from among real producers or consumers, and the cells of the dummy column (row) last. This will allow you to get a plan closer to the optimal one. The same result can be achieved by setting

$$c_{m+1,j} > \max_{i,j} c_{ij}, \quad c_{i,n+1} > \max_{i,j} c_{ij},$$

respectively.

Example 4. Solve a linear programming transportation problem: $a = (30, 40, 70, 60)$, $b = (35, 80, 25, 70)$,

$$C = \begin{pmatrix} 1 & 9 & 7 & 2 \\ 3 & 1 & 5 & 5 \\ 6 & 8 & 3 & 4 \\ 2 & 3 & 1 & 3 \end{pmatrix}$$

◄ Since $a_1 + a_2 + a_3 + a_4 = 200$, $b_1 + b_2 + b_3 + b_4 = 210$, we introduce a fictitious manufacturer A_5 with production volume $a_5 = 10$ and transportation costs $c_{51} = c_{52} = c_{53} = c_{54} = 0$. We obtain a balanced transportation problem, whose initial reference plan is found by the minimum element method and its optimality is checked by the method of potentials.

	35	80	25	70	u_i
30	1 **30**	9	7	2	0
40	3	1 **40**	5	5	-1
70	6	8	3	4 **70**	2
60	2 **5**	3 **30**	1 **25**	3 **0**	1
10	0	0 **10**	0	0	-2
v_j	1	2	0	2	

Hence, we have an optimal solution to the original linear programming transportation problem:

$$X^* = \begin{pmatrix} 30 & 0 & 0 & 0 \\ 0 & 40 & 0 & 0 \\ 0 & 0 & 0 & 70 \\ 5 & 30 & 25 & 0 \end{pmatrix},$$

$$f\left(X^*\right) = 475.$$

At the same time, 10 units of products are short delivered to consumer B_2.
▶

Example 5. Solve a linear programming transportation problem: $a = (30, 70, 50)$, $b = (10, 40, 20, 60)$,

$$C = \begin{pmatrix} 2 & 7 & 3 & 6 \\ 9 & 4 & 5 & 7 \\ 5 & 7 & 6 & 2 \end{pmatrix}.$$

◀ Since $a_1 + a_2 + a_3 = 150$, $b_1 + b_2 + b_3 + b_4 = 130$, we introduce a fictitious consumer B_5 with demand $b_5 = 20$ and transportation costs $c_{15} = c_{25} = c_{35} = 0$. In the resulting balanced transportation problem, the initial reference plan is found by the minimum element method and its optimality is checked by the method of potentials.

a_i \ b_j	10	40	20	60	0	u_i
30	2 **10**	7	3 **20**	6	0	0
70	9	4 **40**	5 **0**	7 **10**	0 **20**	2
50	5	7	6	2 **50**	0	-3
v_j	2	2	3	5	-2	

The optimal solution to the original linear programming transportation problem is:

$$X^* = \begin{pmatrix} 10 & 0 & 20 & 0 \\ 0 & 40 & 0 & 10 \\ 0 & 0 & 50 & 0 \end{pmatrix},$$

$$f\left(X^*\right) = 410.$$

At the same time, the manufacturer A_2 has 20 units of unsold products. ▶

CHAPTER 3

ELEMENTS OF MATRIX GAME THEORY

Game theory originated in the early twentieth century. Only in 1944, after the publication of the work of John von Neumann and Oskar Morgenstern "Theory of Games and Economic Behavior", it became a separate science.

§1. Basic concepts

A matrix game is defined by the following rules. There are two players *A* and *B*. The first player chooses one of his possible strategies A_i, $i = 1,...,m$, and the second player chooses one of his strategies B_j, $j = 1,...,n$. The players make their choices simultaneously and independently of each other.

Let $\varphi_1(A_i, B_j)$ be the payoff of player *A* if he chooses strategy A_i and player *B* chooses strategy B_j, and let $\varphi_2(A_i, B_j)$ be the payoff of player *B* if he chooses strategy B_j and player *A* chooses strategy A_i. We consider zero-sum games:

$$\varphi_1(A_i, B_j) + \varphi_2(A_i, B_j) = 0, \text{ that is } \varphi_1(A_i, B_j) = -\varphi_2(A_i, B_j).$$

If $\varphi_1 > 0$, then φ_1 is the payoff of player *A* and it is equal to the loss of player *B*. If $\varphi_1 < 0$, then the payoff of player *B* is φ_2 and it is equal to the loss of player *A*.

Let $\varphi_1(A_i, B_j) = c_{ij}$, then the matrix $C = (c_{ij})_{i=\overline{1,m}}^{j=\overline{1,n}}$ is called the **payoff matrix** or the payoff matrix of player *A*. Player *A* is also called a **row player**, and player *B* is called a **column player**.

So, to define a game, you need to specify the sets of strategies of both players and the payment matrix. To solve a game means to specify the best choice (best strategy) for each player.

Example 1 (coin guessing). Each of the two players independently chooses a certain side of the coin, naming their choice at the same time. If different sides of the coin are chosen, the second player pays the first one currency unit, otherwise the first player pays the second one currency unit.

The payoff matrix of this game is as follows

$$C = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Consider the game from the point of view of player *A*. By choosing strategy A_i, $i = 1,...,m$, he receives at least $\min_{j=\overline{1,n}} c_{ij} = \alpha_i$ from the other player. Since player *A* seeks to maximize his payoff and can choose any row of the matrix *C*, he chooses the one that maximizes α_i. In this case, the guaranteed payoff of player *A* is equal to $\alpha = \max_{i=\overline{1,m}} \min_{j=\overline{1,n}} c_{ij}$, which is called **the lower price of the game**.

Similarly, for player *B*. Realizing that the elements of matrix *C* are payments to player *A*, he determines for each of his strategies B_j the $\beta_j = \max_{i=\overline{1,m}} c_{ij}$ – the value that he cannot lose more than this, and then chooses the strategy (column of matrix *C*) that corresponds to the minimum value of β_j. The value of $\beta = \min_{j=\overline{1,n}} \max_{i=\overline{1,m}} c_{ij}$ is called **the upper price of the game**.

So, player A can guarantee himself a win of at least α, and player B can prevent him from getting more than β. If $v = \alpha = \beta$, i.e.

$$v = \max_{i=\overline{1,m}} \min_{j=\overline{1,n}} c_{ij} = \min_{j=\overline{1,n}} \max_{i=\overline{1,m}} c_{ij}, \tag{1}$$

then player A must realize that he can get v, and his opponent will prevent him from getting more than v. Therefore, the numbers i^*, j^* such that in relation (1) $c_{i^*j^*} = v$, it is natural to call **the optimal pure strategies** of players A and B, respectively. In this case, the matrix game is said to be solvable in pure strategies, and the value of v is called **the price of the game**.

Example 2.

$$C = \begin{pmatrix} -2 & 1 & 1 & 3 & -5 \\ 4 & 2 & -5 & -6 & 2 \\ 5 & 3 & 4 & 5 & 4 \\ 7 & -3 & 5 & 1 & 6 \end{pmatrix} \xrightarrow{\min} \left.\begin{matrix} -5 \\ -6 \\ 3 \\ -3 \end{matrix}\right\} \xrightarrow{\max} 3 = \alpha$$

$$\max \underbrace{7 \quad 3 \quad 5 \quad 5 \quad 6}_{\min\ 3 = \beta}$$

For the game given by the payment matrix C, we obtained $\alpha = \beta = 3$. The game is solved in pure strategies. The optimal strategy of the row player is A_3, the optimal strategy of the column player is B_2. The price of the game is $v = 3$. The row player can always act in such a way that he gains at least 3, and the column player loses at most 3. Obviously, if one player deviates from his optimal net strategy and the other follows it, the situation of the player deviating from the optimal choice can only get worse.

It turns out that the relation (1) does not hold for every game defined by the payment matrix C. Thus, not every game has a solution in pure strategies.

Example 3. For the game „guess the coin” we have

$$C = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \xrightarrow{\min} \left.\begin{matrix} -1 \\ -1 \end{matrix}\right\} \xrightarrow{\max} -1 = \alpha$$

$$\begin{matrix} \max & 1 & 1 \\ & \min & 1 = \beta \end{matrix}$$

This game has no solution in pure strategies.

Let us establish the general conditions under which the relation (1) holds. Consider a real function $f(x, y)$ of two real variables $x \in X$, $y \in Y$.

Definition 1. *A point* (x^*, y^*) *is called* ***a saddle point of a function*** $f(x, y)$ *if for any* $x \in X$, $y \in Y$ *the inequalities*

$$f(x^*, y) \le f(x^*, y^*) \le f(x, y^*) \tag{2}$$

holds.

As a special case, we have: a saddle point of a matrix $C = (c_{ij})_{i=\overline{1,m}}^{j=\overline{1,n}}$ is a pair (i^*, j^*) such that $c_{i^* j} \le c_{i^* j^*} \le c_{ij^*}$ for all $i = 1,...,m$ and $j = 1,...,n$.

A matrix game is said to have a saddle point if its payment matrix has a saddle point.

Lemma 1. *Let there exist* $\min\limits_{y \in Y} f(x, y)$ *and* $\max\limits_{x \in X} f(x, y)$ *for a real function* $f(x, y)$, $x \in X$, $y \in Y$. *Then the inequality*

$$\max_{x \in X} \left\{ \min_{y \in Y} f(x, y) \right\} \le \min_{y \in Y} \left\{ \max_{x \in X} f(x, y) \right\}$$

holds.

For a matrix game with a payoff matrix *C*, the last inequality is $\alpha \le \beta$.

Lemma 2. *Let the conditions of Lemma 1 be satisfied. For the relation*

$$\max_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\max_{x\in X} f(x,y)$$

to hold, it is necessary and sufficient that the function $f(x,y)$ *has a saddle point. At the same time, equality*

$$f(x^*,y^*) = \max_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\max_{x\in X} f(x,y)$$

holds for the saddle point (x^*,y^*).

For a matrix game, Lemma 2 is formulated as follows:

Theorem 1. *A matrix game has a solution in pure strategies if and only if its payoff matrix has a saddle point. In this case, if* (i^*,j^*) *is the saddle point of the matrix C, then the game price is* $v = c_{i^*j^*}$.

If $\alpha = \beta = v$, then the game is said to be solvable in pure strategies. Then there exist such i^*, j^*, that $c_{i^*j^*} = v$.

§2. Optimal mixed strategies

In the previous section, it was shown that for matrix games with a saddle point, the notion of optimal pure strategies of the players can be defined in a reasonable way. At the same time, it is obvious that in the absence of a saddle point in the game's payoff matrix, none of the players should use the same pure strategy all the time.

In this regard, it is quite natural to try to define the concept of optimal strategy for matrix games without a saddle point in the class of so-called mixed strategies.

Definition 2. *The **mixed strategy** of player A is the vector*

$$u=(u_1,...,u_m),\ \ u_i\geq 0,\ \ i=1,...,m,\ \ \sum_{i=1}^{m}u_i=1,$$

and the mixed strategy of player B is the vector

$$w=(w_1,...,w_n),\ \ w_j\geq 0,\ \ j=1,...,n,\ \ \sum_{j=1}^{n}w_j=1.$$

The values u_i, $i=1,...,m$ and w_j, $j=1,...,n$ are interpreted as the probabilities with which players *A* and *B* choose their strategies A_i and B_j, respectively (*i*-th row and *j*-th column of matrix *C*).

It is clear that *i*-th pure strategy of player *A* can be viewed as a special case of his mixed strategy $u=(u_1,...,u_m)$ at $u_i=1$. The same applies to the *j*-th pure strategy of player *B*.

We denote by *U* and *W*, respectively, the sets of mixed strategies of the first and second players, i.e

$$U = \left\{ u = (u_1, ..., u_m) : u_i \geq 0,\ i = 1, ..., m,\ \sum_{i=1}^{m} u_i = 1 \right\},$$

$$W = \left\{ w = (w_1, ..., w_n) : w_j \geq 0,\ j = 1, ..., n,\ \sum_{j=1}^{n} w_j = 1 \right\}.$$

If player *A* uses his mixed strategy $u \in U$ and player *B* uses $w \in W$, then the mathematical expectation of player *B*'s payoff to player *A* (player *A*'s average payoff) is calculated as follows

$$M(u, w) = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} u_i w_j .$$

Reasoning similarly to the case of pure strategies, we conclude that player *B* can ensure an average loss of no more than

$$\min_{w \in W} \max_{u \in U} M(u, w),$$

and player *A* can secure an average win of at least

$$\max_{u \in U} \min_{w \in W} M(u, w).$$

The minimax and maximin problems are the problems of finding guaranteed mixed strategies by column and row players, respectively.

If, for some mixed strategies $u^* \in U$, $w^* \in W$ the inequalities

$$M(u^*, w) \leq M(u^*, w^*) \leq M(u, w^*)$$

hold for all $u \in U$, $w \in W$, i.e., if (u^*, w^*) is a saddle point of function $M(u, w)$, then

$$M(u^*, w^*) = \max_{u \in U} \min_{w \in W} M(u, w) = \min_{w \in W} \max_{u \in U} M(u, w).$$

The components u^* and w^* of the saddle point (u^*, w^*) of the function $M(u, w)$ are called the **optimal mixed strategies** of players *A* and *B*, respectively, and $v = M(u^*, w^*)$ is the price of the game. It is said that the matrix game has a solution in **mixed strategies**.

That is, the optimal mixed strategy of a row player is such a strategy u^* that by abandoning it, he will reduce his average payoff. On the other hand, if the row player follows the optimal strategy, he guarantees himself an average payoff regardless of the actions of the column player, which can be increased by the careless actions of the column player.

A strategy w^* is optimal for a column player if, under any actions of the other player, he guarantees himself a loss that can be reduced if the row player abandons his optimal strategy.

Theorem 2. *Any matrix game has a solution in mixed strategies.*

Properties of mixed strategies.

1^0. If player *B* uses his optimal mixed strategy, then player *A*'s average payoff will be highest when *A* uses his optimal mixed strategy.

2^0. If player *A* uses an optimal mixed strategy, then player *B*'s average loss will be the smallest if *B* uses his optimal mixed strategy.

3^0. If player *A* uses the optimal mixed strategy $u^* = (u_1, ..., u_m)$, and player *B* uses any of his pure strategies, then player *A*'s payoff is at least the price of the game:

$$\sum_{i=1}^{m} c_{ij} u_i \geq v, \quad j = 1, ..., n, \tag{3}$$

$$u_i \geq 0, \; i = 1, ..., m, \; \sum_{i=1}^{m} u_i = 1.$$

4^0. If player B uses the optimal mixed strategy $w^* = (w_1, ..., w_n)$, and player A uses any of his pure strategies, then player B's loss will not exceed the price of the game:

$$\sum_{j=1}^{n} c_{ij} w_j \le v, \;\; i = 1,...,m, \qquad (4)$$

$$w_j \ge 0, \;\; j = 1,...,n, \;\; \sum_{j=1}^{n} w_j = 1.$$

5^0. If the average payoff of player A (when he uses the optimal mixed strategy and player B uses the pure strategy B_1) is greater than the game price, i.e. $\sum_{i=1}^{m} c_{i1} u_i > v$, then this pure strategy B_1 is used with zero probability, i.e. $w_1 = 0$.

Similarly for other pure strategies and for another player.

6^0. If each element of the game matrix is increased (decreased) by the same number, the optimal mixed strategies will not change.

A similar property is true when multiplying each element of the matrix C by some positive number.

These properties also apply to games with a saddle point.

Solving a matrix game can be reduced to solving a pair of dual linear programming problems. Consider a game with a price $v > 0$. Let's introduce the notation

$$x_i = \frac{u_i}{v}, \; i = 1,...,m.$$

Dividing (3) by v, we get

$$\sum_{i=1}^{m} c_{ij} x_i \ge 1, \;\; j = 1,...,n.$$

It is obvious that $x_i \geq 0$, $i = 1,...,m$. Let's find $F = x_1 + ... + x_m$.

$$F = \frac{u_1}{v} + ... + \frac{u_m}{v} = \frac{1}{v}.$$

Since the player *A* seeks to maximize the game price *v*, the inverse of $\frac{1}{v}$ will be minimized.

Thus, finding the optimal mixed strategy of player *A* is reduced to solving the following linear programming problem:

$$F = x_1 + x_2 + ... + x_m \to \min$$

$$\sum_{i=1}^{m} c_{ij} x_i \geq 1, \ j = 1,...,n, \tag{5}$$

$$x_i \geq 0, \ i = 1,...,m.$$

Similarly, to determine the optimal strategy of player *B*: by dividing (4) by *v* and introducing the notation

$$y_j = \frac{w_j}{v}, \ j = 1,...,n$$

we take into account that player *B* seeks to minimize the loss. Therefore we get the problem

$$f = y_1 + y_2 + ... + y_n \to \max$$

$$\sum_{j=1}^{n} c_{ij} y_j \leq 1, \ i = 1,...,m, \tag{6}$$

$$y_j \geq 0, \ j = 1,...,n.$$

Problems (5), (6) form a pair of dual linear programming problems.

Thus, to solve a matrix game of size $m \times n$, you need to:

1. Reduce the dimensionality of the game's payment matrix by eliminating disadvantageous strategies in advance.

2. Determine the upper and lower prices of the game, check the game matrix for a saddle point. If there is a saddle point, then the corresponding strategies will be optimal, the game price will coincide with the upper and lower prices of the game.

3. In the absence of a saddle point, the solution must be sought among mixed strategies by reducing the matrix game to a pair of dual linear programming problems.

4. Solve one of the pair of dual problems by the simplex method.

5. Write the solution of the matrix game in mixed strategies.

§3. Games of the order $2\times n$ and $m\times 2$. Dominance

Item 3.1. Games of order 2×2

Consider a game with a 2×2 payment matrix

$$C=\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

that has no saddle point. Assume that $c_{11}\neq c_{12}$. The optimal mixed strategies u and w must satisfy the following inequalities

$$\begin{cases} c_{11}w_1+c_{12}w_2\leq v, \\ c_{21}w_1+c_{22}w_2\leq v, \\ w_1+w_2=1, \\ w_1\geq 0, \\ w_2\geq 0, \end{cases} \tag{7}$$

$$\begin{cases} c_{11}u_1+c_{21}u_2\geq v, \\ c_{12}u_1+c_{22}u_2\geq v, \\ u_1+u_2=1, \\ u_1\geq 0, \\ u_2\geq 0. \end{cases} \tag{8}$$

It can be shown that the signs of inequalities in (7) and (8) cannot be strict. That is, (7) and (8) are written as follows:

$$\begin{cases} c_{11}w_1 + c_{12}w_2 = v, \\ c_{21}w_1 + c_{22}w_2 = v, \\ w_1 + w_2 = 1, \\ w_1 \geq 0, \\ w_2 \geq 0, \end{cases} \tag{9}$$

$$\begin{cases} c_{11}u_1 + c_{21}u_2 = v, \\ c_{12}u_1 + c_{22}u_2 = v, \\ u_1 + u_2 = 1, \\ u_1 \geq 0, \\ u_2 \geq 0. \end{cases} \tag{10}$$

In system (10), let's subtract the first equation from the second:

$$u_1(c_{12} - c_{11}) + u_2(c_{22} - c_{21}) = 0$$

Hence, at $c_{11} \neq c_{12}$ we have

$$\frac{u_1}{u_2} = \frac{c_{22} - c_{21}}{c_{11} - c_{12}}.$$

Since $u_1 \geq 0$, $u_2 \geq 0$, then

$$\frac{u_1}{u_2} = \frac{|c_{22} - c_{21}|}{|c_{11} - c_{12}|}.$$

Hence

$$\frac{u_1}{1 - u_1} = \frac{|c_{22} - c_{21}|}{|c_{11} - c_{12}|}.$$

Then

$$u_1 = \frac{|c_{22} - c_{21}|}{|c_{11} - c_{12}| + |c_{22} - c_{21}|} \tag{11}$$

and also

$$u_2 = \frac{|c_{11} - c_{12}|}{|c_{11} - c_{12}| + |c_{22} - c_{21}|}. \qquad (12)$$

Similar formulas are obtained for w_1 and w_2:

$$w_1 = \frac{|c_{22} - c_{12}|}{|c_{11} - c_{21}| + |c_{22} - c_{12}|}, \; w_2 = \frac{|c_{11} - c_{21}|}{|c_{11} - c_{21}| + |c_{22} - c_{12}|}. \qquad (13)$$

We will show that if a row player applies his optimal mixed strategy, the mathematical expectation of winning is equal to the game price regardless of the actions of the column player, i.e. in this case, player *B* cannot play better or worse – any of his actions leads to the same result.

Let player *A* choose the optimal mixed strategy u^* and player *B* choose any mixed strategy *w*. Then the mathematical expectation of player *A*'s payoff is

$$M(u^*, w) = w_1(c_{11}u_1^* + c_{21}u_2^*) + w_2(c_{12}u_1^* + c_{22}u_2^*) = w_1 v + w_2 v = v.$$

Similarly, it can be shown that if player *B* applies his optimal mixed strategy, then the mathematical expectation of player *B*'s loss is equal to the game price regardless of player *A*'s actions.

Remark. *For games of a larger order than 2×2, such conclusions are generally incorrect.*

Item 3.2. Games of order $2\times n$ and $m\times 2$

Consider an $m\times n$ game with a payment matrix $C=(c_{ij})_{i=\overline{1,m}}^{j=\overline{1,n}}$. If $c_{ij}>c_{kj}$, $j=1,...,n$, then strategy A_i is said to **dominate strategy** A_k. Similarly, if $c_{il}<c_{is}$, $i=1,...,m$, strategy B_l is said to dominate strategy B_s.

Let $u*$ be one of the optimal mixed strategies of player *A*. If $u_i^*\neq 0$, then strategy A_i is called a **useful (active) strategy**. It can be shown that an $m\times n$ game has at most min(*m*, *n*) useful strategies, so the $2\times n$ and $m\times 2$ games have at most two useful strategies for each player. Other strategies should not be used in the optimal mixed strategy. It is not always possible to identify their dominance.

Let's consider a graphical method for solving matrix games.

Example 4. Solve the game with a payoff matrix

$$C=\begin{pmatrix}2 & 3\\ 7 & 1\end{pmatrix}.$$

◄ There is no saddle point in this game. For mixed strategies, we write systems (9), (10):

$$\begin{cases}2w_1+3(1-w_1)=v,\\ 7w_1+(1-w_1)=v,\\ 0\leq w_1\leq 1,\end{cases}\qquad \begin{cases}2u_1+7(1-u_1)=v,\\ 3u_1+(1-u_1)=v,\\ 0\leq u_1\leq 1.\end{cases}$$

Hence

$$\begin{cases}5u_1+v=7,\\ 2u_1-v=-1,\\ 0\leq u_1\leq 1.\end{cases}$$

In the u_1Ov coordinate system, we draw the corresponding lines (Fig. 1). Their intersection point determines the coordinates u_1^* and v^*.

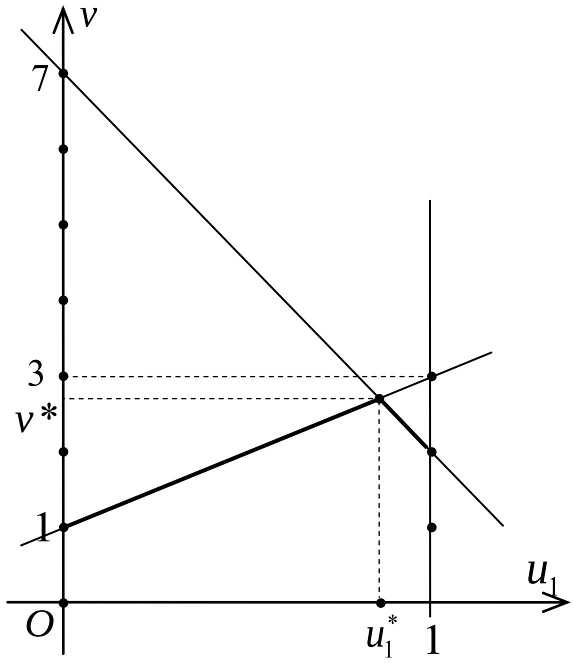


Fig. 1.

We get $u_1^* = \frac{6}{7}$. Then $u_2^* = \frac{1}{7}$ and $v^* = \frac{19}{7}$.

Applying formulas (11)-(13) from the previous paragraph, we obtain a similar result:

$$u_1^* = \frac{|1-7|}{|2-3|+|1-7|} = \frac{6}{7}; \quad u_2^* = \frac{|2-3|}{|2-3|+|1-7|} = \frac{1}{7};$$

$$w_1^* = \frac{|1-3|}{|2-7|+|1-3|} = \frac{2}{7}; \quad w_2^* = \frac{|2-7|}{|2-7|+|1-3|} = \frac{5}{7}$$

and the price of the game $v^* = 2\cdot\frac{2}{7} + 3\cdot\frac{5}{7} = \frac{19}{7}$. ►

Example 5. Solve a game with a payoff matrix

$$C=\begin{pmatrix}-6 & -1\\ 7 & -2\\ 1 & 6\\ 4 & 3\\ 3 & -2\end{pmatrix}.$$

◀

$$C=\begin{pmatrix}-6 & -1\\ 7 & -2\\ 1 & 6\\ 4 & 3\\ 3 & -2\end{pmatrix}\xrightarrow{\min}\left.\begin{matrix}-6\\ -2\\ 1\\ 3\\ -2\end{matrix}\right\}\xrightarrow{\max} 3=\alpha$$

$$\max \quad \underbrace{7 \qquad 6}_{\min\ 6=\beta}$$

This game has no saddle point since $\alpha \neq \beta$, and therefore has no solution in pure strategies. We can solve the problem by reducing it to a pair of linear programming problems. Let's write out one of them:

$$\begin{cases}-6w_1-(1-w_1)=v,\\ 7w_1-2(1-w_1)=v,\\ w_1+6(1-w_1)=v,\\ 4w_1+3(1-w_1)=v,\\ 3w_1-2(1-w_1)=v,\end{cases}$$

$$\begin{cases}5w_1+v=-1,\\ 9w_1-v=2,\\ 5w_1+v=6,\\ w_1-v=-3,\\ 5w_1-v=2,\end{cases}$$

$$0\le w_1\le 1.$$

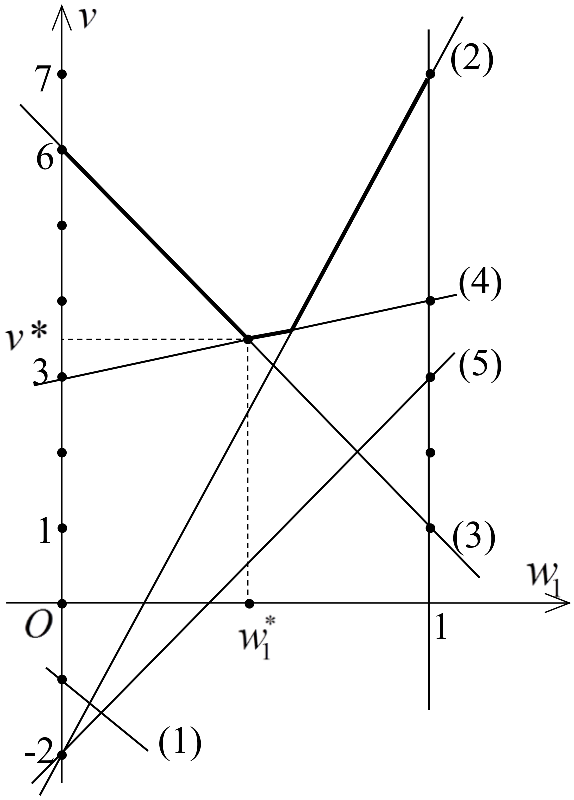


Fig. 2.

Let's draw the corresponding lines in the w_1Ov coordinate system (Fig. 2). The point of intersection of the third and fourth lines defines w_1^* and v^*.

$$\begin{cases} 5w_1 + v = 6, \\ w_1 - v = -3. \end{cases}$$

Hence $6w_1 = -3$, $w_1^* = \dfrac{1}{2}$, $w_2^* = \dfrac{1}{2}$, $v^* = \dfrac{7}{2}$.

The active strategies of player A are the third and fourth (this can also be seen using dominance). Therefore, we have a matrix

$$\begin{matrix} A_3 \\ A_4 \end{matrix} \begin{pmatrix} 1 & 6 \\ 4 & 3 \end{pmatrix}$$

for u_3 and u_4. Applying formulas (11)-(13) from the previous paragraph, we obtain

$$u_3^* = \frac{|3-4|}{|1-6|+|3-4|} = \frac{1}{6}; \quad u_4^* = \frac{|1-6|}{|1-6|+|3-4|} = \frac{5}{6};$$

$$w_1^* = \frac{|3-6|}{|1-4|+|3-6|} = \frac{3}{6} = \frac{1}{2}; \qquad w_2^* = \frac{1}{2}.$$

Then

$$u^* = \left(0;\ 0;\ \frac{5}{6};\ \frac{1}{6};\ 0;\ 0\right), \qquad w^* = \left(\frac{1}{2};\ \frac{1}{2}\right)$$

and the price of the game $v^* = 1 \cdot \frac{1}{2} + 6 \cdot \frac{1}{2} = \frac{7}{2}$. ▶

Example 6. Solve a game with a payoff matrix

$$C = \begin{pmatrix} 3 & -4 & 2 & -1 & -3 & 5 & 1 \\ 1 & 2 & -1 & 3 & 4 & 0 & -3 \end{pmatrix}.$$

◀ Strategy B_3 dominates strategy B_1, so $w_1 = 0$. In addition, strategy B_2 dominates strategy B_4, so $w_4 = 0$. Then the linear programming problem has the form

$$\begin{cases} -4u_1 + 2(1-u_1) = v, \\ 2u_1 - (1-u_1) = v, \\ -3u_1 + 4(1-u_1) = v, \\ 5u_1 \qquad\qquad = v, \\ u_1 - 3(1-u_1) = v, \\ 0 \le u_1 \le 1, \end{cases}$$

$$\begin{cases} 6u_1 + v = 2, \\ 3u_1 - v = 1, \\ 7u_1 + v = 4, \\ 5u_1 - v = 0, \\ 4u_1 - v = 3. \end{cases}$$

In the u_1Ov coordinate system, let's draw the corresponding lines (Fig. 3). The intersection of the lines corresponding to strategies B_2 and B_7 is determined by u_1^* and v^*.

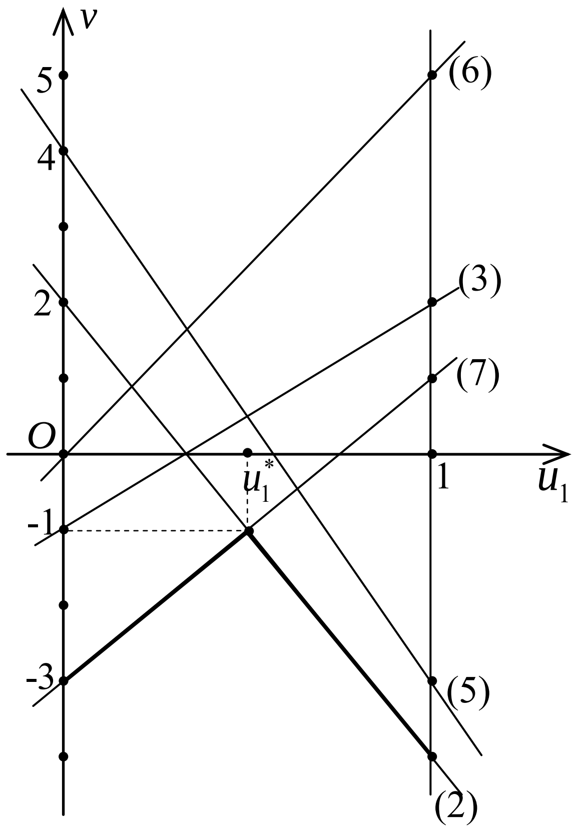


Fig. 3.

Player *B*'s active strategies are the second and seventh (this can also be seen using dominance). Therefore, for w_2 and w_7 we have the matrix

$$\begin{matrix} B_2 & B_7 \end{matrix}$$
$$\begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix}.$$

Applying formulas (11)-(13) from the previous paragraph, we obtain

$$u_1^* = \frac{|-3-2|}{|-4-1|+|-3-2|} = \frac{5}{10} = \frac{1}{2}; \qquad u_2^* = \frac{1}{2};$$

$$w_2^* = \frac{|-3-1|}{|-4-2|+|-3-1|} = \frac{4}{10} = \frac{2}{5}; \qquad w_7^* = \frac{|-4-2|}{|-4-2|+|-3-1|} = \frac{3}{5}.$$

Then

$$u^* = \left(\frac{1}{2}; \frac{1}{2}\right), \qquad w^* = \left(0; \frac{2}{5}; 0; 0; 0; 0; \frac{3}{5}\right)$$

and the price of the game $v^* = 2 - 6 \cdot \frac{1}{2} = -1$. ►

CHAPTER 4

NONLINEAR PROGRAMMING

§1. Problem statements

In Chapter 1, the mathematical programming problem was written as follows:

$$f(x) \to extr, \qquad (1)$$

$$f_i(x) \le 0, \ i = 1,...,m, \ x \in \mathbb{R}^n \qquad (2)$$

If at least one of the functions $f(x)$, $f_i(x)$, $i = 1,...,m$, is nonlinear, then the problem (1)-(2) is called **a nonlinear programming problem**.

The function $f(x)$ is called the **objective function**, and the functions $f_i(x)$, $i = 1,...,m$ are called the **functions of the conditions** of the nonlinear programming problem. The set

$$D = \left\{ x \in \mathbb{R}^n \mid f_i(x) \le 0, i = 1,...,m \right\}$$

is called the **admissible set**, and the points $x \in D$ are called **admissible solutions** of the nonlinear programming problem.

An admissible solution $x^* \in D$, where the function $f(x)$ reaches an extreme (minimum or maximum) value, is called an **optimal solution** to problem (1)-(2):

$$x^* = \arg\min_{x \in D} f(x) \ \text{(or } x^* = \arg\max_{x \in D} f(x)\text{)}.$$

Since the problem of maximizing the function $f(x)$ is equivalent to the problem of minimizing the function $-f(x)$, the constraint $f_i(x) \geq 0$ is equivalent to the constraint $-f_i(x) \leq 0$, the equality $f_i(x) = 0$ is equivalent to the system of two inequalities $f_i(x) \leq 0$ and $-f_i(x) \leq 0$, when formulating the nonlinear programming problem (1)-(2), we restricted ourselves to the case of minimizing the function $f(x)$ under the conditions $f_i(x) \leq 0$, $i = 1,...,m$, $x \in \mathbb{R}^n$. This problem will be written in the form

$$\min\left\{ f(x) : x \in \mathbb{R}^n, f_i(x) \leq 0, i = 1,...,m \right\} \tag{3}$$

In many nonlinear programming problems, an additional condition is imposed in the form of

$$x \in X \text{ , where } X \subset \mathbb{R}^n,$$

for example,

$$X = \left\{ x \in \mathbb{R}^n : x_j \geq 0,\ j = 1,...,n \right\}. \tag{4}$$

Then the nonlinear programming problem is written in the form:

$$\min\left\{ f(x) : x \in \mathbb{R}^n,\ f_i(x) \leq 0,\ i = 1,...,m,\ x \in X \right\}. \tag{5}$$

Nonlinear programming problems are divided into classes depending on the properties of the functions and types of constraints used in the problem formulation.

1. Classical optimization problems

Classical optimization problems are also called conditional extremum problems. They are characterized by the fact that the constraints are written in the form of inequalities. Therefore, they are formulated as follows:

$$f(x) \to extr,$$

$$f_i(x) = 0,\ x \in \mathbb{R}^n,\ i = 1,...,m,\ m \le n.$$

When $m = 0$, we obtain the classical problem of unconditional extremum of the function $f(x)$, $x \in \mathbb{R}^n$.

In classical optimization problems, the condition of existence and continuity of the partial derivatives of functions $f(x)$ and $f_i(x)$ at least up to and including the 2nd order.

Problems of this class can be solved by classical methods using the apparatus of differential calculus. However, the computational difficulties that arise in this case are quite significant, and therefore, to solve practical problems, other methods have to be used.

2. Problems with nonlinear objective function and linear constraints

Problems of this class are written as follows:

$$f(x) \to \min$$

$$f_i(x) = \sum_{j=1}^{n} a_{ij} x_j - b_i \le 0,\ \ i = 1,...,m,$$

$$x_j \ge 0,\ \ \ j = 1,...,n.$$

An important property of these problems is that their admissible set is a polyhedral set.

3. Problems of quadratic programming

In this class of problems, the objective function is quadratic and the constraints are linear. They can be written as follows:

$$f(x) = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{n} \sum_{j=1}^{n} d_{ij} x_i x_j \ \to \ \min$$

$$f_i(x) = \sum_{j=1}^{n} a_{ij} x_j - b_i \le 0, \ \ i = 1, \dots, m,$$

$$x_j \ge 0, \ \ \ j = 1, \dots, n,$$

with the objective function $f(x)$ being convex downward.

Quadratic programming problems can be attributed to both – the previous class and the class of convex programming problems. However, they are allocated to a separate class due to the specifics of the objective function.

4. Problems of convex programming

The class of convex programming problems includes nonlinear programming problems written in the form (3) or (5), in which the objective function $f(x)$ is convex downward and the admissible set is convex. Methods for solving these problems are the most developed in nonlinear programming, and their generalization to other problems of optimization methods.

5. Problems of separable programming

In separable programming problems, a characteristic feature is that both the objective function $f(x)$ and the condition functions, denoted by $g_i(x)$, are additive functions. They can be written as follows:

$$f(x) = \sum_{j=1}^{n} f_j(x_j),$$

$$g_i(x) = \sum_{j=1}^{n} g_{ij}(x_j), \quad i = 1,...,m.$$

The specifics of these problems define a special class of methods for solving them, which are applicable and effective only for such problems.

Most numerical methods for solving nonlinear programming problems allow you to find only approximate solutions or require an infinite number of steps to achieve an exact solution. These are, for example, the group of gradient methods or penalty function methods. In addition, the solutions obtained are only local extrema (methods for finding global extrema constitute a separate important class of optimization methods). This is the fundamental difference between nonlinear programming methods and methods for solving linear programming problems, which can determine whether a solution to a linear programming problem exists in a finite number of steps and find it in case of existence.

§2. Geometric interpretation of nonlinear programming problems

Let us consider the geometric interpretation of nonlinear programming problems and analyze the differences from linear programming problems.

Example 1 : Solve a linear programming problem

$$f = 0,5x_1 + 2x_2 \to \max,$$

$$\begin{cases} x_1 + x_2 \le 6, \\ x_1 - x_2 \le 1, \\ x_1 - 2x_2 \ge -8, \end{cases}$$

$$x_1 \ge 0,\ x_2 \ge 0.$$

◄ Using the graphical method (see Fig. 1), we get

$$x^* = (4/3, 14/3), \quad f^* = 10. \quad ►$$

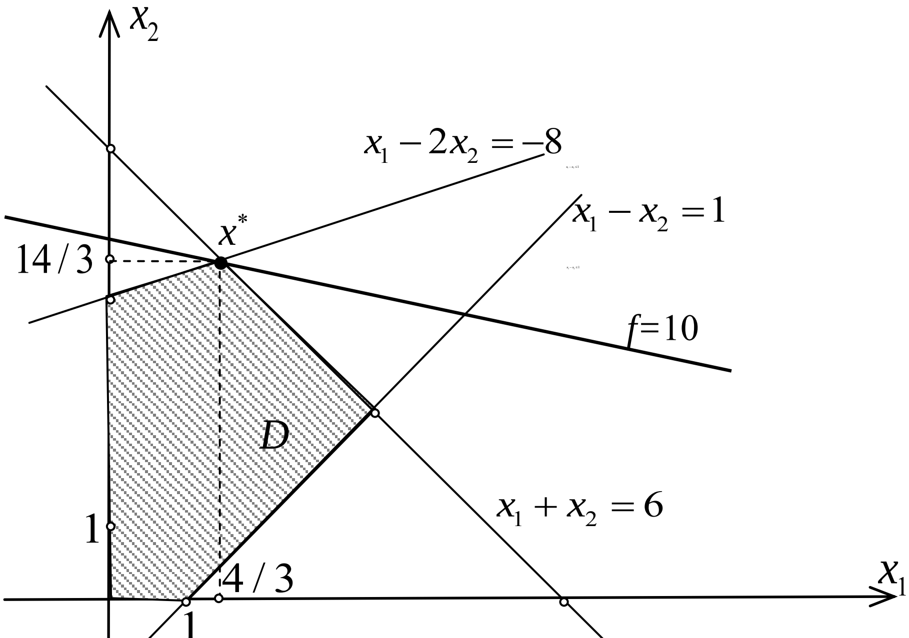


Fig. 1

As we can see, the admissible set of a linear programming problem is a convex polyhedral set with a finite number of vertices. The lines of the objective function of a linear programming problem are straight lines. Those lines that correspond to different values of the level constant are parallel to each other. The obtained local maximum in a linear programming problem is both global and local and is achieved at the vertex of the admissible set.

Let's consider examples of nonlinear programming problems in which these properties do not apply.

Example 2 : Solve a nonlinear programming problem

$$f = 10(x_1 - 3,5)^2 + 20(x_2 - 4)^2 \to \min$$

on the admissible set of Example 1.

◄ The objective function in the problem is quadratic, and the constraints are linear, so the problem is a quadratic programming problem. To solve it, we will use the geometric interpretation (Fig. 2) of the problem. As we can see, the optimal solution x^* is the point of contact between

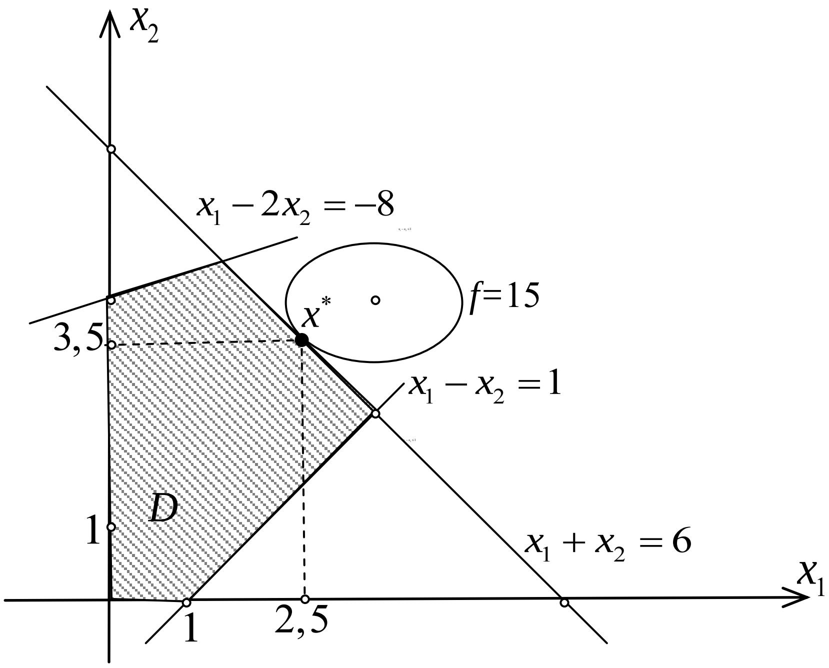


Fig. 2

the line of the objective function level (ellipse) and the line $x_1 + x_2 = 6$. To determine the coordinates of the point x^*, we have the equation

$$x_1^* + x_2^* = 6. \qquad (6)$$

The second equation is obtained by taking into account that the angular coefficient of the tangent line to the level line at the point x^* is equal to the angular coefficient of the line $x_1 + x_2 = 6$, i.e., -1. Considering x_2 to be an implicit function of x_1, defined by the relation

$$F(x_1, x_2) = 10(x_1 - 3{,}5)^2 + 20(x_2 - 4)^2 - f = 0,$$

where f is the parameter that defines the level constant, we obtain, according to the rule of differentiation of the implicit function

$$F'_{x_1} + F'_{x_2}\frac{dx_2}{dx_1} = 0,$$

the second equation for finding the coordinates of the point x^*. We have

$$\frac{dx_2}{dx_1} = -\frac{F'_{x_1}}{F'_{x_2}} = \frac{-20(x_1 - 3{,}5)}{40(x_2 - 4)} = -\frac{x_1 - 3{,}5}{2(x_2 - 4)}.$$

On the other hand, $\frac{dx_2}{dx_1} = -1$, as the angular coefficient of the line $x_2 = -x_1 + 6$. Therefore, at the point x^* equality

$$-\frac{x_1 - 3{,}5}{2(x_2 - 4)} = -1 \qquad (7)$$

is fulfilled. By solving the system of equations (6)-(7), we get $x^* = (2{,}5;\, 3{,}5)$, and then we find $f^* = 15$. ►

Note that, unlike a linear programming problem, the optimal solution of a nonlinear programming problem is not the vertex of its admissible set, although it is achieved on its boundary.

Example 3. Solve the nonlinear programming problem

$$f = 10(x_1 - 2)^2 + 20(x_2 - 3)^2 \rightarrow \min$$

$$\begin{cases} x_1 + x_2 \le 6, \\ x_1 - x_2 \le 1, \\ x_1 - 2x_2 \ge -8, \end{cases}$$

$$x_1 \ge 0,\ x_2 \ge 0.$$

on the admissible set of Example 1.

◄ The function f is a convex quadratic function that takes on only non-negative values. Therefore, it takes its smallest value f^*, which is zero, at the point $x^* = (2;\,3)$, which lies inside the domain D (see Fig. 3).

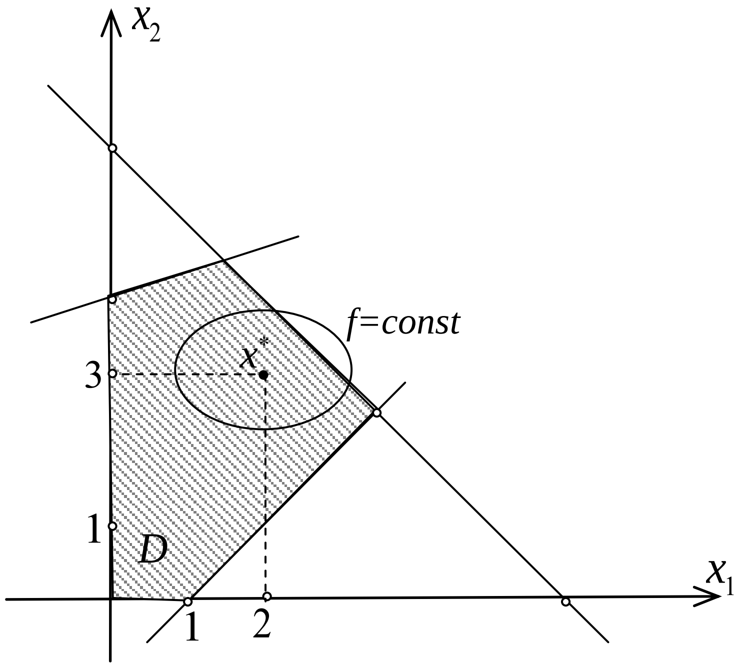


Fig. 3

►

Thus, the optimal value of the objective function can be achieved at an internal point of the admissible domain of the nonlinear programming problem.

Example 4. Solve a nonlinear programming problem

$$f = 25(x_1 - 2)^2 + (x_2 - 2)^2 \to \max$$

$$\begin{cases} x_1 + x_2 \ge 2, \\ x_1 - x_2 \ge -2, \\ x_1 + x_2 \le 6, \\ x_1 - 3x_2 \le 2, \end{cases}$$

$$x_1 \ge 0,\ x_2 \ge 0.$$

◄ From the geometric interpretation, it follows that the feasible set of this problem is the quadrilateral $V_1V_2V_3V_4$ (see Fig. 4), and the level lines are ellipses. The function *f* has four maxima on the admissible set, which are achieved at all vertices of the quadrilateral $V_1V_2V_3V_4$. The maxima at the points V_1 ($f^* = 4$), V_2 ($f^* = 100$), V_3 ($f^* = 4$) are local. The maximum at V_4 ($f^* = 226$) is global. ►

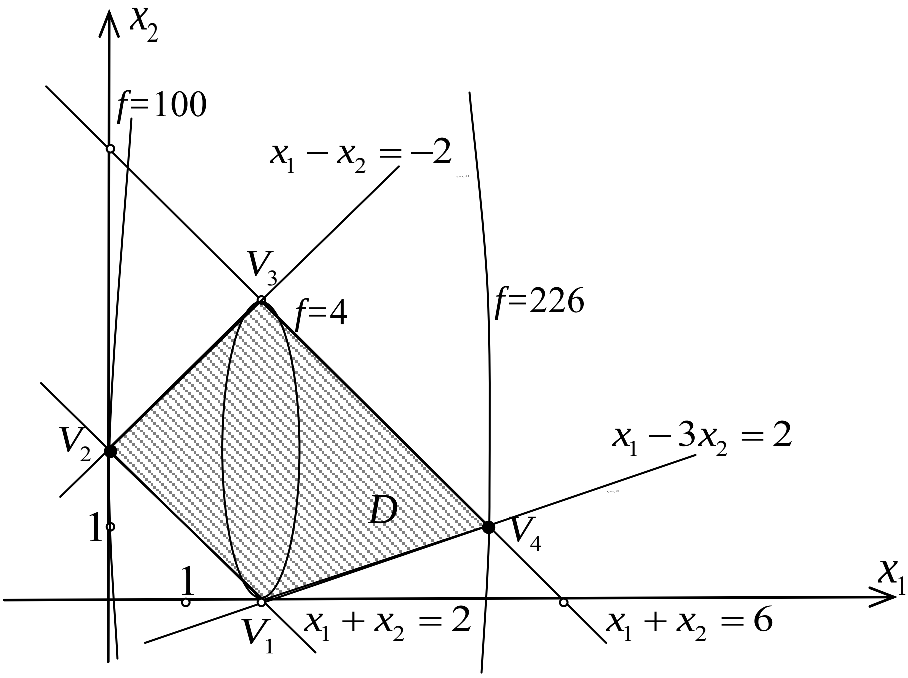


Fig. 4

Thus, a nonlinear programming problem can have several optima, and not necessarily only one of them will determine its optimal solution, unlike a linear programming problem.

In the examples above, the feasible sets were convex because they were determined by systems of linear inequalities. In general, when the constraints of a nonlinear programming problem include nonlinear ones, its feasible set may be neither convex nor connected.

Example 5. Analyze the feasible set of a nonlinear programming problem given the following system of constraints:

$$\begin{cases} (x_1 - 1)x_2 \le 1, \\ x_1 + x_2 \ge 4, \end{cases}$$

$$x_1 \ge 0, \;\; x_2 \ge 0.$$

◀ Let us graphically represent the given system of constraints (see Fig. 5). The admissible set *D* is formed by the points of the plane bounded by the hyperbola branch $x_2 = \dfrac{1}{x_1 - 1}$, the line $x_1 + x_2 = 4$, and the coordinate axes. As can be seen from the geometric interpretation, the admissible set is not only not convex, but not even a connected set. This is in contrast to linear programming problems, where the admissible set is a convex polyhedral set. ▶

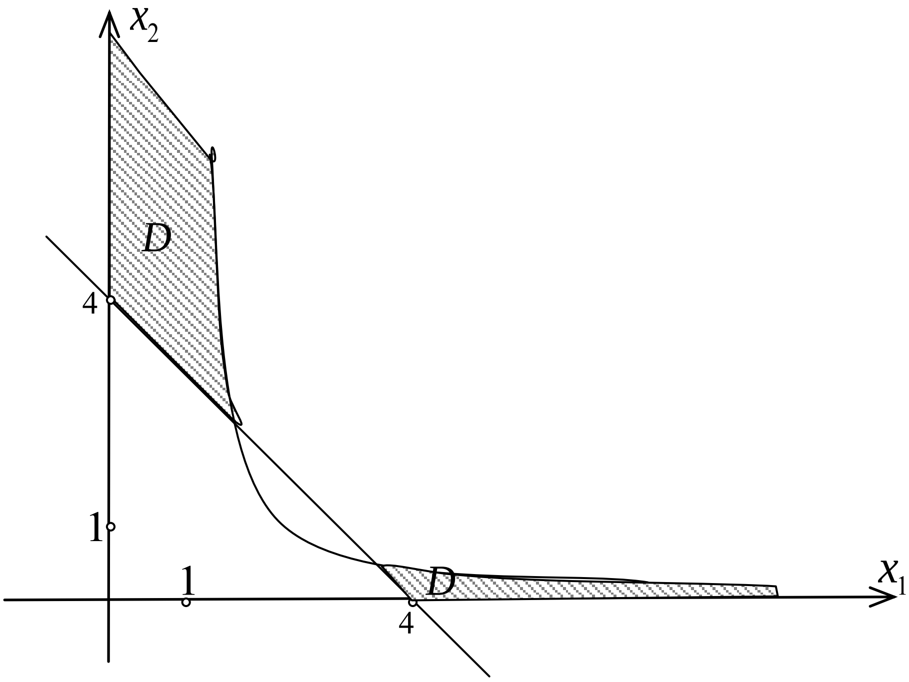


Fig. 5

§3. One-dimensional optimization problems

Item 3.1. Statement of the problem

In general, a one-dimensional optimization problem is formulated as follows: find the extremum (maximum or minimum) of a function of one variable $y = f(x)$ $(x \in \mathbb{R}^1)$ on the interval $[a, b]$.

Such problems are important in themselves, but they also have to be solved when optimizing functions of many variables.

The classical methods of differential calculus, which consist in finding the roots of the equation $f'(x) = 0$, cannot always be applied in practical problems when solving one-dimensional optimization problems. This is due to the fact that the function $y = f(x)$ is not always differentiable, and the problem of solving the equation $f'(x) = 0$ is not computationally simpler than the original problem. Therefore, there is a need to study methods for solving one-dimensional optimization problems that do not use derivatives.

Let's consider several methods for one-dimensional optimization of unimodal functions.

Definition 1. The *function* $y = f(x)$ *defined on the interval* $[a,b]$ is *called* ***unimodal*** *if it has a unique minimum point* $x^* \in [a,b]$ *on this interval and satisfies the condition*

for any x_1, x_2 *such that* $a \le x_1 < x_2 \le x^*$, *the inequality* $f(x_1) > f(x_2)$ *holds;*

for any x_1, x_2 *such that* $x^* \le x_1 < x_2 \le b$, *the inequality* $f(x_1) < f(x_2)$ *holds.*

This definition is formulated for the case of a minimization problem. It is naturally reformulated for the case of a maximization problem.

The definition does not impose the condition of continuity or convexity on the function $y = f(x)$. Therefore, a unimodal function may not be continuous and may not be convex.

Numerical methods for minimizing (or maximizing) a unimodal function are based on its main property, which directly follows from its definition (see Fig. 6).

Suppose that the function $y = f(x)$, which is unimodal on the interval $[a,b]$, has a minimum at the point x^* and points l and r of this interval such that $a < l < r < b$. Then:

$$\text{if } f(l) > f(r), \text{ then } x^* \in [l,b],$$

$$\text{if } f(l) < f(r), \text{ then } x^* \in [a,r],$$

$$\text{if } f(l) = f(r), \text{ then } x^* \in [l,r].$$

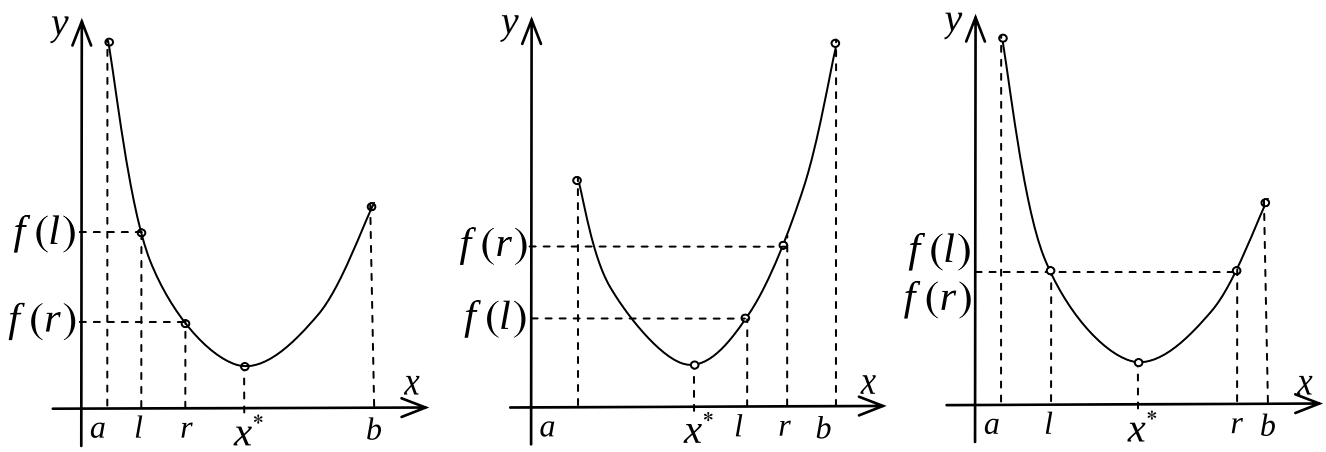


Fig. 6.

The main property of unimodal functions allows you to build sequential algorithms that reduce the minimum search interval at each step.

Let's consider some of these methods.

Item 3.2. Methods of one-dimensional optimization

1. Method of dichotomy or dividing segments in half

Let x^* be the minimum point of the function $y = f(x)$ on the interval [*a*,*b*]. The initial search interval is [*a*,*b*]. Set

$$a^0 = a,\ b^0 = b.$$

Suppose that after *k* steps of the algorithm, the segment $[a^k, b^k]$ containing x^* is constructed. Consider the *k+1* step.

Let's divide the segment $[a^k, b^k]$ in half and for some sufficiently small number $\delta > 0$ plot the points

$$l^k = (a^k + b^k - \delta)/2 \text{ and } r^k = (a^k + b^k + \delta)/2.$$

According to the basic property of the unimodal function, let us put:

$$a^{k+1} = l^k,\ b^{k+1} = b^k, \text{ if } f(l^k) > f(r^k),$$

$$a^{k+1} = a^k,\ b^{k+1} = r^k, \text{ if } f(l^k) < f(r^k),$$

$$a^{k+1} = l^k,\ b^{k+1} = r^k, \text{ if } f(l^k) = f(r^k).$$

We get the segment $[a^{k+1}, b^{k+1}]$ containing the point x^*.

After performing *n* steps of the method, the segment $[a^n, b^n]$ will be constructed, containing the minimum point x^* and the length of the segment is

$$b^n - a^n = (b-a)/2^n + (1-2^{-n})\delta.$$

We take the middle of the last segment $x^* \approx (b^n + a^n)/2$ as the approximate value of the minimum point, with the error in the calculation of x^* not exceeding the number $\varepsilon = (b^n - a^n)/2$. We take the number $y^n = f\left((b^n + a^n)/2\right)$ to be the minimum value of the function $y = f(x)$ on the interval $[a, b]$.

At each step of the described method of dividing segments in half, you need to calculate the value of the function at two points.

More efficient from a computational point of view are the golden ratio and Fibonacci methods.

2. The golden ratio method

The golden ratio of a line segment is the ratio of its length to the length of the larger part of the segment, as the length of the larger part is to the length of the smaller part. There are two points on the segment that make up its golden ratio. Let us have a line segment $[a, b]$. Let us denote by r the point of its golden ratio for which the condition $r - a > b - r$ is satisfied. That is, the segment $[a,r]$ is greater than the segment $[r,b]$. Let x denote the length of the segment $[a,r]$, then the length of the segment $[r,b]$ is $b - a - x$. The length of the segment $[a,r]$ is found from the equation

$$\frac{b-a}{x} = \frac{x}{b-a-x}.$$

We get a quadratic equation $x^2 + (b-a)x - (b-a)^2 = 0$ with solutions $x_{1,2} = (b-a)\dfrac{\pm\sqrt{5}-1}{2}$, of which only the positive root $x = (b-a)\dfrac{\sqrt{5}-1}{2}$ is suitable. So, $r = a + (b-a)\dfrac{\sqrt{5}-1}{2}$.

The second point of the golden ratio of the segment [*a*,*b*] is denoted by *l*. It is located at a distance *x* from point *b*, i.e. $l = b-(b-a)\frac{\sqrt{5}-1}{2}$.

It can be shown that point *l* is the golden ratio of the segment [*a*,*r*], and point *r*, in turn, is the golden ratio of the segment [*l*,*b*].

Let us describe the algorithm of the golden section method.

Let x^* be the minimum point of the function $y = f(x)$ on the interval [*a*,*b*]. At the beginning of the calculation, let's set $a^0 = a$, $b^0 = b$.

At the *k-th* step, we define the values

$$l^k = b^k - \tau(b^k - a^k),$$

$$r^k = a^k + \tau(b^k - a^k),$$

where constant $\tau = \frac{\sqrt{5}-1}{2} \approx 0{,}618033989$. Let's put

$$a^{k+1} = l^k,\ b^{k+1} = b^k,\ \text{if}\ f(l^k) > f(r^k),$$

$$a^{k+1} = a^k,\ b^{k+1} = r^k,\ \text{if}\ f(l^k) \le f(r^k).$$

The method is iterated until the condition

$$b^n - a^n \le \varepsilon,$$

is met, where $\varepsilon > 0$ is a given number that determines the permissible error of the problem solution.

Unlike the dichotomy method, the golden section method calculates the value of the function $f(x)$ at each step at only one point, since one of the points of the golden section at the previous step is the golden section of the segment at the next step.

We take the middle of the last segment $x^* \approx (b^n + a^n)/2$ as the approximate value of the minimum point. We take the number $y^* = f(x^*)$ to be the minimum value of the function $y = f(x)$ on the interval $[a,b]$.

When solving the problem of maximizing the function $f(x)$, you need to move on to the problem of minimizing the function $-f(x)$.

Example 6. Find the minimum of the function $f(x) = e^{-x} - 2\cos x$ on the interval [0,1] with an accuracy of $\varepsilon \le 0,05$ using the golden ratio method.

◄ The results are shown in the following table.

k	a^k	b^k	l^k	r^k	$f(l^k)$	$f(r^k)$	sign	ε
0	0,000	1,000	0,382	0,618	–1,1773	–1,0911	$<$	0,309
1	0,000	0,618	0,236	0,382	–1,1548	–1,1733	$>$	0,191
2	0,236	0,618	0,382	0,472	–1,1733	–1,1576	$<$	0,118
3	0,236	0,472	0,326	0,382	–1,1729	–1,1733	$>$	0,073
4	0,326	0,472	0,382	0,416	–1,1733	–1,1697	$<$	0,045

The required accuracy is achieved on the fourth iteration, so we take $x^* = (0,326 + 0,416)/2 = 0,371$ as the minimum point and the minimum value of the function $f(x^*) = f(0,371) = -1,1739$. ►

§4. Classical optimization methods

Item 4.1. The problem of unconditional minimization

The problem of minimizing the function $y = f(x)$ without any restrictions on the arguments of the function is called the problem of unconditional optimization and is written

$$f(x) \to \min, \; x = (x_1, ..., x_n) \in \mathbb{R}^n. \qquad (8)$$

The gradient of the function $y = f(x)$ is denoted by $\nabla f(x)$, i.e.,

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, ..., \frac{\partial f(x)}{\partial x_n} \right),$$

and the matrix of second partial derivatives of the function $y = f(x)$, also called the Hessian matrix (or Hessian of the function), is denoted by $H_f(x)$, i.e.,

$$H_f(x) = \begin{pmatrix} \frac{\partial f^2(x)}{\partial x_1 \partial x_1} & \frac{\partial f^2(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial f^2(x)}{\partial x_1 \partial x_n} \\ \frac{\partial f^2(x)}{\partial x_2 \partial x_1} & \frac{\partial f^2(x)}{\partial x_2 \partial x_2} & \cdots & \frac{\partial f^2(x)}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f^2(x)}{\partial x_n \partial x_1} & \frac{\partial f^2(x)}{\partial x_n \partial x_2} & \cdots & \frac{\partial f^2(x)}{\partial x_1 \partial x_n} \end{pmatrix}.$$

Theorem 1. (Necessary conditions for minimum). *Suppose the function* $y = f(x)$ *has a minimum (local or global) at the point* x^0. *Then:*

1) if $f(x) \in C^1$, *then at point* x^0 $\nabla f\left(x^0\right) = 0$;

2) if $f(x) \in C^2$, *then at point* x^0 $\nabla f\left(x^0\right) = 0$ *and the Hessian* $H_f\left(x^0\right)$ *is a nonnegatively definite matrix, i.e. for any* $y \in \mathbb{R}^n$ $y^T H_f\left(x^0\right) y \geq 0$.

Definition 2. *The point* x^0, *that satisfies the condition* $\nabla f\left(x^0\right) = 0$ *is called* ***the stationary point*** *of a differentiable function* $y = f(x)$.

It can be shown that x^0 is a stationary point of the differentiable function $y = f(x)$, if for any $y \neq 0$ the equality $\nabla^T f\left(x^0\right) y = 0$ holds.

Remark. *The necessary conditions for the maximum of the function* $f(x) \in C^2$ *at point* x^0 *are the condition of stationarity of the point* x^0, *i.e.,* $\nabla f\left(x^0\right) = 0$ *and the nonnegative definiteness of its Hessian at this point, i.e., for any* $y \in \mathbb{R}^n$

$$y^T H_f\left(x^0\right) y \leq 0.$$

Theorem 2 (Sufficient condition for local minimum). *If the conditions*

1) $f(x) \in C^2$, *for* $x \in \mathbb{R}^n$,

2) at the point $x^0 \in \mathbb{R}^n$ *the condition of stationarity* $\nabla f\left(x^0\right) = 0$ *is fulfilled or for any* $y \in \mathbb{R}^n$ *the condition* $\nabla^T f\left(x^0\right) y = 0$ *is satisfied,*

3) the Hessian $H_f\left(x^0\right)$ *is a positive definite matrix*

are fulfilled, then x^0 *is the point of strict local minimum of the function* $y = f(x)$.

Remark. *Sufficient conditions for a strict local maximum of the function* $f(x) \in C^2$ *at point* x^0 *are the condition of stationarity of the point* x^0*, i.e.,* $\nabla f\left(x^0\right) = 0$ *and the negative definiteness of its Hessian at this point, i.e., for any* $y \in \mathbb{R}^n$

$$y^T H_f\left(x^0\right) y < 0.$$

To check the positive definiteness or negative definiteness of the Hessian $H_f\left(x\right)$ we use the Sylvester criterion. Let $\Delta_1(x),...,\Delta_n(x)$ – denote consecutive principal minors of the matrix $H_f\left(x\right)$.

Sylvester's criterion. *For the matrix* $H_f\left(x\right)$ *to be positively definite, it is necessary and sufficient that for any* $x \in \mathbb{R}^n$ *the conditions*

$$\Delta_1(x) > 0,...,\Delta_n(x) > 0$$

are satisfied.

For the matrix $H_f\left(x\right)$ *to be negatively definite, it is necessary and sufficient that for any* $x \in \mathbb{R}^n$ *the conditions*

$$\Delta_1(x) < 0,\ \Delta_2(x) > 0,...,(-1)^n \Delta_n(x) > 0$$

are satisfied.

Item 4.2. Conditional minimization problem

We write the classical optimization problem in the following form:

$$f^0(x) \to \min, \tag{9}$$

$$f^i(x) = 0,\ i = 1,...,m, \tag{10}$$

where $x \in \mathbb{R}^n$, $f^i(x) \in C^1$, $i = 1,...,m$.

Problems (9)-(10) are called **conditional minimization problems**. A general approach to the study of the problem of finding a conditional extremum of a differential function is provided by the Lagrange method. This method consists in replacing the problem of conditional extremum (9)-(10) with the problem of unconditional extremum for the Lagrange function of problem (9)-(10).

Let us introduce the **Lagrange function** of problem (9)-(10)

$$L(x,\lambda) = f^0(x) + \sum_{i=1}^{m} \lambda_i f^i(x) \tag{11}$$

of variables $(x_1, x_2, ..., x_n, \lambda_1, \lambda_2, ..., \lambda_m) = (x,\lambda) \in \mathbb{R}^{n+m}$.

The following theorem holds.

Theorem 3 (necessary conditions for conditional extremum). *If x^* is a point of local minimum or maximum of the function $f^0(x)$ under the condition $f^i(x) = 0$, $i = 1,...,m$, then there must exist variables $(\lambda_1^*, \lambda_2^*, ..., \lambda_m^*) = \lambda^* \neq 0$, called Lagrange multipliers, such that*

$$\frac{\partial L(x^*,\lambda^*)}{\partial x_j} = \frac{\partial f^0(x^*)}{\partial x_j} + \sum_{i=1}^{m} \lambda_i^* \frac{\partial f^i(x^*)}{\partial x_j} = 0,\ j = 1,\dots,n \tag{12}$$

or

$$\nabla f^0(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f^i(x^*) = 0,$$

i.e., the vectors $\nabla f^0(x^*)$, $\nabla f^1(x^*)$*,...,* $\nabla f^m(x^*)$ *are linearly dependent.*

Thus, only those points x, for which there are exist multipliers $\bar{\lambda} = (\bar{\lambda}_1, ..., \bar{\lambda}_m) \neq 0$, such that the point $(\bar{x}, \bar{\lambda}) \in \mathbb{R}^{n+m}$ satisfies the system of $m+n$ equations

$$\begin{cases} \dfrac{\partial f^0(x)}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \dfrac{\partial f^i(x)}{\partial x_j} = 0, \ j = 1, ..., n, \\ f^i(x) = 0, \ i = 1, ..., m, \end{cases} \tag{13}$$

can be suspicious for a conditional extremum.

Conditions (13) define a system of $n+m$ equations with $n+m$ unknowns $(x, \lambda) = (x_1, x_2, ..., x_n, \lambda_1, \lambda_2, ..., \lambda_m)$. By solving it, we will find the points $\bar{x}$, suspected of being conditional extremes and the corresponding Lagrange multipliers $\bar{\lambda} = (\bar{\lambda}_1, ..., \bar{\lambda}_m) \neq 0$.

To find out whether the points $\bar{x}$ will actually have a minimum or a maximum, we need to apply **sufficient conditions for minimum (maximum)** to the Lagrange function in the variable x, which can be formulated as follows:

Theorem 4. *Let:*

1) $(\bar{x}, \bar{\lambda})$ *satisfies the system (13);*

2) the function $L(x, \lambda)$ *in the vicinity of the point* $\bar{x}$ *is twice differentiable in* x *and has continuous all second-order partial derivatives at the point* $\bar{x}$ *itself;*

3) the Hessian $H_L(\bar{x}, \bar{\lambda})$ *on the variable* x *of the Lagrange function* $L(x, \lambda)$ *at the point* $(\bar{x}, \bar{\lambda})$ *is a positively (negatively) defined matrix.*

Then the point $\bar{x}$ *is the point of local minimum (maximum) of the function* $f^0(x)$ *under conditions (10).*

Example 7. Suppose you need to find the points of extremum of the function $f^0(x) = x_1^2 - x_2^2$ on the set $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid 2x_1 - x_2 - 6 = 0\}$.

◀ We have a problem of finding the conditional extremum of a function with constraints such as equality.

Let's apply the method of Lagrange multipliers. We construct the Lagrange function of the problem

$$L(x_1, x_2, \lambda) = x_1^2 - x_2^2 + \lambda(2x_1 - x_2 - 6)$$

and write down the necessary conditions for the extremum and find the stationary points of the Lagrange function

$$\begin{cases} \dfrac{\partial L}{\partial x_1} = 2x_1 + 2\lambda, \\ \dfrac{\partial L}{\partial x_2} = -2x_2 - \lambda, \\ \dfrac{\partial L}{\partial \lambda} = 2x_1 - x_2 - 6, \end{cases} \quad \begin{cases} \dfrac{\partial L}{\partial x_1} = 0, \\ \dfrac{\partial L}{\partial x_2} = 0, \\ \dfrac{\partial L}{\partial \lambda} = 0, \end{cases} \quad \begin{cases} 2x_1 + 2\lambda = 0, \\ -2x_2 - \lambda = 0, \\ 2x_1 - x_2 - 6 = 0, \end{cases} \quad \begin{cases} x_1 = 4, \\ x_2 = 2, \\ \lambda = -4, \end{cases}$$

We check the fulfillment of the sufficient condition $X^* = (4; 2)$ for $\lambda^* = -4$. Find the second partial derivatives

$$\begin{cases} \dfrac{\partial^2 L}{\partial x_1^2} = 2, \\ \dfrac{\partial^2 L}{\partial x_2^2} = -2, \\ \dfrac{\partial^2 L}{\partial x_1 \partial x_2} = 0, \end{cases}$$

Then we have $\Delta_1(x) = 2 > 0$, $\Delta_2(x) = -4 < 0$ and according to Sylvester's criterion, the Hessian matrix is negative definite, and therefore $X^* = (4; 2)$ is the maximum point of the function $f^0(x)$ and $f^0_{\max} = 16 - 4 = 12$. ▶

§5. Methods for solving optimization problems

Item 5.1. Gradient methods

Consider the problem of unconditional minimization

$$f(x) \to \min, \qquad (8)$$

$x \in \mathbb{R}^n$, the function $f(x) \in C^1$ is differentiable.

In principle, this problem can be solved by classical methods. These methods are called indirect methods because they use the necessary conditions for the extremum of $\nabla f(x) = 0$. However, it should be noted that for real problems, solving this system is no less difficult than solving the original problem. Indirect methods are used mainly when the solution to an extreme problem needs to be found in an analytical form. For solving complex practical problems, direct methods are usually used, which involve the direct comparison of a function at two or more points.

Let us have a point $x^s \in \mathbb{R}^n$. Let's figure out how to move to a new point x^{s+1} when solving problem (8) so that the inequality $f(x^{s+1}) < f(x^s)$ is satisfied. Let's write x^{s+1} in the form

$$x^{s+1} = x^s + \rho d,$$

where the vector $d = (d_1,...,d_n)^T$ determines the direction of displacement, and the number $\rho > 0$ is the step of displacement from point x^s to point x^{s+1}.

Definition 3. *The **direction** d is said to be **suitable** (for the minimization problem) if there exists $\rho > 0$ such that*

$$f(x^s + \rho d) < f(x^s).$$

When moving from point x^s to point $x^{s+1} = x^s + \rho d$, $\rho > 0$, the direction *d* is suitable if the derivative along the *d-direction* of the function $f(x)$ at point x^s is negative:

$$D_d f(x^s) = \nabla^T f(x^s)d = (\nabla f(x^s), d) < 0,$$

i.e., if the condition

$$\nabla^T f(x^s)d < 0$$

is met.

The geometric interpretation of the approaching direction (see Fig. 1) is that the vector *d*, with origin at point x^s, can be taken as any vector that forms an acute angle with an antigradient $-\nabla f(x^s)$.

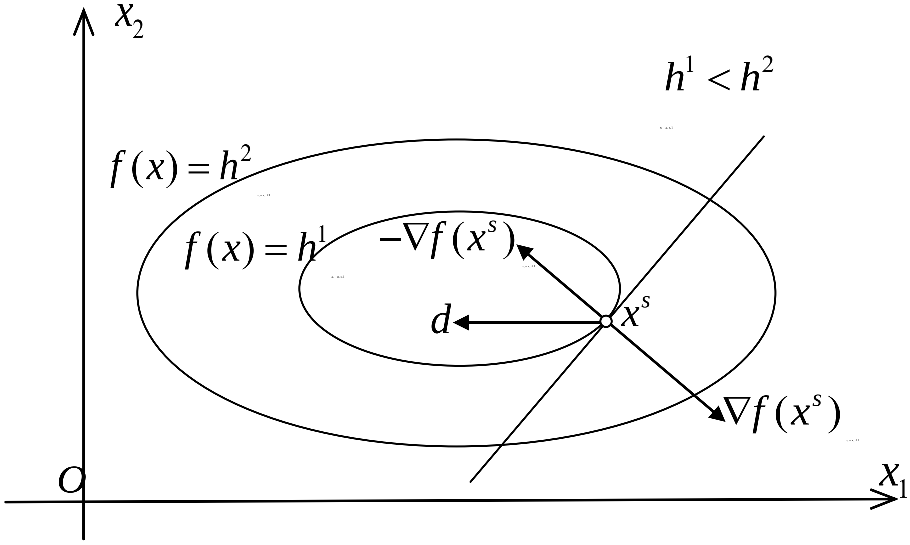


Fig. 1.

If an antigradient is used as the direction *d* that is suitable for minimizing the function $f(x)$ (a gradient is used for maximizing it), then the corresponding method is called a gradient method. The starting point x^0 in the gradient method is chosen arbitrarily, and all other successive approximations to the minimum point are calculated using the formula

$$x^{s+1} = x^s - \rho_s \nabla f(x^s),\ s = 0,1,2,... \qquad (14)$$

Different methods have been developed to adjust the step ρ_s at any iteration of the gradient method. Let's consider two of them – with step fractionation and the fastest descent.

In the gradient **method with step fractionation**, a sufficiently small step $\rho_0 > 0$ is fixed and, starting from the point x^0, the procedure (14) is implemented a number of times. At each iteration, the value of the function $f(x^s)$ is calculated. Procedure (14) is continued until $f(x)$ decreases. In this case (see Fig. 2), the point x^s usually tends to be in the neighborhood of a local minimum whose size is of the same order as ρ_0.

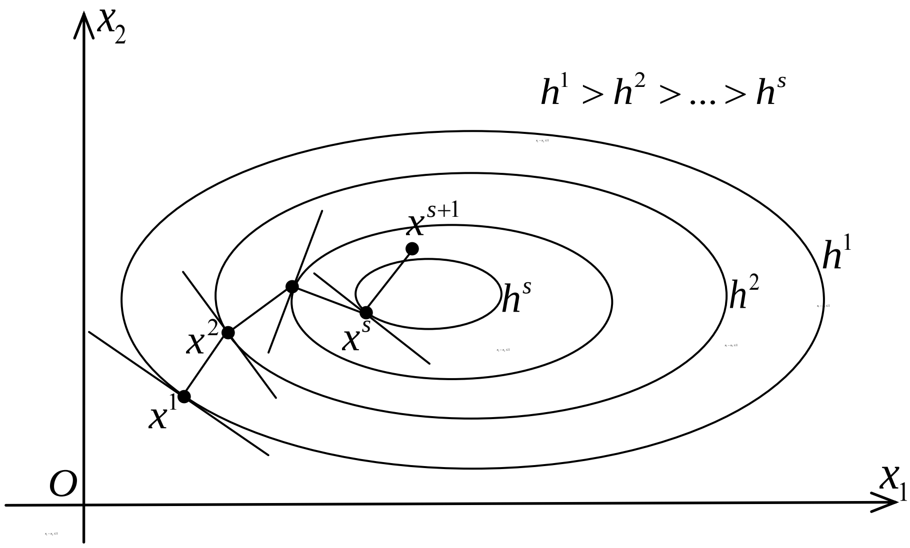


Fig. 2.

If the achieved accuracy is insufficient, the step is reduced, i.e., $0 < \rho_1 < \rho_0$ is chosen, and the iterations continue with a new step according to rule (14) until the point x^s falls in the neighborhood of the local minimum, the size of which is not greater than the specified error.

In the **fastest descent method**, the value of the step ρ_s in procedure (14) is chosen by the rule

$$\rho_s = \arg\min_{\rho>0} f(x^s - \rho_s \nabla f(x^s)).$$

Remark. *The problem of finding the step at each iteration of the fastest descent method is a one-dimensional optimization problem and can be solved by one of the one-dimensional optimization methods discussed earlier.*

With a fixed step ρ, there must be a stop at the point x^{s+1} at each iteration, despite the fact that the direction $-\nabla f(x^s)$ still leads to a decrease in the objective function (see Fig. 3).

In the fastest descent method, moving after the point x^{s+1} in the direction of the antigradient $-\nabla f(x^s)$ no longer leads to smaller values of the objective function. Therefore, the method of fastest descent belongs to the so-called full-step methods.

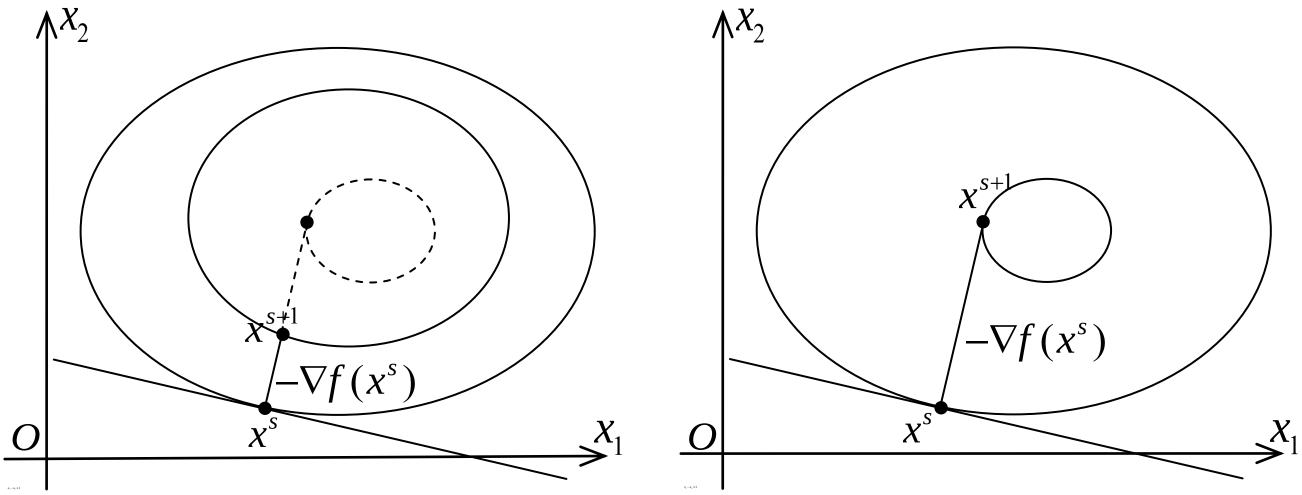


Fig. 3.

For both variants of the gradient method, convergence theorems have been proved.

Since the convergence of the fastest descent method will not be finite in general, the criteria for terminating iterations are determined:

1) $\max\limits_{i=\overline{1,n}} \left| \dfrac{\partial f}{\partial x_i} \right| < \varepsilon$, ($\varepsilon > 0$ given)

or

2) $\|\nabla f(x)\|^2 = \sum\limits_{i=1}^{n} \left(\dfrac{\partial f}{\partial x_i} \right)^2 < \varepsilon$, ($\varepsilon > 0$ given)

or

3) $| f(x^{s+1}) - f(x^s) | < \varepsilon$, ($\varepsilon > 0$ given).

Disadvantages of gradient methods.

The main disadvantage of the gradient method is that it at best ensures that the sequence $\{x^s\}$ converges only to the point of local minimum of the function $f(x)$.

In general, the sequence $\{x^s\}$ converges to a stationary point x' of the function $f(x)$, where $\nabla f(x') = 0$. A complete guarantee of the convergence of $\{x^s\}$ to the global minimum point of $f(x)$ can be given, for example, by the requirement of convexity of $f(x)$.

One of the significant drawbacks of the fastest descent method is that for some types of functions, its convergence may be slow.

Another disadvantage of gradient methods is that they cannot be directly applied to the optimization of nondifferentiable functions or conditional optimization of differentiable functions.

Example 8. Find the minimum of the function

$$f(x) = -4x_1 - 2x_2 + x_1^2 + x_2^2$$

using the gradient method, starting the iteration process from the point $x^0 = (4;5)$.

◀ Finding partial derivatives

$$\frac{\partial f}{\partial x_1} = -4 + 2x_1; \quad \frac{\partial f}{\partial x_2} = -2 + 2x_2.$$

First iteration. Calculate the gradient of the function $f(x)$ at the point x^0:

$$\nabla f(x^0) = (4;8).$$

It is different from zero, so we plot the ray $x^1(\rho)$, coming from the point x^0 in the direction of the antigradient

$$x^1(\rho) = x^0 - \rho\nabla f(x^0) = (4;5) - \rho(4;8) = (4-4\rho;5-8\rho),\ \rho > 0.$$

Find the minimum of the function $f(x^1(\rho))$ with respect to ρ:

$$f(x^1(\rho)) = -4(4-4\rho) - 2(5-8\rho) + (4-4\rho)^2 + (5-8\rho)^2,$$

$$f(x^1(\rho)) = 16 + 16 + 2(4-4\rho)\cdot(-4) + 2(5-8\rho)\cdot(-8) = 160\rho - 80,$$

$f'_\rho = 0$: $160\rho = 80$, from here $\rho = 0,5$.

Since $f''_\rho = 160 > 0$, then $\rho = 0,5$ is the minimum point of the function $f(x^1(\rho))$. Then

$$x^1 = x^0 - \rho_0\nabla f(x^0) = (4;5) - 0,5(4;8) = (2;1).$$

The second iteration. Calculate the gradient of the function $f(x)$ at point x^1:

$$\nabla f(x^1) = (-4 + 2\cdot 2; -2 + 2\cdot 1) = (0;0).$$

Since it is zero, the point x^1 s a stationary point of the function $f(x)$. In addition, $f(x)$ is convex downward, so x^1 is the point of global minimum of $f(x)$. ►

Item 5.2. Zontendijk's method of feasible directions

Let us consider the nonlinear programming problem

$$\min\left\{ f_0(x):\ f_i(x) \le 0, i = 1,...,m,\ x \in \mathbb{R}^n \right\} \qquad (15)$$

provided that the functions $f_i(x)$ are differentiable: $f(x) \in C^1$, $i = 1,...,m$. Note that nonlinear programming problems with nonnegativity conditions $x_j \ge 0$, $j = 1,...,n$ can be easily reduced to the form (15). To do this, it is enough to include in the general system of constraints the conditions $-x_j \le 0$, $j = 1,...,n$.

Let the point $x^s \in X$, where

$$X = \left\{ x \in \mathbb{R}^n :\ f_i(x) \le 0, i = 1,...,m,\ x \in \mathbb{R}^n \right\}.$$

The **constraint** $f_i(x) \le 0$ is called **active** at point x^s, if $f_i(x^s) = 0$. We denote by $I_s = \{i : f_i(x^s) = 0\}$ the set of indices of active constraints at point x^s. It is obvious that only these constraints determine the direction of movement from a valid point x^s to another valid point x^{s+1}.

We represent the point $x^{s+1} \in \mathbb{R}^n$ as

$$x^{s+1} = x^s + \rho r^s,$$

where r^s – is an arbitrary vector of displacement direction from point x^s, and $\rho > 0$ – is the offset step.

The **direction** r^s is called **feasible** if there exists $\rho > 0$ such that the point x^{s+1} satisfies the condition

$$x^{s+1} \in X \ \text{ or } \ f_i(x^{s+1}) \le 0,\ i = 1,...,m. \qquad (16)$$

Since x^s is an admissible point, conditions (16) are equivalent to conditions

$$f_i(x^{s+1}) \le f_i(x^s),\ i = 1,...,m. \tag{17}$$

Further, since only the constraints active at point x^s affect the choice of a feasible direction, in conditions (17) we should assume that $i \in I_s$.

We require r^s to be a feasible direction, then it must be determined by conditions

$$f_i(x^{s+1}) \le f_i(x^s),\ i \in I_s \tag{18}$$

and a suitable direction, then it must satisfy inequality

$$f_0(x^{s+1}) < f_0(x^s) \tag{19}$$

for sufficiently small $\rho > 0$. And (as written above) the direction r^s is suitable if the derivative along the direction r^s of the function $f_0(x)$ at the point x^s is negative:

$$D_{r^s} f_0(x^s) = (\nabla f_0(x^s), r^s) < 0. \tag{20}$$

Similarly, we conclude that conditions (18) will be satisfied at sufficiently small $\rho > 0$ only for those directions r^s for which the derivatives $D_{r^s} f_i(x^s)$, $i \in I_s$ are not positive:

$$D_{r^s} f_i(x^s) = (\nabla f_i(x^s), r^s) \le 0,\ \ i \in I_s. \tag{21}$$

To limit the lengths of the vectors r^s, a normalization condition is usually added to the system of conditions (20)-(21), which defines all feasible and suitable directions, such as the following:

$$-1 \le r_j^s \le 1,\ \ j = 1,...,n. \tag{22}$$

Finally, to obtain a feasible and suitable direction r^s, we obtain the following linear programming problem:

$$D_{r^s} f_0(x^s) = (\nabla f_0(x^s), r^s) \to \min$$

$$D_{r^s} f_i(x^s) = (\nabla f_i(x^s), r^s) \le 0, \; i \in I_s, \qquad (23)$$

$$-1 \le r_j^s \le 1, \; j = 1, \ldots, n.$$

If the optimal value of the problem (23) is nonnegative, then x^s is a stationary point of the function $f_0(x)$ under the conditions $f_i(x) \le 0$, $i = 1, \ldots, m$, $x \in \mathbb{R}^n$; otherwise, the vector r^s determines a feasible and suitable direction. Then, in the found direction r^s we construct a ray $x(\rho) = x^s + \rho r^s$ ($\rho > 0$) and, substituting $x(\rho)$ into all inactive constraints, find the number ε that bounds the step ρ from above: $0 < \rho \le \varepsilon$. We define the specific value of ρ_s as

$$\rho_s = \arg \min_{\rho \in (0; \varepsilon]} f_0(x^s - \rho r_s)$$

using some one-dimensional optimization procedure.

Note that, as with gradient methods, the method of feasible directions does not guarantee anything more than the convergence of x^s to a stationary point of the function $f_0(x)$. You can take any valid point $x \in X$ as an initial approximation x^0. If the constraints of problem (15) are linear, then x^0 can be taken as an arbitrary basic solution of the constraint system of problem (15).

The proofs of the theorems formulated in this section, the justification of the convergence of the described numerical methods, and examples of their application can be found in many textbooks on optimisation methods, including those listed in the references.

CHAPTER 5

MULTICRITERIA OPTIMIZATION

Multicriteria optimization is an essential branch of mathematical modeling that is used for analyzing and decision-making in complex systems characterized by multiple interrelated criteria. Unlike single-criteria optimization, which aims to maximize or minimize a single objective function, multicriteria optimization involves considering a set of criteria that may conflict with one another. This makes finding a definitive solution more challenging, as improving one criterion often leads to the deterioration of another.

One of the key concepts in multi-criteria optimization is Pareto optimality, introduced by the Italian mathematician and economist Vilfredo Pareto. Pareto studied resource allocation and the influence of interdependent factors on economic systems. In his approach, optimal solutions are defined as those for which it is impossible to improve one criterion without worsening another. This idea became the foundation for formulating multi-criteria optimization problems across various fields of science and engineering.

This section explores the fundamental principles and approaches to multi-criteria optimization, as well as methods for identifying effective and weakly effective solutions. Special attention is given to the analysis of the set of Pareto-optimal solutions, their properties, and the algorithms for constructing them.

§ 1. Efficient and weakly efficient estimates and solutions

In multicriteria problems, vector scores, i.e., the value of the vector criterion $f=(f_1, f_2,..., f_m)$, are compared by preference. Naturally, it is easiest to compare by preference those vector estimates that differ from each other by only one component. Therefore, information about the preference for changing the value of one partial criterion at fixed values of all other criteria is the most accessible and reliable, and it is this information that should be obtained first and used to analyze the problem.

In general, the values of the criterion f_l may have different preference correlations depending on the values of all other criteria. In other words, for numbers s and t from Y_l it may turn out, for example, that the estimate $(y_1,...,y_{l-1},s,y_{l+1},...,y_m)$ is preferred over the estimate $(y_1,...,y_{l-1},t,y_{l+1},...,y_m)$, but the estimate $(y'_1,...,y'_{l-1},s,y'_{l+1},...,y'_m)$ is less preferred than the estimate $(y'_1,...,y'_{l-1},t,y'_{l+1},...,y'_m)$.

In this case, it is impossible to say which of the values s or t of the criterion f_l is preferable without specifying the values of the other criteria.

The **criterion** f_l, for which this situation occurs is called **dependent** on the other criteria.

However, it is much more common to find criteria for which all their values can be ordered by preference without considering the values of the other criteria. Such **criteria** are called **independent** of the other criteria. More precisely, the criterion f_l is independent in preference from the other $m-1$ criteria if, for any four estimates of the form

$$(y_1,...,y_{l-1},s,y_{l+1},...,y_m),\ (y_1,...,y_{l-1},t,y_{l+1},...,y_m),$$

$$(y'_1,...,y'_{l-1},s,y'_{l+1},...,y'_m),\ (y'_1,...,y'_{l-1},t,y'_{l+1},...,y'_m)$$

the relation

$$(y_1,...,y_{l-1},s,y_{l+1},...,y_m)R(y_1,...,y_{l-1},t,y_{l+1},...,y_m)$$

always implies

$$(y'_1,...,y'_{l-1},s,y'_{l+1},...,y'_m)R(y'_1,...,y'_{l-1},t,y'_{l+1},...,y'_m).$$

Definition 1. *Problems in which all criteria are independent in preference, i.e., each criterion is independent in preference from the set of all others, and the non-strict preference relation on the set of values of each criterion is the relation ≧ ("not less than") are called* ***multicriteria maximization problems****.*

In such problems, it is desirable to have the highest possible value for each criterion, or, as they say, it is desirable to maximize each criterion. If it is desirable to minimize each criterion in the problem, then it is called a **multicriteria minimization problem**.

In the following, we will denote the **multicriteria maximization problem** by

$$\max_{x\in X} f(x), \qquad (*)$$

where $X = \{x \in \mathbb{R}^n \mid g(x) \le 0\}$, $\mathbb{R}^n$ is an n-dimensional Euclidean space, $f(x), g(x)$ are continuously differentiable vector functions on $\mathbb{R}^n$ that perform the mapping:

$$f:\mathbb{R}^n \to \mathbb{R}^m,\ g:\mathbb{R}^n \to \mathbb{R}^r.$$

And the **multicriteria minimization problem** is similar:

$$\min_{x\in X} f(x). \qquad (**)$$

For multicriteria maximization problems on the set Y, a non-strict preference relation $(\geqq)$, two strict preference relations $(\geq),(>)$ and an indifference relation $(=)$ are introduced:

- the relation of non-strict preference ≧ of vector estimates y, y' takes place if the relations

$$y_i \geqq y'_i,\quad i = 1,2,...,m$$

are satisfied;

- strict preference $\geq$, if the relations

$$y_i \geq y'_i, \quad i = \overline{1,m}, \quad y \neq y'$$

are satisfied;
- strict preference $>$, if the relations

$$y_i > y'_i, \quad i = \overline{1,m}$$

are satisfied;
- indifference = if the equalities

$$y_i = y'_i, \quad i = \overline{1,m}$$

are satisfied.

According to the general definition, an estimate $y^* \in Y$ is said to be the best estimate under the ≧ relation in Y if for any estimate of $y \in Y$, $y^* \geqq y$. Since the ≧ relation is a (partial) order, there can be only one such point y^* (see Fig. 1). If in a practical multi-criteria problem there exists the largest achievable value of y^* with respect to the ≧ relation, then it should be considered optimal. Unfortunately, this case is extremely rare: as a rule, the value y^* does not exist. This is because the ≧ order is not complete.

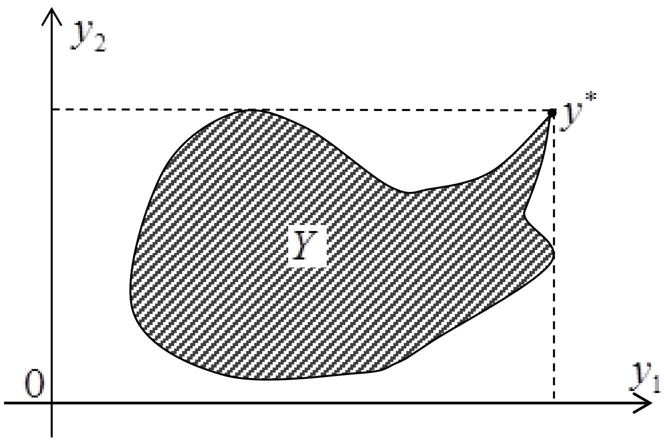


Fig. 1.

For example, if $y_i > y'_j$, then y and y' are not comparable by the ≧ relation. Therefore, depending on the nature of the problem, you have to use the estimates that maximize the relation $\geq$ or $>$.

Definition 2. *An* ***estimate*** $y^0 \in Y$ *is called* ***maximal with respect to the preference relation*** $\geq$ *relative to the set* Y *if there does not exist an estimate* $y \in Y$ *such that* $y \geq y^0$.

This estimate is also referred to as **efficient**, **Pareto optimal**, **Pareto-efficient**, or a **Pareto optimum**.

The set of such estimates from Y, which we will denote by $P(Y)$, is called the **efficient** or **Pareto set**.

Definition 3. *An estimate* $y^0 \in Y$ *is called* ***maximal with respect to the preference relation*** $>$ *relative to the set* Y*, if there does not exist an estimate* $y \in Y$ *such that* $y > y^0$.

This estimate is also referred to as **weakly efficient**, **weakly Pareto optimal**, a **weak Pareto optimum**, or **Slater optimal**.

The set of all such estimates from Y will be denoted by $S(Y)$ and will be called the **weakly efficient set**.

Since the relation $y > y'$ implies $y \geq y'$, every efficient vector estimate with respect to Y is also weakly efficient, so $P(Y) \subseteq S(Y)$. Indeed, if y^0 is not weakly efficient, then for some $y \in Y$ it will be true that $y > y^0$, and therefore $y \geq y^0$, so y^0 cannot be efficient.

Let's give more strict definitions of optimal Pareto and Slater estimates.

Definition 2′. *For problem (*), the set*

$$\left\{ y^* \in Y \mid \min_{1 \leq i \leq m} \left[y_i - y_i^* \right] < 0 \,,\, \forall\ y \in Y \right\}$$

is called the ***set of Pareto optimal estimates***.

Definition 3′. *For problem (*) the set*

$$\left\{ y^* \in Y \mid \min_{1 \leq i \leq m} \left[y_i - y_i^* \right] \leq 0 \,,\, \forall\ y \in Y \right\}$$

is called the ***set of optimal Slater estimates***.

For $m = 2$, we can give a simple and clear geometric interpretation of the sets $P(Y)$ and $S(Y)$. The set $P(Y)$ represents, figuratively speaking, the northeastern boundary of the set Y (without those parts of it that are parallel to one of the coordinate axes or lie in rather steep and deep dips), and $S(Y)$ may additionally include vertical and horizontal sections of the boundary adjacent to $P(Y)$.

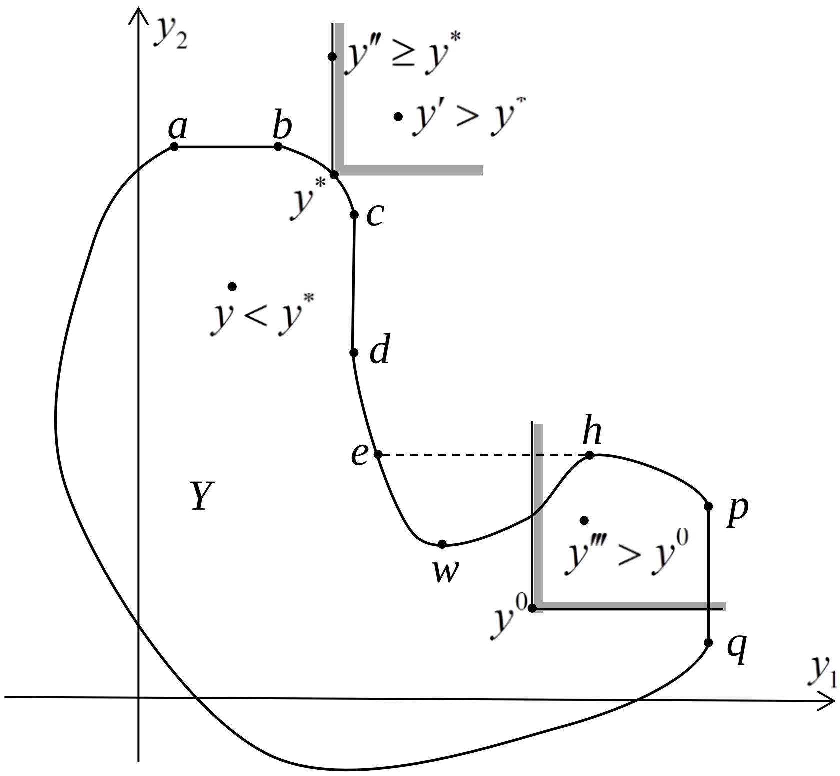


Fig. 2.

In the figure (Fig. 2), the set $P(Y)$ (the effective boundary of Y) is formed by the curves bc, de (without points *d* and *e*) and hp, and $S(Y)$ consists of two parts – $abcde$ (including *e*) and hpq. This can be easily verified by noting that the points better than y^*, in the sense of the ≥ relation, fill a right angle whose sides are parallel to the coordinate axes, and the vertex is y^* (y^* itself is excluded); and the points better than y^*, in the sense of the > relation, make up the interior of this angle.

The relations ≧, ≥, and > defined on the estimate set give rise to similar relations $\gtrapprox_f$, $\gtrsim_f$, and $\succ_f$ in the solution set. For example,

$$x \gtrsim_f x' \leftrightarrow f(x) \geq f(x')$$

The solution that maximizes the relation $\gtrsim_f$ (in terms of $\succ_f$) corresponds to the maximum estimate in *Y* that maximizes the relation ≥ (in terms of >). These solutions are usually given names similar to those of the corresponding estimates. In the following, we will use the terms "efficient" and "weakly efficient", as well as "Pareto optimal" and "weakly Pareto optimal solution".

Definition 4. *A **solution** $x^0 \in X$ is **efficient** if there is no solution $x \in X$ such that $x \gtrsim_f x^0$, i.e., for which $f(x) \geq f(x^0)$.*

This solution is also called a **Pareto efficient solution**, a **Pareto optimal solution**. The set of efficient solutions is denoted by $P_f(X)$.

Definition 5. *A **solution** $x^0 \in X$ is called **weakly efficient** if there is no solution $x \in X$ such that $x \succ_f x^0$, i.e., $f(x) > f(x^0)$.*

This solution is also called a **Slater optimal solution**.

The set of weakly Pareto optimal solutions is denoted by $S_f(X)$. It is obvious that $P_f(X) \subseteq S_f(X)$.

Let's give more strict definitions of optimal solutions.

Definition 4′. *For the problem (*), the set of points in the space $\mathbb{R}^n$ defined as follows*

$$\left\{ x^0 \in X \mid \min_{1 \leq i \leq m} \left[f_i(x) - f_i(x^0) \right] < 0 \, , \forall \, x \in X \right\}$$

*is called the **set of Pareto optimal solutions**.*

Definition 5′. *For the problem (*), the set of points in the space* $\mathbb{R}^n$ *defined as follows*

$$\left\{x^0 \in X \mid \min_{1\le i\le m}\left[f_i(x) - f_i(x^0)\right] \le 0 \,,\ \forall\ x \in X\right\}$$

is called the ***set of weakly Pareto optimal (Slater optimal) solutions****.*

Let us now define the concepts of optimal estimation and optimal solution for problem (**).

Definition 6. *For problem (**), the set*

$$\left\{y^* \in F \mid \max_{1\le i\le m}\left[y_i - y_i^*\right] > 0 \,,\ \forall\ y \in F\right\}$$

is called the ***set of Pareto optimal estimates****.*

Definition 7. *For the problem (**), the set*

$$F_* = \left\{y^* \in F \mid \max_{1\le i\le m}\left[y_i - y_i^*\right] \ge 0 \,,\ \forall\ y \in F\right\}$$

is called the ***set of weakly Pareto optimal estimates, Slater optimal estimates****.*

Definition 8. *For problem (**), the* ***set of Pareto optimal solutions*** *is defined as*

$$\left\{x^* \in X \mid \max\left[f_i^{\,i}(x) - f_i(x^*)\right] > 0\ \ \forall\ x \in X\right\}.$$

Definition 9. *The solution of problem (**) is the* ***set of weakly Pareto optimal (Slater optimal) points*** *in the space* $\mathbb{R}^n$*, defined as follows:*

$$X_* = \left\{x^* \in X \mid \max_{1\le i\le m}\left[f_i(x) - f_i(x^*)\right] \ge 0 \,,\ \forall\ x \in X\right\}.$$

Definition: 9′. *The* ***set of optimal points*** *of problem (**) corresponding to* F_* *is called the the following set*

$$X_* = f^{-1}(F_*) \cap X \,.$$

Geometric interpretation of the Pareto optimal and Slater optimal estimates of the problem (**) in Fig. 3.

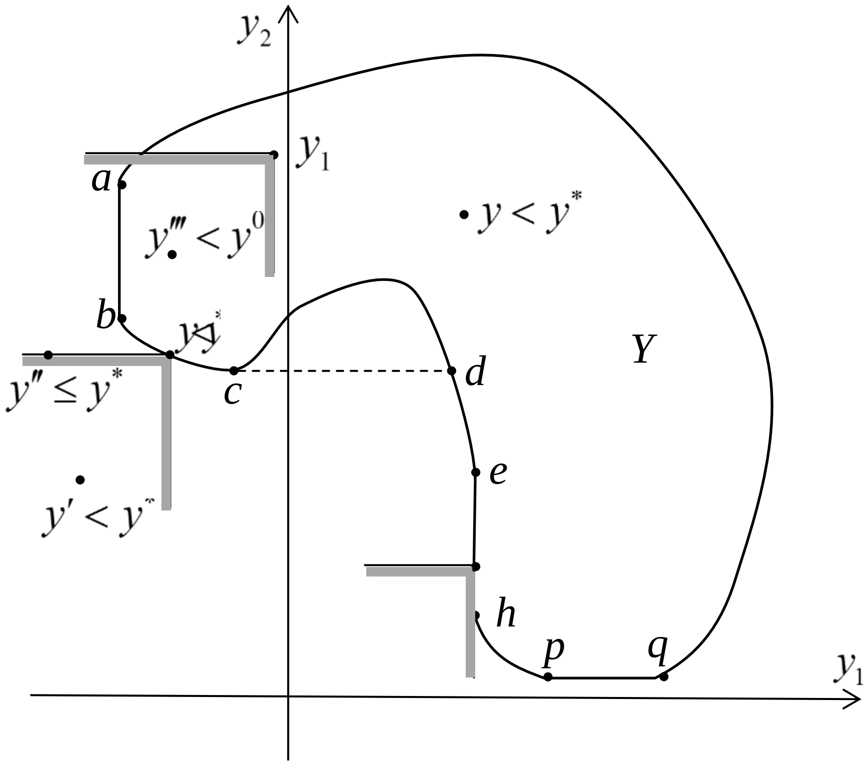


Fig. 3.

The Pareto set includes the plots bc, de (without points d and e), hp. The Slater set is formed from the sections abc, $dehpq$. If there are no points $y \in Y$ inside the right angle with vertex y^*, then y^* belongs to the set of Slater optimal estimates (points from Y can be on the sides of the angle). If there are no points from Y on the sides of the angle, then y^* belongs to the set of Pareto optimal estimates.

Definition 10. *A **set** of efficient estimates $P(Y)$ is called **externally stable** if for any $y \in Y \setminus P(Y)$ there exists an estimate $y^0 \in P(Y)$, such that $y^0 \ge y$.*

Definition 11. *A **set** of weakly efficient estimators $S(Y)$ is called **externally stable** if for any $y \in Y \setminus S(Y)$ there exists an estimate $y^0 \in S(Y)$ such that $y^0 > y$.*

Naturally, we can speak of an externally stable set of efficient (weakly efficient) solutions as the set of solutions that corresponds to an externally stable set of efficient (weakly efficient) estimates. It is more convenient to use a slightly different definition of the externally stable set of efficient (weakly efficient) estimates.

Definition 12. *A set $P(Y)$ is **externally stable** if for any $y \in Y$ there exists $y^0 \in P(Y)$ such that $y^0 \geqq y$.*

These definitions are equivalent.

For example. Let the set Y be a unit square from which the upper right vertex is cut out (Fig. 4). For this set, the set $P(Y)$ is obviously empty, and the set $S(Y)$ is formed by the top and right sides of the square (without the point (1, 1)). The set $S(Y)$ is externally stable: for every point $y \in Y$ with $y_1, y_2 < 1$, there is a corresponding point $y^0 = ((y_1+1)/2\,, 1)$, with $y^0 > y$.

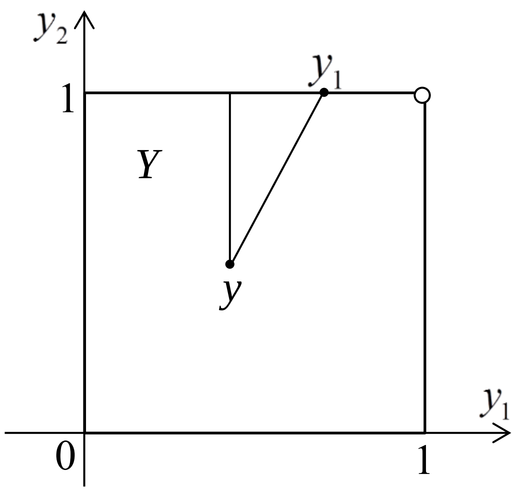


Fig. 4.

In this regard, it is interesting to ask when the external stability of the set $P(Y)$ is equivalent to the same property of the set $S(Y)$?

Statement. *If the set $P(Y)$ is externally stable, then the set $S(Y)$ is also externally stable.*

The following statement can also be proved: if the set $R(y)=\{y' \in Y \mid y' \geqq y\}$ is closed and bounded at any $y \in S(Y)$, then the external stability of $S(Y)$ implies the external stability of $P(Y)$.

The definition of a (weakly) efficient solution is "static" in the sense that it is based on a pairwise comparison of solutions and is not related to the question of whether it is possible to smoothly move from one solution to another, better one, infinitesimally ("at a positive rate") increasing each criterion. The possibility of such a transition in some models is of great interest. An example is the model of pure exchange, in which each consumer participates in the exchange, trying to compile a set of goods of the highest utility, i.e., formally maximizing his or her objective function. This type of model was considered in the XIX century by F. Edgeworth and V. Pareto. An efficient state in the exchange model is a state (distribution of goods among consumers) that cannot be improved by redistributing goods to any of the participants without affecting the interests of some other participants. Thus, Pareto optimality reflects the idea of economic equilibrium: if the state is not efficient, then trade will take place, which will lead to an efficient state.

If the exchange process is viewed as a sequence of small transactions that are beneficial to all participants, then it can be formally described by a smooth curve, along which all criteria increase infinitesimally. Then we can distinguish states from which no smooth curve of this type emerges.

Such states were called Pareto critical points by S. Smale. It is clear that the set of such points (the critical Pareto set) contains the entire set of weakly efficient points, but in general it is wider than the latter (due to the local nature of the definition of a critical Pareto point). Thus, in the figure 2, the critical Pareto set, in addition to all weakly efficient ones, will include solutions whose values lie on the boundary ew.

The definition of the Pareto critical point is a generalization of the stationary (critical) point of a smooth function (i.e., the point at which its gradient becomes zero).

§ 2. Properly and truly efficient estimates and solutions

Studies show that among the efficient ones, there may be estimates (solutions) that are in some sense anomalous.

Example 1. Let the set Y be given as follows:

$$Y = \left\{ y \in \mathbb{R}^2 \mid y_1 \le -\left(y_2\right)^2 \right\}$$

(Fig. 5)**.** The efficient estimates here are part of the parabola $y_1 = -(y_2)^2$, which lies in the second quarter. The efficient estimates include the estimate $y^0 = (0,0)$. The differences in the coordinates of the efficient estimates y and y^0 are equal to

$$\Delta y_2 = y_2 - y_2^0 = y_2 > 0 \qquad \text{and} \qquad \Delta y_1 = y_1 - y_1^0 = -(\Delta y_2)^2 < 0 .$$

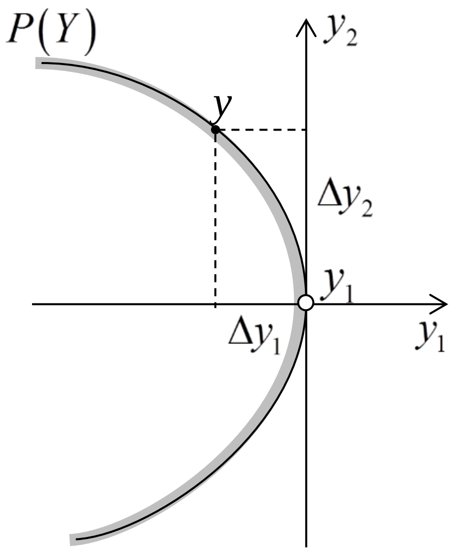


Fig. 5.

Therefore, if you move from point y^0 to an efficient point y that is close enough to it, you will get a first-order smallness gain in the second criterion at the expense of a second-order smallness loss in the first criterion.

If we do not consider the criterion f_1 to be incomparably more important than f_2, then it is natural to agree to a certain increase in f_2, losing an order of magnitude less in f_1. Thus, the estimate y^0 is anomalous: it is not stable in the above sense, and therefore the solution corresponding to it cannot, generally speaking, claim to be optimal.

This example shows that sometimes it makes sense to specifically identify efficient estimates (and solutions) that are free of such undesirable properties. The first definition of this kind of efficient solutions, called proper efficient, was given by H. Kuhn and A. Tucker. However, it was formulated for the differentiable case and was associated with special optimality conditions. For the general case, the definition of proper efficiency was proposed by A. Geoffrion.

Definition 13. *An efficient estimate y^0 is called* ***proper efficient*** *or* ***optimal in the sense of Geoffrion*** *if there exists a positive number θ such that for any $i \in M$, $y \in Y$, satisfying the inequality*

$$y_i > y_i^0 \tag{1}$$

and for some $j \in M$, it holds that

$$y_j < y_j^0, \tag{2}$$

the following inequality is satisfied:

$$\frac{y_i - y_i^0}{y_j^0 - y_j} \leqq \theta . \tag{3}$$

Note that since y^0 is efficient, if there exists an estimate y for which inequality (1) holds for some i, then there will necessarily be a number j for which inequality (2) holds. Therefore, the meaning of the definition is to require the existence of a number θ for which (3) holds under the specified conditions.

Solutions that correspond to property efficient estimates are also called **property efficient or Geoffrion optimal**. The set of such solutions and estimates is denoted by $G_f(X)$ and $G(Y)$.

Example 1. The set $G(y)$ is the upper branch of the parabola $y_1 = -(y_2)^2$ (without the vertex $y^0(0,0)$).

Example 2. The best estimate $y^0 \in Y$ with respect to the $\geqq$ relation is property efficient, since $y^0 \geqq y$ for all $y \in Y$ and inequality (1) does not hold. Thus, a solution that maximizes each of the criteria $f_1, f_2, ..., f_m$ simultaneously is property efficient. In particular, in single-criteria problems, any optimal solution is property efficient.

Example 3. If the set Y is finite, then any efficient estimate is also property efficient estimate. Indeed, if Y is finite, then $P(Y)$ is externally stable. Thus, if there is only one efficient estimate y^0, then it is the best estimate by the $\geqq$ relation and therefore property efficient (Example 2). If $P(Y)$ contains more than one estimate, then the desired positive number θ can be specified using the equality

$$\theta = \max\left\{ \frac{y_i - y_i^0}{y_j^0 - y_j} \mid y \in Y,\ i, j \in M,\ i \neq j,\ y_i > y_i^0,\ y_j < y_j^0 \right\}.$$

Therefore, if Y is finite (and the finiteness of the set X is sufficient for this), then the concepts of efficiency and property efficiency are equivalent.

Definition 14. *An efficient* ***estimate*** *that is not property efficient is called* ***non-property efficient.***

Similar terminology is introduced for solutions.

According to the definition 14, if y^0 is non-property efficient, this means the following: for any large $\theta > 0$, there exist $i \in M$, $y \in Y$ satisfying (1) such that for any j for which (2) holds, the inequality

$$\frac{y_i - y_i^0}{y_j^0 - y_j} > \theta.$$

holds.

Thus, the transition from non-property efficient solution to some other solution provides an increase in at least one partial criterion at the expense of losses of a higher order of smallness for all those criteria whose values decrease. That is, non-property efficient solutions in this sense are anomalous (unstable).

One way to define preference relations in multi-criteria problems is as follows: a certain cone Ω (cone of dominance) is selected in the $\mathbb{R}^m$ space, and it is considered that

$$yR^{\Omega}y', \qquad \text{if } y - y' \in \Omega.$$

It is clear that when $\Omega = \mathbb{R}^m_{\geqq}$, the ratio $\geqq$ is obtained, and when $\Omega = \mathbb{R}^m_{>}$, the ratio $>$ is obtained. Thus, the multi-criteria maximization problem is a special case of the cone optimization problem.

Consider the case when the cone Ω is polyhedral:

$$\Omega = \left\{ y \in \mathbb{R}^m \mid By \geqq 0_{(l)} \right\},$$

where B is a numerical matrix of size $l \times m$. For such a cone, the inclusion of $y - y' \in \Omega$ is equivalent to the fact that $B(y - y') \geqq 0_{(l)}$, i.e. $By \geqq By'$. Thus, the original problem with a vector criterion f, in which preferences are set using a polyhedral cone, after the introduction of a new vector criterion $f^B = (f_1^B, f_2^B, \ldots, f_l^B) = Bf$ turns out to be an "ordinary" multicriteria maximization problem.

The notion of property efficiency (in the sense of Geoffrion) essentially uses the "coordinate nature" of the $\geqq$ relation and therefore does not directly transfer to the more general case of optimization with respect to a cone. In this regard, J. Borwein proposed a definition of property efficiency, which in the case of multicriteria optimization problems is as follows.

First, let's introduce the concept of a tangent cone.

The **tangent cone** $T(A, y^0)$ to the set $A \subseteq \mathbb{R}^m$ at point $y^0 \in \bar{A}$ is the set of all vectors from $\mathbb{R}^m$ that are boundary points of the form $\omega = \lim_{r\to\infty} t_r (y^r - y^0)$, where $\{t_r\}$ is a sequence of nonnegative numbers, and $\{y^r\}$ is a sequence of points from A that converge to y^0.

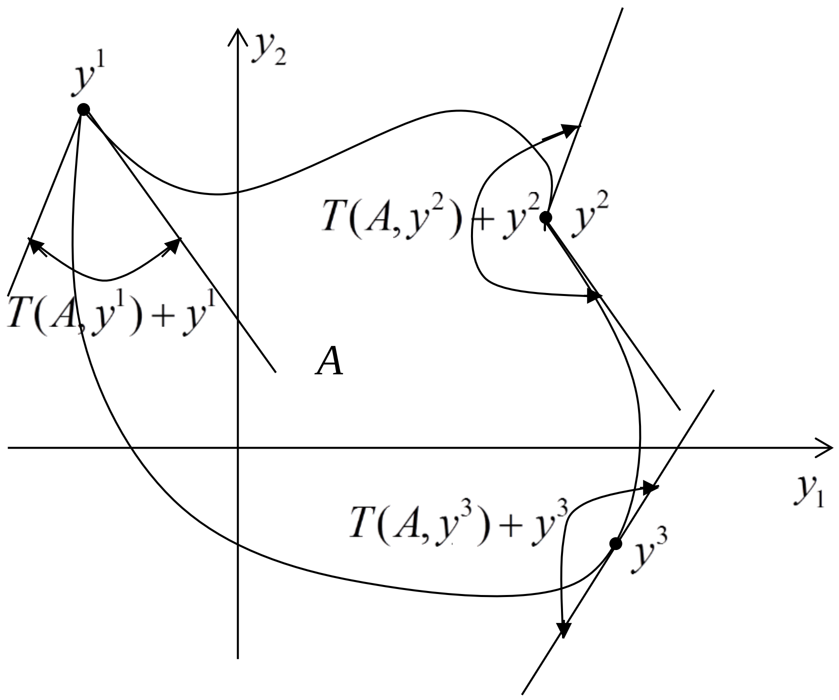


Fig. 6.

Figure 6 shows the tangent cones to the set $A \subset \mathbb{R}^2$ at its three boundary points. Note that for the interior points of A, the tangent cones are the entire space $\mathbb{R}^m$.

The tangent cone $T(A, y^0)$ is one of the approximations of the set A at the point y^0. It is easy to verify that the tangent cone is indeed a cone with a vertex at the origin, and it is closed.

Now for Borwein's definition of propety efficiency.

Definition 15. *An* ***estimate*** $y^0 \in Y$ *is called* ***truly efficient*** *or* ***optimal in the sense of Borwein*** *if it is efficient and satisfies the condition*

$$T(Y_*, y^0) \bigcap \mathbb{R}^m_{\geqq} = \{0_{(m)}\},$$

where

$$Y_* = Y - \mathbb{R}^m_{\geqq} = \left\{ w \in \mathbb{R}^m \mid w = y - e,\ y \in Y,\ e \in \mathbb{R}^m_{\geqq} \right\}.$$

Solutions whose estimates are truly efficient are also called **truly efficient**. The corresponding sets are denoted as follows:

$B(Y)$ – the set of truly efficient estimates.

$B_f(Y)$ – a set of truly efficient solutions.

The definition 15 implies that

$$B(Y) \subseteq P(Y),\ B_f(X) \subseteq P_f(X).$$

Example 4. In example 1

$$Y_* = Y \bigcup \left\{ y \in \mathbb{R}^2 \mid y_1 \geqq 0,\ y_2 \geqq 0 \right\}.$$

Here, every efficient estimate except y^0 is truly efficient (so that $B(Y) \subset P(Y)$).

Statement. *A property efficient estimate is a truly efficient.*

This statement shows that $G(Y) \subseteq B(Y)$ and $G_f(X) \subseteq B_f(X)$. In Example 1, the non-property efficient estimate y^0 is not a truly efficient estimate, so $G(Y) = B(Y)$. But it is also possible to strictly include $G(Y) \subset B(Y)$.

Example 5. Let the set Y be given as follows

$$Y = \left\{ y \in \mathbb{R}^2 \;\middle|\; y_2 = \frac{1}{y_1},\quad y_1 > 0 \right\}.$$

Here, each estimate is efficient and obviously truly efficient (so that $Y = P(Y) = B(Y)$), but not proprety efficient ($G(Y) = \varnothing$). Indeed, for y and $y^t = \left(t,\ \frac{1}{t} \right)$, where $t > y_1$ we get $\lim\limits_{t \to \infty} \frac{y_1^t - y_1}{y_2 - y_2^t} = +\infty$.

Note that moving from an arbitrary point y to a sufficiently close point $y + \Delta y$ results in decreasing and increasing criteria of the same order of smallness.

This example leads to the conclusion that the notion of property efficiency is too “rigid”: it also rejects such efficient solutions that may well “claim” to be optimal. Therefore, Borwein's definition is of independent interest to multicriteria optimization problems.

The relationship between the concepts of efficiency of different types can be schematically represented as follows:

Proper efficiency
(Geoffrion optimality)

↓

Truly efficiency
(Borwein optimality)

↓

Efficiency
(Pareto optimality)

↓

Weakly efficiency
(Slater optimality)

That is, the following inclusions are correct:

$$\begin{gathered} G(Y) \subseteq B(Y) \subseteq P(Y) \subseteq S(Y), \\ G_f(X) \subseteq B_f(X) \subseteq P_f(X) \subseteq S_f(X). \end{gathered} \tag{4}$$

As can be seen from the examples, the inclusions of (4) are generally strict.

§ 3. Conditions for optimality

In this paragraph, depending on the properties of the criteria and the structure of the set of admissible solutions, various necessary and sufficient conditions are formulated to ensure that a given solution or a given estimate is in some sense optimal (efficient) for the problem (*). As in ordinary extreme problems, knowledge of the optimality conditions allows us to develop methods for finding efficient solutions and ways to check the efficiency of the selected solution. In addition, these conditions allow for a deeper understanding of the nature and relationship of different types of efficient solutions, as well as to study the structure and properties of sets of efficient solutions and estimates.

The conclusions and theorems presented in this section are widely covered in numerous textbooks, articles, and monographs dedicated to multicriteria optimization. They form an essential part of the theoretical foundation of this field and serve as a basis for further research and practical applications. These results reflect the core principles and methodologies used in multicriteria decision-making processes.

The theoretical foundations of multicriteria optimization are most fully covered in the textbooks listed in the reference list. These books and resources provide a comprehensive approach: from mathematical foundations to algorithms and applications.

Item 3.1. General conditions for optimality

Let us formulate the optimality conditions for the problem (*) without any significant assumptions about the structure of the set of admissible solutions X and the properties of the vector function $f=(f_1,f_2,...,f_m)$.

Theorem 1. *(Hermeyer). Suppose that* $y^0 \in Y$ *and all* $y_i^0 > 0$. *The estimate* y^0 *is weakly efficient if and only if there exists a vector* $\mu \in \mathcal{M}$ *such that*

$$\min_{i\in M} \mu_i y_i^0 = \max_{y\in Y} \ \min_{i\in M} \mu_i y_i$$

For a weakly efficient estimate $y^0 \in Y$, *we can take* $\mu = \mu^0$, *where* $\mu^0 \in \mathcal{M}$ *is a vector with components*

$$\mu_i^0 = \frac{\lambda^0}{y_i^0}, \quad i=1,2,...,m \ ; \quad \lambda^0 = \frac{1}{\sum_{k=1}^{m} \frac{1}{y_k^0}} \tag{5}$$

and then

$$\max_{y\in Y} \ \min_{i\in M} \mu_i^0 y_i = \lambda^0.$$

In the formulation of the theorem, $\mathcal{M}$ is a set of vectors from $\mathbb{R}^m$ with positive components equal to one in total:

$$\mathcal{M} = \left\{ \mu \in \mathbb{R}_>^m \mid \sum_{i=1}^{m} \mu_i = 1 \right\}, \ M = \{1,2,\ldots,m\}.$$

Geometrically, it is quite obvious that $y^0 \in S(y)$ if and only if no point of the set Y falls inside the orthant $\mathbb{R}_+^m$ shifted to the point y^0. Since the hypersurface $\min_{i\in m} \mu_i y_i = \lambda$ for $\lambda = 0$ and positive μ_i is the boundary of this orthant, the formulated geometric fact is expressed in terms of $\min_{i\in M} \mu_i y_i$.

A useful generalization of theorem 1 is the following theorem.

Theorem 2. *Let $y^0 \in Y$, and ζ_i, $i = 1,2,...,m$ be increasing functions of one variable such that*

$$\zeta_1\left(y_1^0\right) = \zeta_2\left(y_2^0\right) = ... = \zeta_m\left(y_m^0\right).$$

An estimate y^0 is weakly efficient if and only if

$$\zeta_1\left(y_1^0\right) = \max_{y \in Y} \min_{i \in M} \zeta_i(y_i). \tag{6}$$

By choosing the functions ζ_i, we can obtain specificizations of equality (6). For example, if $y^0 \in \mathbb{R}^m_>$, then setting $\zeta_i(y_i) = \mu_i^0 y_i$, where μ_i^0 is defined by formula (5), we arrive at theorem 1. If we assume $\zeta_i(y_i) = y_i - y_i^0$, we get the following consequence.

Consequence 1. *Estimation $y^0 \in Y$ is weakly efficient if and only if*

$$\max_{y \in Y} \min_{i \in M} (y_i - y_i^0) = 0.$$

Definition 16. *A numerical **function** ψ defined on a set A is said to be **increasing** (not decreasing) **in the preference relation** P if $\psi(a) > \psi(b)$ (respectively, $\psi(a) \geqq \psi(b)$) follows from aPb for any $a,b \in A$.*

Now we can formulate the following theorem.

Theorem 3. *If a function $\varphi(y)$ is increasing by (preference relation) > on a set Y, then any of its maximum points on Y is weakly efficient.*

Examples of functions increasing by preference relation > in $\mathbb{R}^m$ are $\min_{i \in M} \zeta_i(y_i)$ and $\max_{i \in M} \zeta_i(y_i)$, where all ζ_i are functions increasing in $\mathbb{R}$ (for example, those mentioned above $\zeta_i = y_i - y_i^0$; $\zeta_i = \mu_i y_i$ when $\mu_i > 0$).

The function $\varphi(y) = y_j$, where *j* is an arbitrary fixed number from *M*, is also increasing in $\mathbb{R}^m$ by preference relation >. By theorem 3, all maximum points on Y of these functions are weakly efficient. Note that the functions $\min_{i \in M} \mu_i(y_i^* - y_i)$ and $\max_{i \in M} \mu_i(y_i^* - y_i)$, where $y^* \in \mathbb{R}^m$, $\mu_i > 0$

are decreasing in $\mathbb{R}^m$ by preference relation >, and therefore their minimum points on Y are weakly efficient.

Definition 17. *A* ***subset*** *$B \subseteq A$ is called* ***closed from above*** *with respect to the preference relation > with respect to A if for any $a \in A$ and $b \in B$, it follows from the condition $a > b$ that $a \in B$.*

Using the results already formulated and constructing subsets of Y closed from above with respect to the preference relation > with respect to Y, we can obtain new conditions of weakly efficiency.

Theorem 4. *Suppose that the function φ_0 – is increasing in preference relation > and the functions φ_j, $j = 1,2,...,p$, are non-decreasing in preference relation > on Y. If the estimate $y^0 \in U$, where*

$$U = \left\{ y \in Y \;\middle|\; \varphi_j(y) \geqq t_j,\ j = 1,2,...,p \right\},$$

and t_j are arbitrary fixed numbers, satisfies condition

$$\varphi_0(y^0) = \max_{y \in U} \varphi_0(y),$$

then it is weakly efficient.

Example 6. Let $j \in M$, $N \subseteq M$ and the point $y^0 \in Y$ satisfy the inequalities $y_i^0 \geqq t_i$, $i \in N$. If $y_j^0 = \max_{y \in U} y_j$, where

$$U = \left\{ y \in Y \middle| \; y_i \geqq t_i,\ \text{for all } i \in N \right\},$$

then by theorem 4, $y^0 \in S(Y)$. Note that for $N = \varnothing$ we have $U = Y$.

Statement. *If the function $\varphi(y)$ is increasing by preference relation > on the set*

$$U = \left\{ y \in Y \;\middle|\; y = y_0 \ \text{ or } \ y > y^0 \right\},$$

then the estimate $y^0 \in Y$ is weakly efficient if and only

$$\varphi(y^0) = \max_{y \in U'} \varphi\left(y\right).$$

Item 3.2. Properties of efficient estimates

Theorem 5. *(Podinovsky).*

An estimate $y^0 \in Y$ *is efficient if and only if for every* $i \in M$

$$y_i^0 = \max_{y \in Y^i} y_i , \qquad (7)$$

where

$$Y^i = \left\{ y \in Y \mid y_j \geqq y_j^0 ; j = 1, 2, ..., m; j \neq i \right\} .$$

If $y^0 \in Y$ *is efficient, then it is the only point in Y that satisfies (7) for every* $i \in M$ *.*

For $y^0 \in Y$, let us introduce the set

$$Y^{(i)} = \left\{ y \in Y \;\middle|\; y_j = y_j^{\,0},\; j = 1, 2, ..., m,\; j \neq i \right\} .$$

Since $Y^{(i)} \subseteq Y^i$, the following consequence follows from theorem 5.

Consequence 2. *If* y^0 *is efficient, then* $y_i^0 = \max_{y \in Y^{(i)}} y_i$ *for every* $i \in M$ *.*

Theorem 6. *The estimate* $y^0 \in Y$ *is efficient if and only if*

$$\max_{(y,\varepsilon) \in \mathrm{T}} \sum_{i=1}^{m} \varepsilon_i = 0 ,$$

where

$$T = \left\{ (y, \varepsilon) \in Y \times \mathbb{R}_{\geq}^{m} \;\middle|\; y - \varepsilon = y^0 \right\} .$$

Theorem 7. *Let the function* $\varphi(y)$ *be non-decreasing in the preference relation* $\geq$ *on* Y *and let* y^0 *be its maximum point on* Y *. For* y^0 *to be efficient, one of the following conditions must be met:*

φ *is increasing in preference* $\geq$ *on* Y *;*

y^0 *is the unique maximum point of* φ *on* Y *.*

Examples of specific types of functions whose maximization (or minimization) leads to effective points.

Example 7. The function $\varphi(y)=\sum_{i=1}^{m}\mu_i y_i$, where $\mu_i>0$ is increasing for each variable y_i on the real line and therefore increases by the relation $\geq$ on $\mathbb{R}^m$. Therefore, any of its maximum points on Y is efficient.

Example 8. The function $\varphi(y)=\left[\sum_{i=1}^{m}\mu_i y_i^{\,s}\right]^{\frac{1}{s}}$, where $s>0$ and $\mu_i>0$ is increasing for each variable y_i on the set of nonnegative integers, and therefore increases by the relation $\geq$ on $\mathbb{R}^m_{\geqq}$. So, if y^0 is the maximum point of the function $\varphi(y)$ on $Y\subset\mathbb{R}^m_{\geqq}$, then $y^0\in P(Y)$. The same function $\varphi(y)$ at $s<0,\ \mu_i>0,\ i\in M$ is increasing for each variable y_i on the set of positive numbers, and therefore increases by the relation $\geq$ on $\mathbb{R}^m_{>}$. So, if y^0 is the maximum point of the function φ on the set $Y\subset\mathbb{R}^m_{>}$, then it is efficient.

Example 9. Let $y_i^*=\sup_{y\in Y} y_i$ for all $i\in M$, and let the function $\varphi(y)$ increase with preference relation $\geq$ on $\mathbb{R}^m_{\geqq}$. Then, as is easy to see, the function $\varphi(y^*-y)$ decreases by the preference relation $\geq$ on Y. By theorem 7, any of its minimum points on Y is efficient.

The function $\varphi(y^*-y)$, according to example 8, can be the function $\left[\sum_{i=1}^{m}\mu_i\left(y_i^*-y_i\right)^s\right]^{\frac{1}{s}}$, when $s>0,\ \mu_i>0,\ i\in M$. If $y_i^*>\sup y_i,\ i\in M$, then this function may also have $s<0$.

The function $\max_{i\in M} \mu_i\left(y_i^* - y_i\right)$, where y_i^* are arbitrary fixed numbers and $\mu_i \geqq 0$, is nonincreasing in preference relation $\geq$ on $\mathbb{R}^m$. If its minimum point on Y is unique, it is efficient.

The analog of theorem 4 for efficient estimates is the following theorem.

Theorem 8. *Let φ_j, $j = 0,1,..., p\left(p \geq 1\right)$ be non-decreasing by a preference relation $\geq$ functions on Y. If the point $y^0 \in U$, where*

$$U = \left\{ y \in Y \;\middle|\; \varphi_i\left(y\right) \geqq t_j,\ j = 1,2,..., p \right\},$$

and t_j are arbitrary fixed numbers, satisfies condition

$$\varphi_0\left(y^0\right) = \max_{y\in U} \varphi_0\left(y\right),$$

then one of the following conditions is sufficient for its efficients:

φ_0 increases by a preference relation $\geq$ on U;

y^0 is a unique maximum point of φ_0 on U.

Example 10. If in example 6 y^0 is the unique point that satisfies all the conditions specified there, then it is efficient.

Consequence 3. *Let $y^0 \in Y$, $U^0 = \left\{ y \in Y \mid y \geqq y^0 \right\}$, and let the function φ increase by the preference relation $\geq$ on U^0. The estimate y^0 is efficient if and only if $\varphi\left(y^0\right) = \max_{y\in U^0} \varphi\left(y\right)$.*

Example 11. Let $\mu_i > 0, i \in M$. The estimate $y^0 \in Y$ is efficient if and only if

$$\sum_{i=1}^{m} \mu_i y_i^0 = \max_{y\in U^0} \sum_{i=1}^{m} \mu_i y_i .$$

§ 4. Methods of solving multicriteria problems

From the mathematical point of view, multicriteria optimization problems are a natural generalization of conventional optimization problems.

Methods for solving multicriteria problems allow us to significantly expand the applicability of already available methods of nonlinear programming, unconditional minimization, and globally-balanced optimization. This is an important quality, since the apparatus for solving ordinary optimization problems has been developed quite well, many methods have been created, and proven programming software is available. Therefore, it is reasonable to apply the available resources to the solution of multicriteria problems relying on some or other methods of transforming such problems into ordinary optimization problems.

Let $\mathbb{R}^n$ be a n-dimensional Euclidean space, $f(x)$ and $g(x)$ are continuous-differentiable vector-functions on $\mathbb{R}^n$, realizing the mappings:

$$f:\mathbb{R}^n \to \mathbb{R}^m \text{ and } g:\mathbb{R}^n \to \mathbb{R}^r.$$

The multicriteria minimization problem will be denoted by

$$\min_{x\in X} f(x),\ X=\{x\in\mathbb{R}^n \mid g(x)\le 0\} \tag{8}$$

By the solution of problem (8) we mean the set of weakly Pareto-optimal (Slater-optimal) points in the space $\mathbb{R}^n$ defined as follows:

$$X_* = \left\{x^* \in X \mid \max_{1\le i\le m}\left[f_i(x) - f_i(x^*)\right] \ge 0\ ,\ \forall\ x\in X\right\}.$$

It corresponds to the set of weakly Pareto optimal (Slater optimal) estimates $F_* = f(X_*)$. Here the symbol $f(B)$ denotes the image of the set $B \subseteq \mathbb{R}^n$ under the reflection f .

Most of the known approaches to solving a multicriteria optimisation problem are based on its reduction to a nonlinear programming problem. One of the main and the first methods to be applied is the **convolution method**. It uses as an auxiliary function

$$R(x) = \sum_{i=1}^{r} \alpha_i f_i(x), \text{ where } \alpha_i \geq 0, \ \sum_{i=1}^{r} \alpha_i = 1$$

and the minimisation problem is solved

$$\min_{x \in X} R(x) \tag{9}$$

If some conditions are fulfilled, the points that deliver the minimum in problem (9) belong to the set X_*. To obtain different points from the set X_*, one should solve problem (9) each time with different sets of weight coefficients α_i.

Multiplicative convolutions are also used

$$R(x) = \prod_{i=1}^{r} \alpha_i f_i(x),$$

as well as **convolutions of normalised criteria** of the form

$$R(x) = \sum_{i=1}^{r} \left[\frac{f_i(x) - f_i(x_*)}{f_i(x_*)} \right]^p .$$

Another approach, in which an auxiliary problem has to be solved repeatedly to obtain different points from X_*, is methods that use constraints on the criteria. For example, the **constraint method** is based on selecting a main criterion, e.g. $f_k(x)$. The other criteria are used as constraints: $f_i(x) \leq b_i$, $1 \leq i \leq r$, $i \neq k$. The problem of minimising $f_k(x)$ in the presence of these constraints is then solved.

Assuming there is an ordering of the criteria in terms of importance we can use the concession method. First, the 'most important' criterion is minimised, e.g. $f_1(x)$, i.e. $F_*^1 = \min_{x \in X} f_1(x)$ is found. Next, the possible concession ε according to this criterion is indicated and the constraint $f_1(x) \le F_*^1 + \varepsilon$ is added to the system of constraints $x \in X$. On the resulting set, the second most important criterion is minimised and so on. Under fairly general conditions, this method finds a point from X_*.

The next approach to solving problem (8) consists of **goal programming methods**. They assume the existence of a target point $Y = \{y_i,\ i = 1, 2, ..., r\}$. Using the target values y_i and weights α_i, for each criterion $i = 1, 2, ..., r$, problem (8) is transformed into a goal programming problem, which can be represented as minimising the distance between the set of achievable evaluations F and the target point Y:

$$\min_{x \in X} d(F, Y) = \left[\sum_{i=1}^{r} \alpha_i \,|\, f_i(x) - y_i \,|^p \right]^{\frac{1}{p}}.$$

Another direction of solving multicriteria optimisation problems includes methods based on finding a compromise solution. These methods, use the principle of guaranteed result. The methods based on human-machine decision-making procedures have been widely developed. They use a dialogue mode between a human being, who understands the physical essence of the problem to be solved, and a computer processing information about the criteria preferences. This information serves to set up a new optimisation problem and obtain the next intermediate solution. This results in an interactive procedure for selecting the optimal solution (**interactive programming**).

Recently, there have appeared works that develop methods on a somewhat different basis. These methods do not make a preliminary transition to the problem of nonlinear programming in explicit form. In fact, all of them are generalisations of known local and global methods of nonlinear programming for problems with several criteria.

Thus, the **method of searching on an uneven grid**, which is designed for global optimisation, is used to solve multicriteria problems. This is a constructive method for finding the ε-optimal solution to the multicriteria problem (8). For this purpose, it is necessary to cover the set X with balls whose union contains the entire admissible set. These constructions are carried out using the technique of uneven coverings, as is done in global optimization methods. It is guaranteed that the set of feasible points obtained is an ε-optimal solution to the multicriteria problem (8). The effectiveness of search methods on an uneven grid largely depends on how close the current record value is to the optimal one. Therefore, it is advisable to use local methods, such as the modified Lagrange function method, as auxiliary procedures, which can significantly speed up the process of finding good record estimates.

The work of B. Pshenichnyj and A. Sosnovskyi is devoted to the modification of the well-known **linearisation algorithm** for solving a multicriteria problem. Here we consider as an auxiliary problem of quadratic programming of the form

$$\min_{p,\xi}\left\{\xi+\|p\|^2/2 \;\mid\; \left\langle f_i^{+}(x),\, p\right\rangle \le \xi,\;\; i=1,2,\dots,r,\right.$$

$$\left.\left\langle g_j^{+}(x),\, p\right\rangle + g_j(x) \le 0,\quad j=1,2,\dots,m\right\}.$$

The proposed method allows us to identify weakly Pareto-efficient solutions when the Cottle regularity condition is fulfilled. It is also shown that, if the generalised regularity condition is additionally satisfied, we can also obtain proper efficient solutions of problem (8). For the convex case of problem (8) the above conditions are sufficient.

To collapse all partial criteria into a single criterion, it was proposed to use penalty functions, which are commonly used in nonlinear programming to collapse constraints. Then problem (8) is reduced to a parametric problem of nonlinear programming, where some threshold levels in the space of criteria $y \in \mathbb{R}^r$ act as parameters.

This approach made it possible to use **penalty function methods** to solve problem (8), which reduces to the problem of unconditional minimisation $\min H(x, y, t)$. Different rules for choosing penalty functions $H(x, y, t)$, levels y and penalty coefficients t lead to both the method of external penalty functions and the method of internal penalty functions.

Different classes of convolution functions $Q(h(x, y))$ can be used. They lead to three classes of methods corresponding to **inner centre methods, outer centre methods and joint inner and outer centre methods**. In both the penalty function methods and the centre methods, the choice of different levels leads to different points from X_*.

On the basis of the Lagrange function

$$L(x, u, v, y) = \langle u, h(x, y) \rangle + \langle v, g(x) \rangle, \tag{10}$$

a method for solving the multicriteria optimisation problem (8) is constructed, which is a generalisation of the **method of modified Lagrange function** known in nonlinear programming. The proposed modification of function (10) improves the behaviour of the Lagrangian function in the vicinity of solutions. The method of modified Lagrangian functions for solving multicriteria optimisation problems retains all the good properties that are inherent in the method for solving nonlinear programming problems, namely: linear convergence rate on dual variables and existence of exact local minima of the auxiliary function in the vicinity of solutions of the problem.

Extremely rich information about the properties of the Pareto set is provided by the results of works obtained in nonlinear programming on the theory of sensitivity functions.

Let us consider a generalisation of a number of iterative methods for solving nonlinear programming problems to the case of solving problem (8). In these methods, the main leading process is the process in the space of initial variables. Simultaneously with it, estimates of optimal values of criteria are calculated.

First, let us consider a generalisation of the **method of feasible directions**. This method, proposed by G. Zoutendijk in 1959, was one of the first to be used for solving convex programming problems. Later it was shown that it can be used also for solving nonlinear programming problems. Different versions of the method of E. Polak, D. Topkis and A. Veinott, G. Meyer, D. Mayne, etc. differ among themselves both in the type of auxiliary problem, type of normalising constraints, choice of descent step, and different ways of dealing with the 'zigzag' motion. By combining the methods of possible directions with the methods of internal and external penalty functions, hybrid type algorithms suitable for solving problems with nonlinear constraints of the equality type are obtained. Possible direction methods of the second order and combined 'two-phase' methods are developed.

Another approach is also possible, when the method is interpreted as a method of minimisation of some auxiliary function, which depends not only on the initial direct variables, but also on the estimation from above of the optimal value of the target function. This interpretation allows us to construct a generalisation of the method to the case of solving a multi-criteria optimisation problem.

Let us describe two variants of the method, which are joint 'two-phase' in the sense that the initial point can be chosen arbitrarily.

Consider the multicriteria optimisation problem (8), in which the vector functions $f(x)$ and $g(x)$ defining it are assumed to be convex. The constraints in problem (8) are assumed to satisfy Slater's condition, i.e., the set

$$X^0 = \{x \in \mathbb{R}^n \mid g(x) < 0\}$$

is not empty. Take the vector $y \in \mathbb{R}^n$, denote $l = r + m$ and consider an l-dimensional vector function:

$$h(x, y) = [\, f_1(x) - y_1, ..., f_r(x) - y_r, g_1(x), ..., g_m(x) \,].$$

Let us compose the auxiliary function

$$H(x, y) = \max_{1 \le i \le l} h_i(x, y).$$

The **special points of the function** $H(x, y)$ are those $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}^n$, for which the conditions

$$H(x^*, y^*) = 0, \tag{11}$$

$$x^* \in \operatorname{Arg}\min_{x \in \mathbb{R}^n} H(x^*, y^*) \tag{12}$$

are fulfilled.

Lemma 1. *For vector* $x^* \in \mathbb{R}^n$ *to be a solution of problem (8) (i.e.,* $x^* \in X_*$*), it is necessary and sufficient that there exists a vector* $y^* \in \mathbb{R}^n$ *such that conditions (11), (12) are satisfied.*

In fact, lemma 1 reduces the solution of the original problem (8) to finding special points of the function $H(x, y)$.

For this purpose, we consider the linear programming problem

$$\max_{s, \sigma} \sigma \tag{13a}$$

$$\left\langle \left(h_i(x,y)\right)_x, s \right\rangle + \sigma \le 0, \quad i \in I_\varepsilon(x, y) \tag{13b}$$

$$\left|s_j\right| \le 1, \quad j = 1, 2, ..., n, \tag{13c}$$

where

$$I_\varepsilon(x, y) = \left\{1 \le i \le l \mid h_i(x, y) \ge H(x, y) - \varepsilon\right\}$$

is the **set of indices of the** ε**-active components** of the function $h(x, y)$. The solution of this problem sets the direction of decreasing of the function $H(x, y)$.

Since the function $H(x, y)$ is non-smooth, the numerical method uses the ε-algorithm to find special points of the function $H(x, y)$. The initial values $x^0 \in \mathbb{R}^n$, $y^0 \in F_+$, direction $e \in \mathbb{R}^r_+$, $\varepsilon > 0$ are set. The next iteration point is constructed by the rule

$$y^{k+1} = y^k - \beta(x^k, y^k, e) \cdot e, \tag{14a}$$

$$x^{k+1} = x^k + \alpha_k \cdot s_\varepsilon(x^k, y^{k+1}), \tag{14b}$$

where $\beta(x, y, e) = \min\limits_{1 \le i \le r} \dfrac{y_i - f_i(x)}{e_i}$, and the parameter ε varies according to the standard, for the method of feasible directions, scheme. The set of limit points of the sequence $\{x^k\}$ worked out by the iterative process (14) is contained in the set X_*.

Let us consider a generalisation of another version of the method of feasible directions, namely the Topkis-Weynott version. The difference between this version of the method and the version considered above is that the linear programming problem here is of the form:

$$\max_{s,\sigma} \sigma \tag{15a}$$

$$\left\langle \left(h_i(x,y)\right)_x, s \right\rangle + h_i(x,y) - H(x,y) + \sigma \le 0, \quad i = 1, 2, ..., l, \tag{15b}$$

$$\left|s_j\right| \le 1, \quad j = 1, 2, ..., n, \tag{15c}$$

If in the problem (13) there are only ε-active components of the vector function $h(x, y)$, then the constraints (15b) include all components. In the numerical method, iterations are carried out according to scheme (14) with the solution at each step of the auxiliary problem (15). It should be noted that in the proposed variants of the method, the parameter y plays the role of estimating the values of the vector function $f(x)$. Here, the values of y^k obtained at each iteration belong to a single ray originating from the point y^0 and penetrating the criterion space in a given direction e. The methods result in a one weakly Pareto-optimal estimate. To obtain different points from the Pareto set, one should either change the initial vector y^0 and move in the same direction e, or vice versa, fixing the initial vector y^0 to vary the direction e. If $y_i^0 > \max\limits_{x \in X} f_i(x)$ for all $i = 1, 2, ..., r$, then by varying the direction e within the positive orthant $\mathbb{R}^r_+$, we can obtain any point from X_*.

Now let us consider the **linearisation method**, which was proposed by B. Pshenichnyj for solving the general problem of nonlinear programming. At each step in this method, the functions defining the original problem are linearised and an auxiliary quadratic programming problem is solved. Numerous versions of the linearisation method of R. Wilson, U. Garcia Palomares and 0. Mangasarian, S. Han differ from each other by the rules of constructing the auxiliary problem and determining the step of descent. Further studies by B. Pshenichnyj, Y. Danilin, V. Zhadan and other authors are devoted to various modifications of the method, as well as to questions of its convergence. The linearisation method is also applied to the problem of finding the minimax. B. Pshenichnyj and A. Sosnovskyi generalised it to the case of solving multicriteria optimization problems.

Let us consider a slightly different generalisation of this method, which differs by the fact that the auxiliary problem is a linear programming problem. And the estimates in it, as well as in the method of possible directions, belong to some ray. This makes it possible to precisely determine the optimal point of the Pareto set in the criteria space, which corresponds to the intersection of the ray with the boundary of the set of achievable values of the criteria in this space.

For problem (8) we introduce an auxiliary function

$$M(x, y, t) = \varphi_e(x, y) + t\psi(x), \tag{16}$$

where the functions $\varphi_e(x, y)$ and $\psi(x)$, for the chosen direction $e \in \operatorname{int} \mathbb{R}^r_+$, are defined as follows

$$\varphi_e(x, y) = \max_{1 \le i \le r} \frac{f_i(x) - y_i}{e_i}, \quad \psi(x) = \max_{1 \le j \le m} g_j(x).$$

Let set $W = \{[x, y] \in \mathbb{R}^{n+r} \mid \varphi_e(x, y) = 0\}$, and let W_* be the set of points $w^* = [x^*, y^*] \in W$ where the necessary first-order conditions are satisfied. We will call the point $[x, y, t] \in \mathbb{R}^{n+r+1}$ a **special point of function** (16) if

$$[x, y] \in W, \quad \psi(x) = 0 \quad \text{and} \quad \inf_{\|s\|=1} \frac{\partial_x M(x, y, t)}{\partial s} \geq 0 .$$

Lemma 2. *If* $[x^*, y^*] \in W_*$ *then* $[x^*, y^*, t]$ *is a special point of the function* $M(x, y, t)$ *and vice versa: if* $[x^*, y^*, t]$ *is a special point of the function* $M(x, y, t)$ *then* $[x^*, y^*] \in W_*$.

Thus, solving problem (8) is equivalent to finding special points of the function $M(x, y, t)$. The numerical method of finding them involves solving at each iteration some auxiliary linear programming problem: find

$$\min_{s,\eta,\sigma} \left(\eta + P(x, y) \cdot \sigma \right) \tag{17a}$$

$$\left\langle \left(f_i(x) \right)_x, s \right\rangle + f_i(x) \leq y_i + e_i \eta, \quad i \in I_\delta(x, y), \tag{17b}$$

$$\left\langle \left(g_i(x) \right)_x, s \right\rangle + g_i(x) \leq 0, \quad i \in J_\delta(x), \tag{17c}$$

$$\left| s_j \right| \leq 1 + \sigma, \quad j = 1, 2, ..., n, \quad \sigma \geq 0, \tag{17d}$$

where $s \in \mathbb{R}^n$, $\delta > 0$, $P(x, y)$ is an arbitrary continuous function on $\mathbb{R}^n \times \mathbb{R}^r$ such that

$$P(x, y) \geq 1 + \sqrt{n} \cdot \max_{i \in I_\delta(x,y)} \left\| \left(f_i(x) \right)_x \right\| / e_i .$$

The index sets are defined as follows

$$I_\delta(x, y) = \left\{ 1 \leq i \leq r \mid h_i^e(x, y) \geq \varphi_e(x, y) - \delta \right\},$$

$$J_\delta(x) = \left\{ 1 \leq j \leq m \mid g_j(x) \geq \psi(x) - \delta \right\},$$

here

$$h_i^e(x, y) = \left[f_i(x) - y_i \right] / e_i \quad \text{for all} \quad 1 \leq i \leq r .$$

If the admissible set in problem (17) is not empty, then its solution exists and is finite. If the point $[x, y] \in W$, then the direction $s(x, y)$, found from the solution of problem (17), is the direction of decreasing of the function $M(x, y, t)$ in x for sufficiently large t. In addition, the existence of a special point of the function $M(x, y, t)$ is equivalent to the presence of a zero solution among the solutions of problem (17).

In the numerical method of finding the special points of the function $M(x, y, t)$, the iterative process is carried out according to the following formulas: let $x^0 \in \mathbb{R}^n$, $y^0 \in \mathbb{R}^n$, the step $\alpha > 0$ and the parameter $0 < \beta < 1$ are chosen, then

$$y^{k+1} = y^k + \lambda(x^k, y^k, e) \cdot e \tag{18a}$$

$$x^{k+1} = x^k + \alpha_k \cdot s_\varepsilon(x^k, y^{k+1}) \tag{18b}$$

where

$$\lambda(x, y, e) = \max_{1 \le i \le r} \frac{f_i(x) - y_i}{e_i},$$

$s(x^k, y^{k+1})$ is the solution of problem (17) at the point $[x^k, y^{k+1}]$.

The iterative process (18) is constructed in such a way that all points $[x^k, y^{k+1}]$ belong to the set W, so that at each step, except for the descent in the variable x, a correction is made by fulfilling the equality $\varphi_e(x^k, y^{k+1}) = 0$. If the point $[x^k, y^{k+1}] \notin W_*$, then the step down from this point is always strictly positive and can be obtained by dividing the initial step α in half by a finite number of times.

Under fairly general and natural conditions, the set of boundary points of the sequence $\{x^k\}$ formed by the process (18) is contained in the set X_*.

The method also results in one weakly optimal estimate. To obtain different estimates, either y^0 or the direction e should be varied.

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