Mathematical Reasoning and Investigation

Mathematical Reasoning and Investigation

SIMON JAMES; CHRIS RAWSON; AND ILLUSTRATED BY ERIN CHEFFERS, DEAKIN UNIVERSITY

DEAKIN UNIVERSITY

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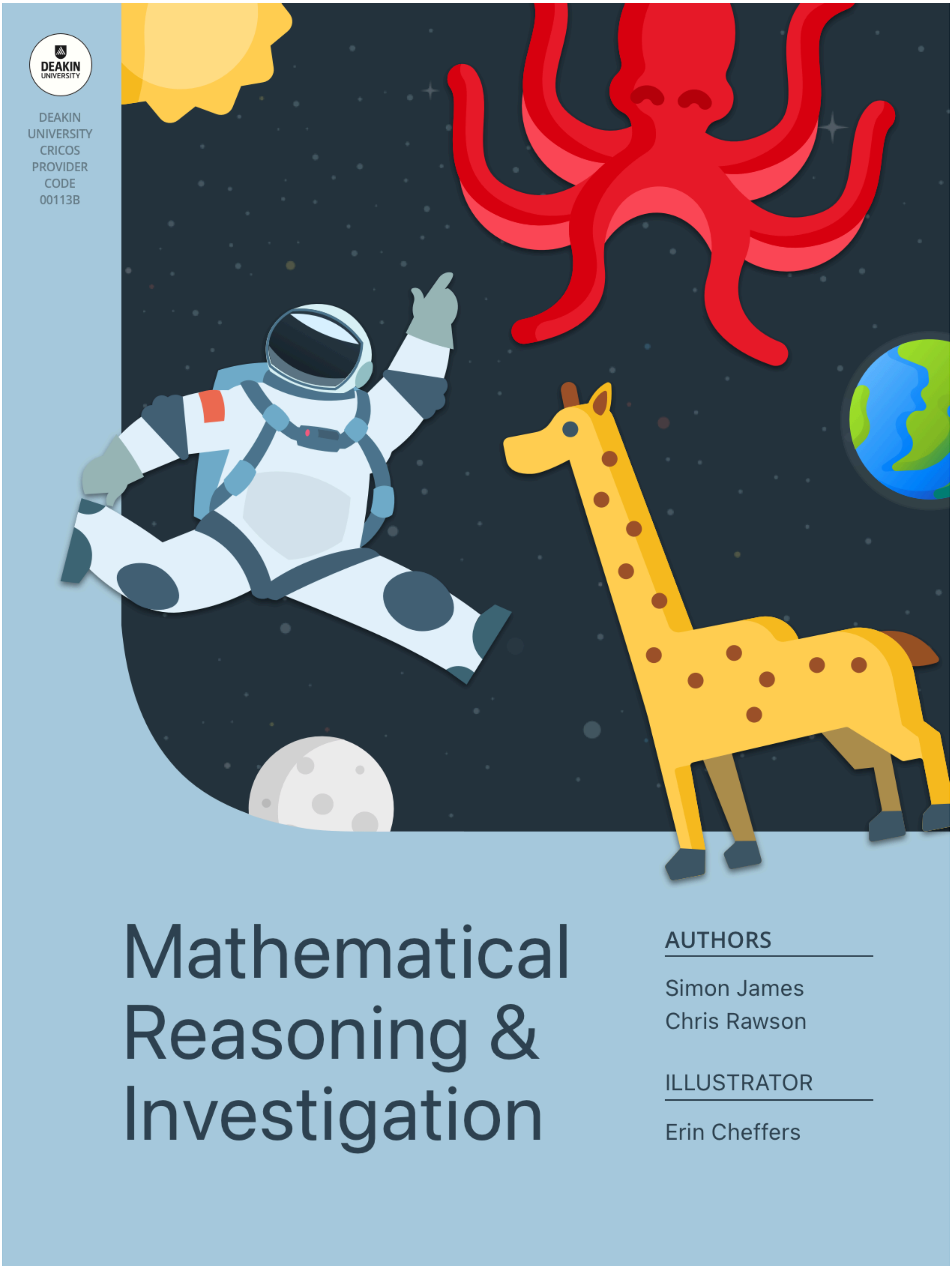
Front Cover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Mathematical Reasoning & Investigation
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We would also like to thank the Shell Center for Mathematical Education, who not only granted us permission to use and modify their problems for use within this textbook (particularly in the section on interpreting visual information), but also, generously allowed us to license them under the CC BY NC 4.0 to help us in making this textbook open for re-use by the public.

A great many individuals came together to make this book possible. It would never have happened without the support of the people we're about to shout out to.

Our illustrator, Erin Cheffers, has done an outstanding job and has really brought the book up to the next level. Thank you so much Erin, it was such a pleasure working with you and we hope to do so again in the near future!

The whole CloudFirst team deserve a shout out, and in particular we would like to acknowledge the fantastic advice and support we received from Lea Piskiewicz, Steven Brown and Darci Taylor. Lea's influence is stamped like a fingerprint on every banner in the book and a great many of the other images, too. I would say that her assistance with finding an illustrator, copyright clearance and helping us to think through the visual elements of the book has been exceptional, except that all of Lea's work is exceptional. So, let's just say that this incredibly high level of work and support is what we've come to expect of Lea. Likewise, Darci Taylor and Steven Brown's wise guidance, support and advice have helped us to deal with many tricky scenarios and we have appreciated their counsel.

From the Deakin University Library, we would like to acknowledge Angie Williamson, who has been a calming presence and a source of truth and stability throughout the entire process. We would also like to acknowledge the meticulous and detailed efforts of Astrid Bovell. We may have a few more grey hairs, but the book is so much the better for it and she has been a good sport throughout. We appreciate her timely and proactive support and advice – we couldn't ask for more from either Angie or Astrid.

Finally, the biggest thanks of all goes from Chris Rawson to Associate Professor Simon James. Simon's work has greatly improved the way I understand mathematics and its relationship to 'the real world.' Simon was pulled on to this project with a promise that it would be quick and easy; it has been anything but. Nonetheless he has been generous with his time and attentive to the minute details of the work and done so cheerfully. Simon is a man of many hidden talents. I admire him as much for his modesty as for his enormous abilities in teaching and in mathematics.

Double finally, Simon equally thanks Chris for our collaboration over the last few years. Having your work to be scrutinised by a learning designer is a very daunting and intimidating thing at first, but Chris has always been encouraging and supportive, while at the same time always offering ideas, criticism and insight that help me improve my teaching. I hope to continue working with him on more projects in the future!

Acknowledgement of country

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Acknowledgement of Country

The authors and Deakin University acknowledge the Traditional Custodians of all the unceded lands, skies and waterways on which Deakin students, staff and communities come together. As we learn and teach through virtually and physically constructed places across time, we pay our deep respect to the Ancestors and Elders of Wadawurrung Country, Eastern Maar Country and Wurundjeri Country, where Deakin's physical campuses are located; as well as the Traditional Custodians of all the lands on which you may be learning and teaching, where education has taken place for many thousands of years.

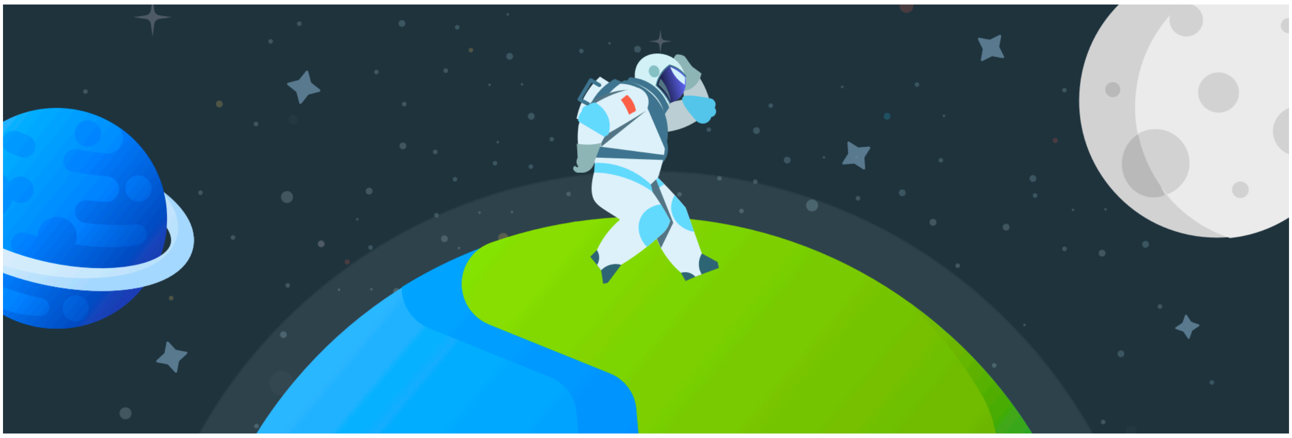
Accessibility Information

We've designed this book with accessibility needs in mind.

We've done our best to ensure that the book has been optimized for people who use screen-reader technology and are navigable using a keyboard.

- We've tried to ensure that content is organized under headings and subheadings, which are used sequentially.
- There are many graphs, charts and images in this book. We've attempted to ensure that all such media include alt text descriptions where this is helpful, and/or can be understood from the contextual detail surrounding the image.
- Our links have been written descriptively, and all links open in a new window.
- Any videos that we have produced include captions and a full transcript, both of which have been checked for accuracy. We cannot make the same assurance for third party videos that have been embedded in the book.
- This is a maths book, so you can expect to encounter a lot of mathematical expressions (equations and symbols, etc). These have all been written in LaTex and render using MathJax, making them compatible with MathsML. In short, your screen reader should be able to decipher them.

While we have done our best to make this resource as useful and accessible as possible, we are not experts! If you have problems accessing this resource, please contact the authors and we will endeavour to resolve the issue.



Welcome to *Mathematical Reasoning and Investigation.*
Or, as I like to call it, mathematics for people who think they're bad at it.

Why mathematics is important

Mathematics can be a powerful way to investigate, analyse and make sense of the world around us. Most of us are aware that mathematical reasoning underpins a great deal of research in science, medicine and technology, but are less familiar about the ways in which it can be used to solve the kinds of problems that confront us every day in our workplaces and society more generally.

So many of these issues can be informed by mathematical analysis. Learning about mathematics can help us to make better-informed decisions and develop critical insights into effects of the decisions made on our behalf. Indeed, some of our biggest unresolved social questions can, and should, be informed by mathematical reasoning and investigation. Questions like how we can live in a more equal world, how we can best use and distribute the resources around us, and how we should understand and respond to changes in the climate all rely on numbers to point us in the right direction. We're not suggesting for a moment that numbers are the only thing that should inform our decisions. However, when you're trying to tackle a difficult problem, being able to reason mathematically is an important tool to have at your disposal.

Mathematical reasoning

This is what makes the capability of *mathematical reasoning* so essential. People who have developed the ability to reason mathematically are able to make reasoned conjectures, test them and then come to reasoned conclusions. The process of doing so will prompt us to challenge our assumptions, keep a mind that is open and strategic at the same time, justify our own thought processes and think about what it means to solve problems in the first place. If there's one thing that practicing mathematical reasoning can teach us, it's the skill of patiently working through problems creatively, systematically and methodically.

Who is this book for?

This book has been written for

1. Adults who want to improve their own skills in mathematical reasoning and investigation
2. Teachers who want ideas about how to teach mathematical reasoning using a problems-based

methodology.

If you are in the first category, try not to stress (see the section below). We all need to start from somewhere, and this book makes very few assumptions about the base knowledge that people will come in with. Anyone with a basic level of mathematical knowledge should be able to start at the beginning of the book and work their way through. If you can use a basic calculator, add, subtract, multiply and divide then you'll be good to go. As with any area of mathematics, and most things in life, the more you know the easier it will be for you. If you're comfortable with algebra, that's a great start. But if not, we'll walk you through it and you'll be using it to solve problems in no time.

If you are a teacher, you may be able to adapt sections of this book to your classroom and, perhaps more importantly, use it to inspire learning activities that explicitly develop a student's ability to reason mathematically, by taking a problems-based approach. As you'll see, often when taking a problems-based approach, getting to the correct answer (if there is one) is a secondary objective to developing the ability to plan, test conjectures and generalise findings. Working through the book may also help give you the grounding in mathematical reasoning you need to be confident setting activities for your students.

Starting out with mathematics can make you nervous

If we can agree that maths is a tool that can help us to make better informed decisions, then Western societies in general have a problem. The thought of doing mathematics makes us feel a little nervous.

> I (Chris Rawson) believed I was bad at mathematics well into my adult years. I believed that I wouldn't be able to do it, that I didn't like it, that it was fundamentally procedural and even that it dampened creativity. It was this more than anything else that held me back from attempting to learn more. But by working through the content that forms the bedrock of this book, I was able to move beyond my own anxiety and false assumptions.
>
> Once I gave mathematics a go, I discovered that even if I'm not a maths wizard like Simon James, I am able to competently tackle some quite tricky problems. This book is based on Simon's methods – the overwhelming majority of it is Simon's work. One of the core reasons I encouraged its publication as an open resource was because working through this content had taught me so much and made me feel more confident and able to tackle problems that previously I would simply not have attempted. It's been liberating, and that is something I want to share with others.

According to research published in 2018, 93% of adults in the USA report that sometimes the thought of doing mathematics makes them feel anxious. The problem isn't confined to the United States, and it's not just adults either. According to the OECD's 2012 PISA report that looked at 34 different nations, 59% of students aged 15 and 16 often worry that maths class will be difficult for them.

This isn't helped by some of the stereotypes and media portrayals of what it looks like to be good at maths. Many people in the movies who are good at maths are born with a gift with numbers and can imagine complex patterns emerging (sometimes quite literally) where most of us see nothing. Frankly, they are geniuses and the rest of us will never get near them. The assumption: mathematical ability is a gift that you're born with. People with this gift, in the movies, also happen to be almost exclusively white males.

It's important we don't make these same assumptions for ourselves, our students or those around us. Mathematical reasoning is a skill that can be developed like anything else, it just takes practice. It's a little bit like starting to learn a musical instrument: it's normal to think 'I'll never be able to do this' at the beginning, but with persistence you will get there. It's interesting to recognise that this advice doesn't only hold true for people who started out thinking that, like Chris, they were allergic to mathematics. It's just as true of people with a deep mathematical knowledge and affinity for numbers, like Simon James. One of the most powerful moments in this book is when Simon describes starting out on geometry, struggling to solve problems, feeling frustrated when he has trouble and then cheating by looking up the answer on YouTube (then promptly forgetting the answer). We don't nominate this moment for special attention because we want to cut Simon down to size – rather it's to point out that even someone who has a PhD in mathematics can struggle when they encounter something for the first time. The trick is knowing when to persist, and when to try another strategy.

Mathematics is more than just methodology

Mathematical reasoning is a process of making reasoned conjectures, testing them and seeing if we can justify our solutions to sometimes complex or fairly undefined problems. This can be an incredibly creative and satisfying process. Many people don't associate mathematics with creativity – indeed some people consider it the *opposite* of creativity, seeing it as a fundamentally rigidly structured process of memorising abstract rules. Worse, some people get the impression that mathematics is purely abstract, a secret code only understood by scientists and with otherwise little real-world application. This is a very sorry state of affairs.

Perhaps some of this comes down to the way we were taught mathematics. Memorising complex methods by rote, solving abstract equations and checking to see if we got the right answer is bound to seem abstract, difficult and ultimately irrelevant to most students. While these kinds of activities remain a fundamental and necessary part of developing numeracy in children, teachers should not resile from the task of demonstrating *how* and *why* mathematics can be used to solve issues both in our day-to-day life and, as children get older and more socially aware, our society in general. However, if you're not confident doing this on your own, it's difficult to teach it to others.

The kind of problem-based learning modelled throughout this book is a great place to start. We seek to propose an alternative to the back of the book. Instead, we attempt to pose problems that ask you to come up with your own methods, to make conjectures, test how reasonable they are and to justify your answers. Finally, we ask you to work towards the ability to generalise – that is, to be able to answer any similar problems that come up.

Just to emphasise the point, the authors are not suggesting that there is no place for teaching mathematical methods explicitly. In fact, we do so ourselves at times throughout the course of the book where it promotes problem solving. Instead, think of learning mathematics as being analogous to learning a new language. In this analogy, methods are like grammar – a series of rules that effectively need to be learned by experience and somehow internalised or at least memorised. But as important as grammar is to language (the words will make no sense without it), you can't learn to communicate and interact with others or achieve your goals just because you know all the grammatical rules. Similarly, the ability to reason mathematically goes above and beyond knowing mathematical methods in an abstract sense – we need to show that we can apply it to help us navigate our lives in the real world and achieve our objectives. As many of us ask ourselves as teenagers: what's the point of algebra if we can't do anything with it?

About the authors

Simon James is Associate Professor of Mathematics at Deakin University. Chris Rawson assisted in the development of this book in the capacity of Senior Educational Designer, also at Deakin.

Conclusion

We hope that you and your students enjoy working through the problems in this book. More importantly, we hope it helps you to see new possibilities for mathematical reasoning and investigation.

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What can you learn from the book?

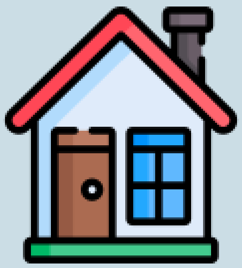
What is mathematical reasoning and investigation?

Mathematical reasoning is the ability to develop and apply techniques for solving a specialised problem and, where possible, to abstract a principle from the specialised problem which might form a general rule or approach to problem solving. In this unit, that includes the ability to do so with visual information (eg, geometry).

In other words, the ability to reason using mathematics means that you can take a problem, figure out some kind of solution that you feel confident that you can justify, and then see if that solution might be applicable in another situation, setting or context.

Take a well-known example.

Painting



Alma can paint a house in 3 hours, Marvin can paint a house in 5 hours. How long does it take them to paint a house if they work together?

If you can work out the solution to this, you can probably work out the solution when Alma takes 12 hours and Marvin takes 29 hours – or even when Alma takes **A** hours and Marvin takes **B** hours, but a solution that only helps us solve specific painting problems might not be too interesting. At the surface level, Alma and Marvin's painting problem doesn't look very similar to the "point of no return" problem.

The point of no return

A plane that travels 300 km/h in still air takes off with 4 hours of fuel. On the outward journey, it is helped along by a 50km/h tail wind which increases its speed to 350km/h. After cruising for some time the pilot realises that, on the return journey, the plane will only be able to travel 250km/h. What is the maximum distance the pilot can travel from the airfield and still have enough fuel to return home?

However, the mathematics underlying some approaches to both these problems is actually the same, because both can be tackled by calculating an average of rates. Once we start making these kinds of connections, we can become very agile problem solvers!

What will you learn?

How can we develop our mathematical reasoning?

We hope that, by engaging with the book, your ability to employ mathematical reasoning to solve problems will be extended. More than that, we want you to reflect on what it is about undertaking problem solving that leads to the development of mathematical reasoning, so that you can assist a similar development with other learners.

Here's how we've broken it down into learning outcomes.

Learning outcomes

Learning outcome	**What this means for you**
Apply results from measurement, geometry, networks, sequences and probability to solve routine problems involving written and visual information.	You'll need to use what you learn in geometry, measurement, networks, sequences and probability to solve routine problems. A routine problem is one with a clearly defined method. That is, there's a set routine when it comes to solving that type of problem.
Apply techniques of mathematical reasoning and problem-solving to solve non-routine problems related to measurement, geometry, networks, sequences and probability.	You are going to see a lot of open-ended problems, where there's no perfect fit for a routine solution. You'll need to use mathematical reasoning to solve the problems, justify your solution and see if you can apply a similar method in new settings.
Write code to develop programming skills and use algorithms and systematic iteration to solve problems and explore data.	You'll be using a program called Scratch to develop code that can solve mathematical problems and explore patterns in data.

These are the overall learning outcomes for the book as a whole. Each section has more specific learning outcomes, as well.

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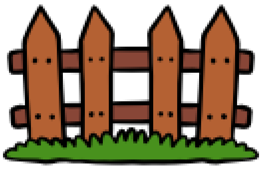
What if I get stuck?

So, you're stuck. What should you do next?

To start with, don't assume that a state of being stuck is a negative thing. We learn a lot more from getting ourselves "unstuck" than from never getting stuck in the first place. On the other hand, you won't learn a lot if you keep retreading old ground or trying random methods without reason. Depending on *how stuck* you are, you can try any of the following.



Put some parameters in place

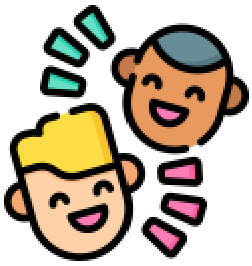
Consider whether there's a "reasonable" initial guess you could make that will guide your thinking and help you understand what mathematics might be involved – are there some "unreasonable" answers that you can take off the table?

Read the question again

Make sure you've read the problem correctly and that you understand all the terms – if you don't understand a term, you could try an internet search or ask someone else in the class (e.g. using the discussion board).

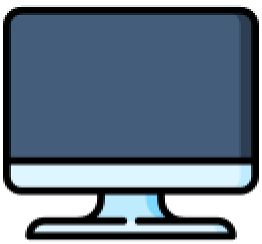
Take a break

Put it on the shelf for a day or two, or continue on to other parts of the topic module. In some cases there will be hints available, or in some cases there are activities in Scratch or elsewhere that can help you verify or check your understanding.



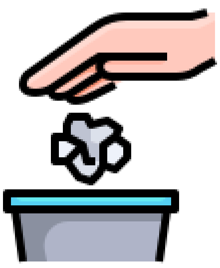
Ask a friend

Doing maths together can be more fun than it might initially sound. Find a friend who is willing to try to solve a problem with you.



Research it online

Conduct an internet search on the problem or concept. For most of the problems we present and all of the mathematical concepts, there is an abundance of examples, analyses and commentary available online. Improving your mathematical resourcefulness is also an important goal of any mathematical study.



Give up

Kidding! But bear in mind that giving up is **not** the same as asking for help, leaving a problem aside for a few days, or even indefinitely shelving it. So rather than thinking of these behaviours in a negative way, think of them as strategies toward fully understanding and being able to tackle the problem you're stuck on 🙂

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PART I
THE HERO'S JOURNEY

This first section is all about the hero’s journey

If there’s one big theme we’ll be playing around with throughout the book, it’s the question of ‘what constitutes good quality mathematical reasoning?’

During the first section, we’ll begin to answer that huge question by starting with a slightly more manageable one: what does a good approach to problem solving look like?

We begin with the topic of mathematical reasoning, where we will give an overview of the three-stage framework of problem solving: entry, attack and review.

Learning outcomes

By the end of the section, you should be able to

- Describe and apply a three-stage framework for solving non-routine problems
- Reflect on the process of solving mathematical problems

1.

What constitutes good quality mathematical reasoning?

If there's one big question we'll be looking at throughout the book, it's this.

During the first section, we'll begin to answer that huge question by starting with a slightly more manageable one: what does a good approach to problem solving look like?

We begin with the topic of mathematical reasoning, where we will give an overview of the three-stage framework of problem solving: entry, attack and review. In the next section, we'll refresh some of the fundamentals of maths and algebra. You may have already come across algebra during your schooling, so here you can put your focus on deep understanding – so that even if you know how to do something, you could comfortably explain it to someone else and understand how to approach it from different angles.

The sub-topics of problem solving and algebra will lay the foundation for how we approach future topics. In general, problem solving has the potential to draw on all of your mathematical learning. However, it should also bring your understanding together so that mathematics progresses from being a series of unrelated tasks to a much more cohesive framework for understanding the world.

Problem solving can make you a bit of a hero

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=5#oembed-1

Transcript

As the video discusses, there appears to be a deep structure to many of the stories we tell ourselves about what makes someone a hero. The main character (who in this analogy would be you) leaves their comfortable home and sets off on an odyssey in order to solve a problem that troubles a group of people. Along the way, they consult with their forebears and overcome a series of challenges on the path to their ultimate victory. They then return home as a hero, having solved the problem at its source, once and for all. For our purposes, the ultimate victory might look something like a method for solving not only *this* particular problem, but any similar problem.

Let's keep that in mind as we try our first problem.

The handshake problem

You and 19 others are at a networking event. As the event closes, everyone shakes hands with everyone else. How many handshakes take place altogether?

Before you go any further, try to solve the handshake problem. You can use any tools at your disposal to do so.

If you're able to find an answer, can you **generalise** your solution? In this case, generalising might mean coming up with a way of quickly solving the problem, no matter how many people there are at the networking event.

What did you do?

Did you:

- Skip the process of trying and use google to find an answer?
- Work out quickly in your head what would roughly be required and then assumed we'll show you the answer later in the book?
- Remember that you've solved this problem before and so you already knew how to solve it?
- Jump straight into scribbling diagrams and writing sums to try and find the answer?
- Re-read the question to work out what it was actually asking and why it might be considered a problem in the first place?

For the moment, any of these is fine – a problem has to grab you, and depending on your approach to this topic, you'll attempt some of the problems and skip over the details of others. It's just important to bear in mind that what we're *really* interested in when it comes to problem solving is the journey and the opportunity to improve.

What makes a good problem solver?

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=5#oembed-2

Transcript

Stop and think

Do you agree with Simon's description of what a good problem solver looks like?

Do you think that you are a good problem solver? What could you work on to improve your problem solving skills?

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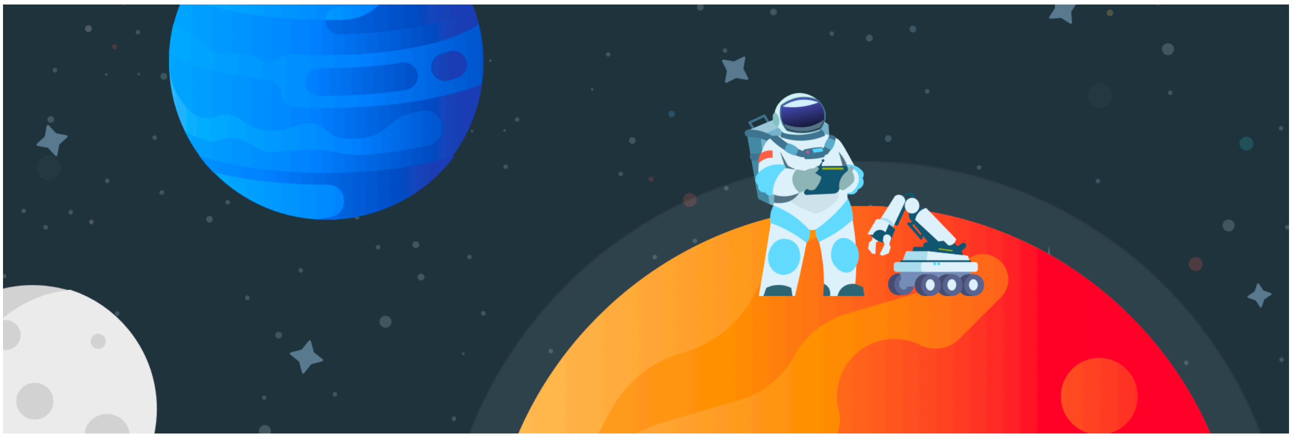
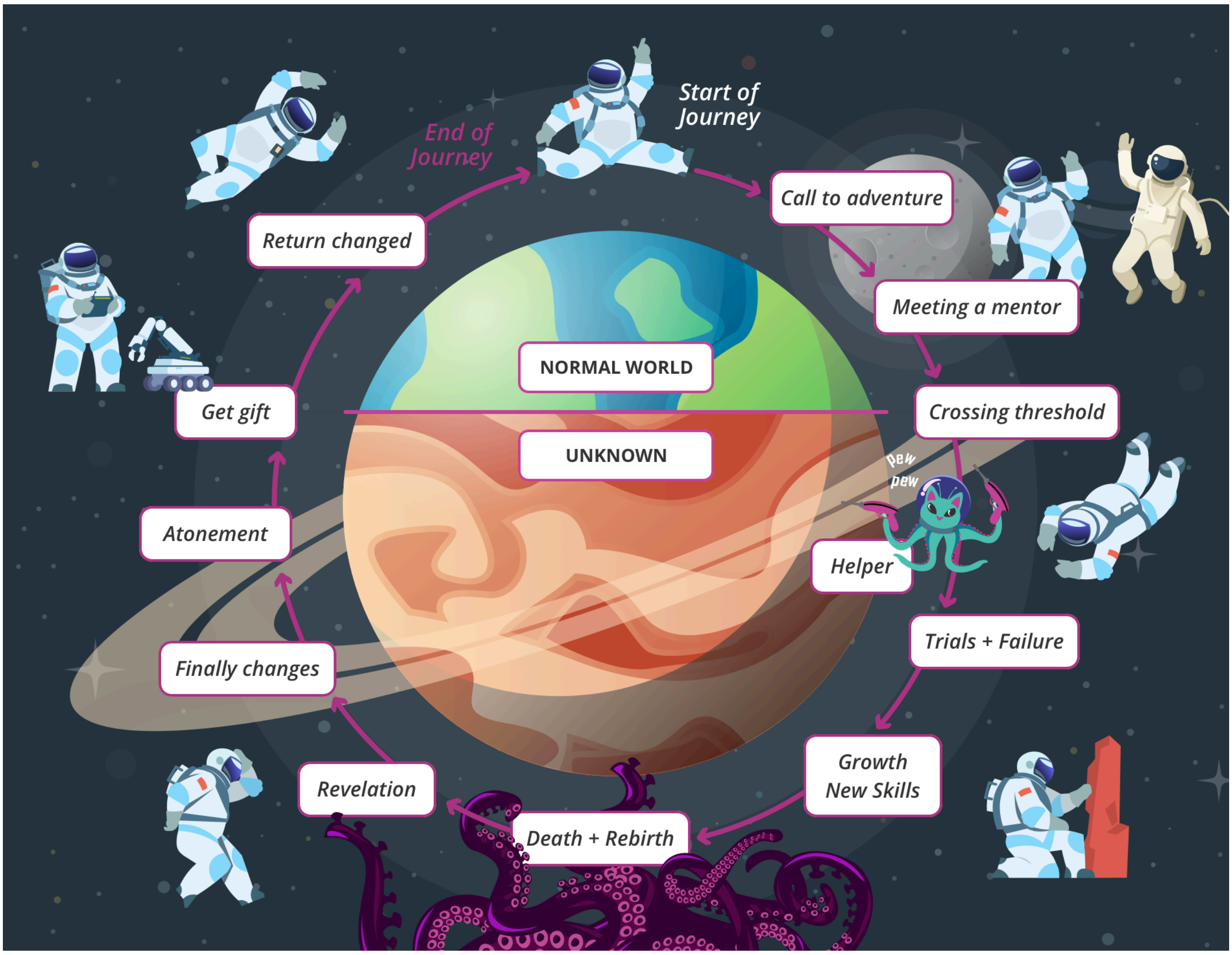
2. Entry



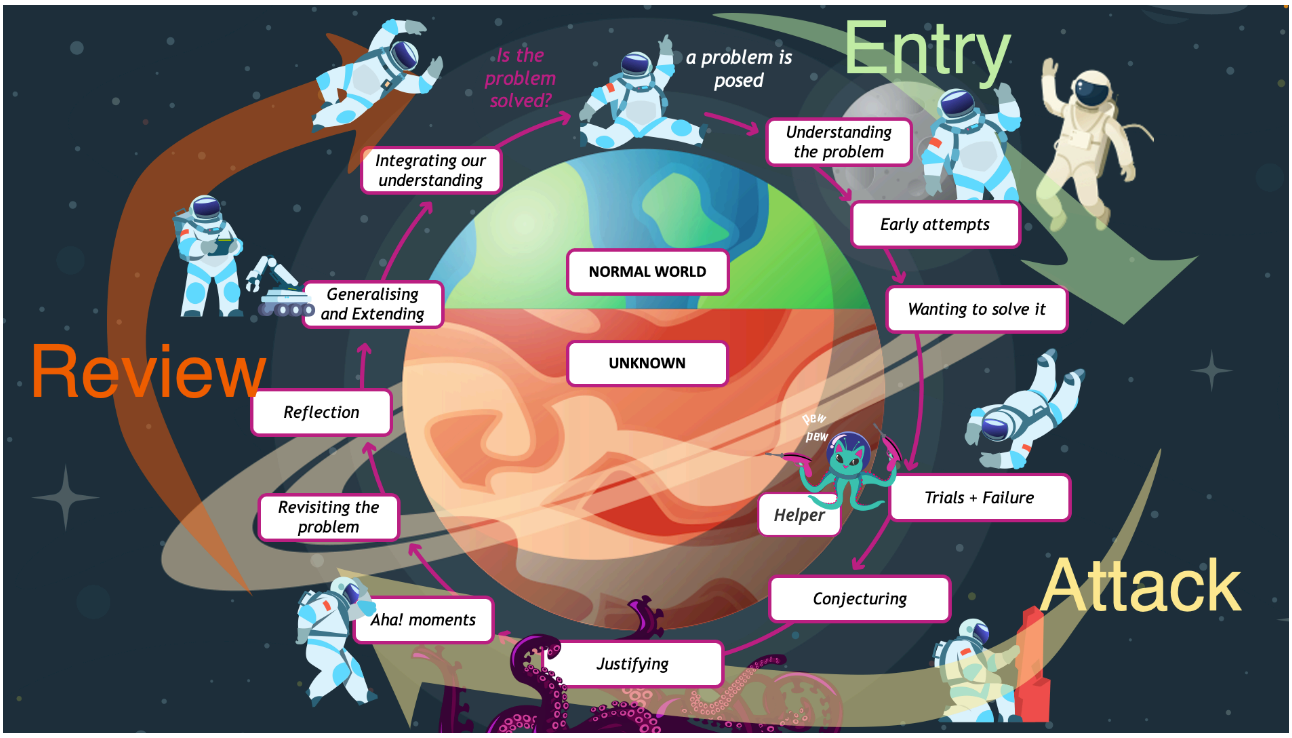
Mathematical problems can usually be approached a number of ways, and require more than just an answer.

Broadly speaking, mathematical problem solving involves the application of logical and mathematical methods to **non-routine** or open problems.



The Hero's Journey

A number of formal frameworks have been identified for undertaking a problem solving task, sometimes with 9 steps, sometimes with 5, however we follow Mason, Burton and Stacey (2010) in reducing them to three phases: Entry, Attack and Review. In terms of our Hero's Journey analogy, the breakup looks something like this.



Entry: taking up the call

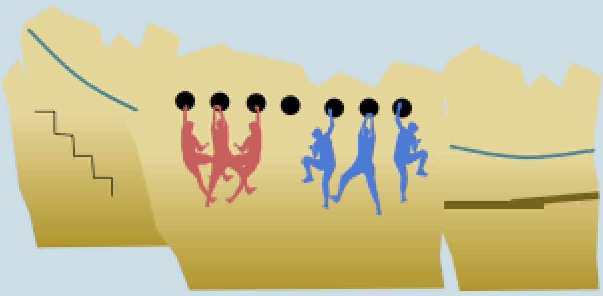
In the entry stage, we usually try to identify the important features of the problem, the information we require, how we interpret the problem, whether the problem is similar to something we have seen before. The video below explores the idea a little more.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=77#oembed-1

[Transcript](#)

Let's practice using the entry phase.

The Huashan problem



In China there is a mountain called Huashan that has a plank walk on the side of a cliff.

There is one section where there are only 7 foot and hand holes (each pair can hold just one person). There are three people trying to go from left to right, and three trying to go from right to left. One person can jump across someone going the opposite direction or move across one space if there is a free hand hole. What is the minimum number of moves that can be made to so that the blue climbers occupy the three spaces on the left and the red climbers occupy the three spaces on the right?

Without trying to solve the problem, have a go at the entry phase for the problem below (don't try to solve it – just see if you can answer the "I know…", "I want…", and the "What Can I Introduce?" aspects).

This problem is based on Leapfrogs from Mason, Burton and Stacey (2010, p 62).

Did you get stuck in starting? Remember that a great way to start is to specialise. Write down your thoughts so far under the headings:

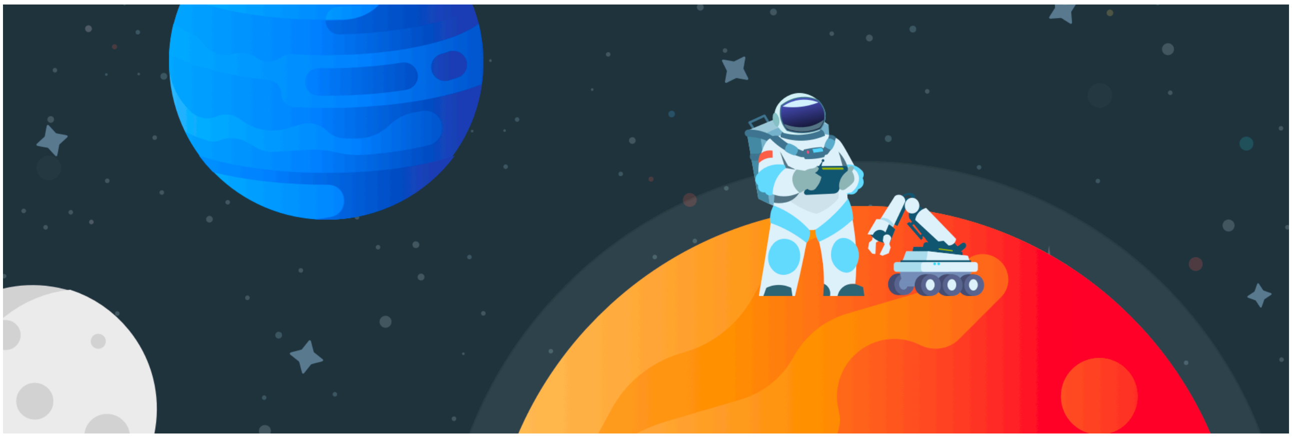
- I know…
- I want…
- What can I introduce?

Now you can either spend some time trying to solve it, or watch this video and see if your entry was similar or different.

References

Mason, J., Burton, L. and Stacey, K., 2010. Thinking Mathematically Second Edition. *England: Pearson Education Limited*.

3. Attack



The attack stage involves our attempts to solve the problem and build a model, or to determine a method for solving it.

It often helps to test some initial hypotheses or use some example values so that we can better understand the underlying mechanics of the problem and whether or not we are missing information (this might have also been established in the Entry stage).

Attack strategies

An interactive H5P element has been excluded from this version of the text. You can view it online here:
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The attack phase constitutes our attempts and working to solve the problem. It's often tempting to just jump straight to this phase without doing enough work in the entry and sometimes this can pay off – especially if it's not a 'real' problem (or if we're really lucky). But if you've spent time solving a problem before you are probably aware that this stage involves much more than just doing some sums and seeing what you come up with.

Specialising and Generalising

There are two sets of processes surrounding the attack that can lead us toward a final solution. As we begin, we have the process of specialising. This usually means choosing a special case. We might try randomly at first, for example, with the handshake problem, we can just draw 5 people and count the handshakes, then 3 people and see if there's any connection. With the Huashan cliff-face, we might first see what happens when we make three moves, or we might try a simpler version where there's only 2 people on each side.

This is fine for getting a handle of the problem and how the mathematics works, but we might then start being **systematic**. In the case of handshakes, we can try with one person, two people, three people in order and look for the pattern.

For some problems, we might also be systematic to eliminate certain cases. With the cliff-face, we only have a limited number of moves, individuals moving across or individuals jumping, we can list the moves and work out which ones lead to dead-ends.

However then eventually we might also start specialising '**artfully**' – what is meant here is that we try specific test cases that will provide specific information. This type of specialising usually comes when we have a conjecture that we want to test out.

For example, we might have a rule for the number of handshakes, and we want to see if it works for 1 people or 0 people, or even 100 people.

What we're usually working towards is a general rule, so while it might be good to know how many handshakes there are for 20 people, we'd much rather a general result that will always work. We might even **generalise** further to the case of both left and right hand-shakes, or to include self-shakes – extending the problem parameters beyond to make our finding as useful as possible.

Conjecturing and Justifying

The process of conjecturing and justifying can be articulated in terms of different levels of scepticism.

We start with a **conjecture**, for example,

Is the number of handshakes equal to two times the number of people?

We check the conjecture on different cases – either building our case or determining that it is false. We then might try to find specific cases that refute it, for example when we try odd cases like 1 person or even 0 people to see if the solution we found works.

We want to get a sense of **why** it is right, or modify it so that it is right.

One way of thinking about it is to first convince yourself, then imagine you are trying to convince a friend, and finally, imagine trying to convince an enemy – well perhaps not an enemy, but someone who likes to be annoying and disagree with you and will exploit any inconsistency in your thinking!

With that in mind, let's take a look at some of the main strategies you can employ to solve problems.

Strategy	Strengths	Weaknesses
Trial and error (sometimes referred to as exhaustive search or brute force attack)	Good if there are only a finite number of possibilities. Good if the "error" can inform the next guess (e.g. higher/lower)	Time consuming. Not very elegant. Can't usually adapt such models to new situations.
Simplify the problem	Can help to understand the dynamics of the situation We might identify patterns Solution to the simplified problem can give a rough idea of what the solution/ model should be in the more difficult case	Elements we remove or simplify may be the crucial elements of the actual problem
Draw a diagram/graph, act out the situation	Representing the problem visually or acting it out can help make sense of the problem Graphs can provide important information that is easier to interpret	Solutions may not always be accurate Models that involve the drawing of a diagram could be time-consuming or difficult to construct
Use a table / Look for a pattern	Organising the information in a readable and organised format can help us see patterns	Some mathematical patterns may not become apparent Sometimes tables can overwhelm us with information
Work backwards	Sometimes results in a sequential process with no 'dead ends' or need to startover	Thinking backwards is sometimes conceptually more difficult
Develop algebraic or simulation models	Usually adaptable to new situations Can generate results quickly	If the model is too complex, can be difficult to explain to others Getting in the habit of solving everything algebraically can blind us to simpler and more elegant solutions

Considerations during the Attack phase

The attack phase can be frustrating or rewarding and is often both. As you work through a problem, it's important to recognise that you will go back and forth between the entry, attack and generalising phases often until you are confident you can justify your solution.

Feeling lost is a normal part of the process. Don't worry, it certainly does not mean you're not good enough and is no reflection on your intelligence or ability. Trust us, even people with PhDs in mathematics can feel lost when trying to solve problems using mathematical reasoning.

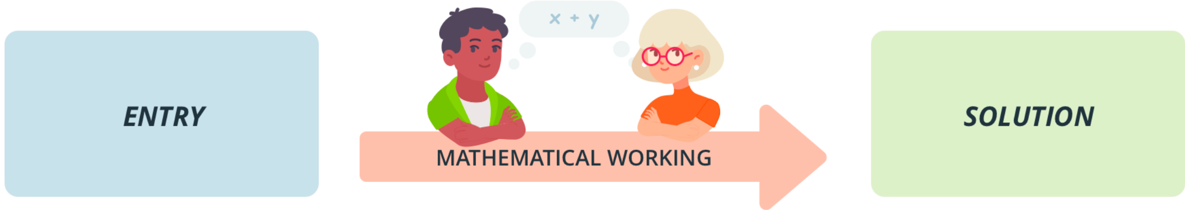
As you practice problem solving, you will develop your intuition around which strategy might work in a certain situation and how far or long to persevere with a method before you consider it a dead end.

Here are some things to keep in mind while you're working through the problems listed in the next chapter.

Textbook answers don't give the whole story

It's a good thing to learn from – and employ the knowledge you've gained from – traditional textbooks that focus on mathematical methods. The more methods you know, the more tools you have at your disposal. However, we need to keep in mind that the process of solving real world problems is often quite different to the process of solving textbook problems.

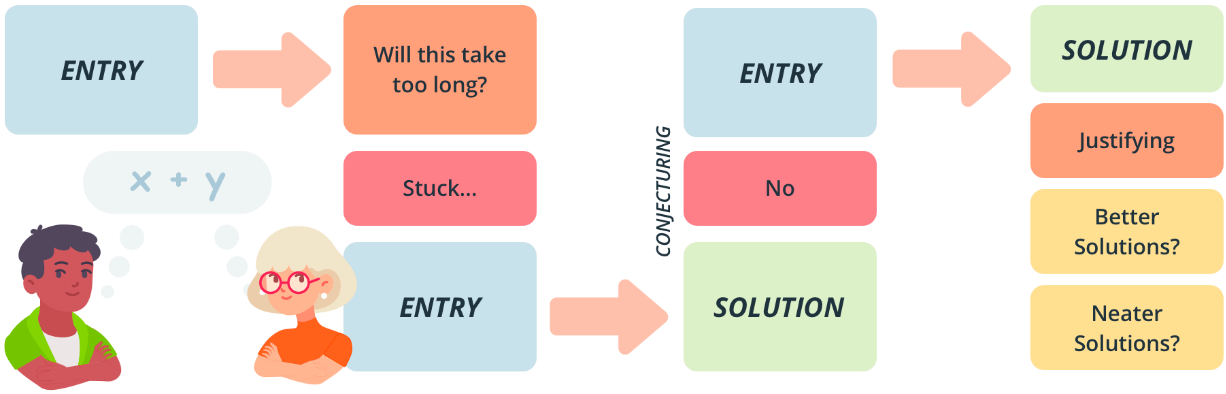
It's a pity that most example questions and answers we're given in some textbooks (even to problem solving problems) don't realistically model the process of entry and attack. We might build up the idea that the attack phase simply involves working away with our chosen method, after which we reach the solution and the problem is solved. If we get a different answer to the back of the book, all we need to do is look back over our working.



Problems that can be solved using mathematical reasoning in the real world are rarely like this and if it is, it's probably more of a routine problem than a problem solving problem.

An attack usually involves multiple attempts, returning to the entry, trying again, re-treading old ground. We might start working it out in one way, but then realise that either this won't work or will take too long! We start to find patterns and we make conjectures, but then sometimes those conjectures turn out to be false and need reworking – then even once we've reached a solution, we may need to try even more methods in order to justify the conclusion we reached.

In the end, we might arrive at a good looking, aesthetically pleasing solution, and it's this solution that we present – often deceptively much simpler than the difficult process we embarked upon.



So the attack phase involves trying out ideas, looking for patterns, asking ourselves why those patterns are occurring, determining whether or not they can be expressed with a neat and concise rule, making conjectures, confirming the conjectures, perhaps changing our point of view, ensuring that we're not missing some piece of information, or even investigating why certain approaches did not work. And it is in this phase that we are most likely to find ourselves stuck.

Getting stuck (and unstuck)

It's ok to be stuck. Whether it's a mathematics problem, or even a computer game, we've all experienced the state of being stuck. But being stuck is okay – in fact being stuck is referred to in *Thinking Mathematically* as "an honourable state" (Mason, Burton & Stacey, 2010, p. 45).

There are some good things about being stuck.

Don't give up just yet...

Firstly, it means we've engaged with the problem and we haven't given up on it yet – because if we've abandoned the problem then we're no longer stuck, and if we don't care about the solution then we're probably not feeling that stuck either. However, if we give up too soon we might have missed an opportunity to learn from that state.

It is only from being stuck and accepting it is possible to learn how to get unstuck that we can overcome problems. In fact, some of the greatest results in mathematics have arisen after either a collective or individual period of being stuck.

So, if you are stuck – acknowledge it, make a note of it in fact. But even more importantly, much later on, take the opportunity to reflect on how you became 'unstuck'. Although we all might do it, being negative is not productive – it's simply not true that the state of being stuck is some reflection on our intelligence or even ability at mathematics.

Try to stay productive

Accepting the state of being stuck does not mean persevering with the same method we've been trying or just randomly leaping to another. Continuing with a bad strategy won't help.

We can examine the reasons we are stuck: are we making certain assumptions? At this point, we might even think back to the times when we were stuck and why. Sometimes lessons and insight can come from unexpected places.

Should I cheat?

Of course, we sometimes have the option to "cheat" too. The internet is fantastic for this. And, indeed, this might be preferable to spending days on something that we're not ready to solve. But even if you do choose to 'cheat', make sure you use the experience as a *learning opportunity*.

It shouldn't just be about how another person chose to solve the problem, but rather what was it that stopped us from seeing this path to the solution:

- Were we under a false assumption?
- Was there mathematics involved that we hadn't considered (or was it actually beyond what we could have expected from ourselves)?
- Now that we have a solution – can we see whether our methods could eventually get there? Can we work using another approach?

Try something else

In terms of different attack approaches, it's good to be open to different methods.

Once we become confident using a powerful method for problem solving such as algebra, we can become over-reliant on it and this sometimes blinds us to more simple, elegant and intuitive solutions.

References

Mason, J., Burton, L. and Stacey, K., 2010. Thinking Mathematically Secon Edition. *England: Pearson Education Limited*.

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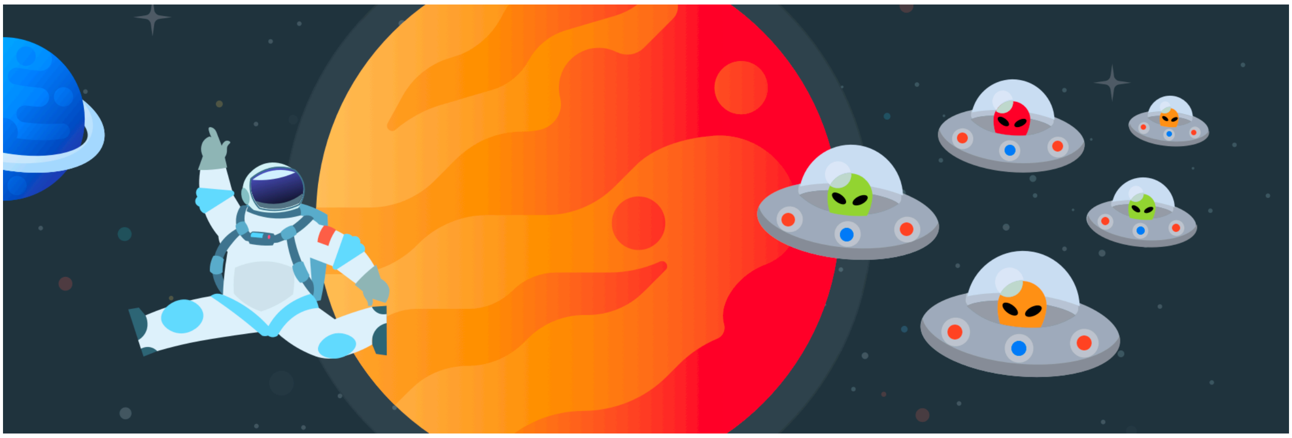
4. Problems!



Some problems for you

Use your entry and attack strategies to see if you can solve any of the following problems. Don't feel obliged to spend more than 15 minutes on any one of them.

Married Couples

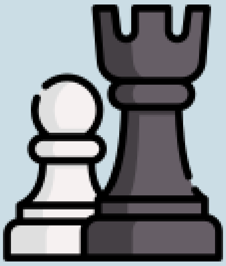
Four mathematicians and their partners went shopping. They found that altogether, they were poorer by $5000. Dr. Hilbert's partner had spent $100, Prof. Fermat's partner had spent $200, Euler's better half had spent $300 and Prof. Galois wife spent $400. As for the mathematicians, Amy spent five times as much as her partner, Ajay four times as much as his, Danny three times as much as his and Idris twice as much as his. What is the last name of each of the mathematicians?

Tips

An interactive H5P element has been excluded from this version of the text. You can view it

online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=163#h5p-41

Chessboard



How many squares are there on a chessboard?

Tips

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Bees

Queen bees share a fascinating relationship with the other bees in the hive.

Male bees hatch from eggs that have not been fertilized, meaning that their mother is a Queen bee and they do not have a father. Female bees, on the other hand, have both a mother and a father.

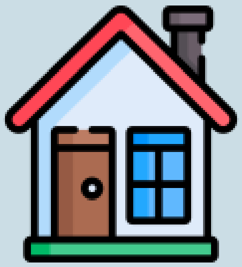
If you trace a male bee's family tree back for 12 generations, how many other bees will be part of that family tree? How many of these bees will be male? Exclude siblings, aunts and uncles, cousins and second cousings, etc.

(problem inspired by Mason, Burton and Stacey, 2010)

Tips

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=163#h5p-43

Painting



Alma can paint a house in 3 hours, Marvin can paint a house in 5 hours. How long does it take them to paint a house if they work together?

Tips

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=163#h5p-44

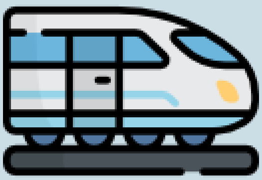
Shopkeeper

You walk into a store and the shopkeeper says "Every time you buy something for $10 dollars, I'll double the amount you have left." You take the shopkeeper up on his offer 3 times and end up with 0 dollars. How much did you start with?

Tips

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=163#h5p-45

Passing trains



At 9pm, Albert leaves Melbourne travelling at a constant speed of 100km/h towards Maryborough. At 10pm, Jean-Paul travels from Maryborough and will arrive in Melbourne at midnight. If the distance is 220km, where will they cross?

Tips

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=163#h5p-46

References

Mason, J., Burton, L. and Stacey, K., 2010. Thinking Mathematically Second Edition. *England: Pearson Education Limited*.

5. Review: the hero's return

The review phase is where the real learning takes place.

The review phase is where we reflect on the whole process.

In some cases, our review might reveal that more work is to be done and we are forced to return to the entry and attack phases. At a base level, we need to check that our solution is consistent and reasonable given the problem posed, however we also might need to see whether or not the method will be valid for different values or under different conditions. We may find that our model leads to unreliable results, requiring us to start the process over again or go back a few steps. We also might think about whether we could go about similar problems in the future in a more efficient or simpler way.

Our thoughts at the review stage should be to:

Check

- Calculations
- Arguments to ensure that computations are appropriate
- Consequences of conclusions to see if they are reasonable
- That the resolution fits the question

Reflect

- On key ideas and moments
- On implications of **conjectures** and arguments
- On your resolutions and **justifications**: can it be made clearer

Extend

- The result to a wider context by generalizing
- By seeking a new path to the resolution
- By altering some of the constraints

This stage provides the opportunity to really understand the problem and know that our solution seems reasonable. Beyond that, the review phase also provides us with an opportunity to think about alternative strategies and approaches along with their relative "elegance".

Checking that you've done things correctly is a fine strategy in an exam or on completion of a project – but remember that this is your main chance to become a 'better' problem solver! Reflect on why you found something difficult – what was the missing knowledge or false assumption that got in the way? Or was this a problem that is hard to solve? Now that you have obtained an answer – can you see a pattern or method that makes sense of it? Reflect on why you found something easy – is there an extension to the problem? Would a slight change make it harder?

In short, the review reflects back on the attack and broadly asks:

- does my solution make sense?
- do I have proof that the answer is correct?
- could I find a simpler solution?
- did I get all possible information out of my attack?
- can I explain my reasoning?

This stage is often ignored with answers given and checked as 'right' or 'wrong'.

Stop and think

Think about your problem solving experiences when you try to explain something or teach someone.

The process of problem solving, conjecturing, and coming up with models allows us to continuously ***challenge*** and then ***revise*** or reject our ways of thinking about a situation. The 'model' one develops can then be used to reinterpret the situation. Some might argue that this is what is at the heart of learning mathematics. Don't forget how important 'the struggle' is in helping you to really learn and absorb something – the journey and the Aha! moments are what reinforce the learning that we return home with, in the process becoming better and wiser problem solvers.

The following video recaps the concepts of generalising and specialising.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=179#oembed-1

Transcript

A last note on problem solving

In solving problems we need to reflect back on the process and make sure that our solution is reasonable, however it can also help to reflect on the entire process of problem solving in order to cultivate modelling abilities in learners. There are a number of mathematical strategies that students can be introduced to for use in problem solving, however in order to help build their mathematical intuition, it is better not to be prescriptive in which strategies to use for which situations. Through more and more practice at creating models to solve non-routine problems, students can become aware of the underlying mathematical similarities that exist between problems and cultivate this intuition.

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PART II

FUNDAMENTAL CONCEPTS IN MATHEMATICAL REASONING

This section is going to be a little different to the others in the book. Throughout most of the book, we're going to pose you a series of problems, and we're confident that by working through those problems you'll develop strong methodologies that can be generalised to support mathematical reasoning and investigation in other areas. That's the whole point of the book, after all!

In this section, we're going to be a fair bit more explicit about teaching methods. This is really just to give you additional support and help bring you up to speed quickly with some of the essential skills surrounding solving mathematical problems. Some of it, like order of operations and describing relationships with variables, is mostly just reminding you of mathematical conventions to ensure that we're interpreting the numbers in the same way.

We'll look at things like the order of operations, the properties of addition and multiplication and how to work with fractions. But the reason we're looking at all of these really boils down to helping you develop the ability to use perhaps the single most powerful problem-solving methods we have: algebra.

Learning outcome

By the end of the section, you should be able to express proposed solutions to mathematical problems using algebra.

6. Algebra

Throughout this section, we'll explore some of the fundamentals that can empower people to reason mathematically. We start with algebra.

When most of us think about algebra, we think about using letters or symbols such as x or a to represent variable quantities. When it comes to problem solving this is actually a pretty good place to start.

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Transcript

If we'd like to define it a little more tightly, algebraic **expressions** (eg, $4x - 7$) are a sequence of operations (like adding, multiplying, etc) performed on quantities and **variables** or **unknowns.** For example, the expression

$$3x + 12$$

means 'start with a number, multiply it by 3 and then add 12,' where x represents the number we started with. You can choose any number to be x and perform the sums on it, which is partly why we call it a variable.

So if $x = 2$, then we have

$$\begin{aligned}&(3 \times 2) + 12\\&= 6 + 12\\&= 18\end{aligned}$$

Isolating an unknown

In the scenarios above, we pick any number that makes sense (it could be almost at random) and then perform a series of operations on it. But one of the other things algebra can be really powerful for is figuring out an unknown number. Simple examples of this might be figuring out how much oil is left in a 44-gallon drum after a fraction of it has been removed, how much weight an iron girder can hold based on its density or guesstimating the amount of jelly beans in a jar based on its dimensions and the dimensions of a jelly bean.

That is to say, algebra is often used when we don't really know what x (or $a, b, c...$) equals, but we'd like to find out.

Typically in these situations, we'll have an **equation**, which differs to an 'expression' in that an equals sign will be involved. This can impose restrictions on some of the variables, which in turn can provide clues about what values they can take. For example, in the above, if we start out with an equation,

3x + 12 = 21

Substituting 2 like we did before won't allow both sides of the equality to stay the same. We then need to think about what *has* happened to the variable we're trying to isolate and then reverse it. This is sometimes referred to as 'solving for x'. It can be tricky at first but with a little practice it becomes second nature.

The main thing to remember when we do this is that anything we do, we have to do on both sides of the equation. This ensures both sides of the equation remain equal. A basic example will help: if $3 \times 2 = 6$, then $3 \times 2 + 2 = 6 + 2$ or 8.

Let's get classic and write it out like you'd find in a textbook, just to give an example (this is the 'foundations' section after all).

$\frac{6x}{5} - 2 = 8$ – find the value of x

I'll set this out in a table so we can solve step-by-step:

Here (right) is our equation which we need to evaluate or solve.	$\frac{6x}{5} - 2 = 8$
Now our task is to think about what is happening to x and how we can reverse it. We'll start with the -2 first. There is no rule for where to start, but over time you will build your intuition for what makes the task most straight forward. Let's reverse the minus 2 by adding 2. Remember we need to do it to both sides of the equations.	$\frac{6x}{5} - 2 + 2 = 8 + 2$ $\frac{6x}{5} = 10$
Next let's deal with the fact that x is being divided by 5. To reverse this, we need to multiply both sides by 5.	$\frac{6x}{5} \times 5 = 10 \times 5$ $6x = 50$
And now there is only one operation left to deal with – let's reverse the times 6 by dividing both sides by 6.	$\frac{6x}{6} = \frac{50}{6}$ $x = \frac{50}{6}$
Finally, we want to 'simplify' this a little. In this case it could be written in a few different ways to make it a little more intuitive for a reader.	$x = \frac{25}{3}$ or $x = 8\frac{1}{3}$ or $x = 8.333...$

Reverse order of operations

In the table above, we've used a method for solving often referred to as the reverse order of operations. Essentially, we've taken the normal order of operations and reversed it. Most of the time when we deal with an equation, we should leave addition and subtraction to the end – in this case we've reversed that order and 'undone' the subtraction first.

Performing the steps in this order is usually the best way to start when we solve equations – but we actually have some choice and could even use multiplication to remove the 5 from the left-hand side in the first step, as long as we make sure we apply the operation correctly to both sides.

In this case, we'd need to remember that the 5 multiplies both terms: the $\frac{6x}{5}$ as well as the -2. It would give us

$$\frac{5(6x)}{5} = 5 \times 8$$
$$6x - 10 = 40$$

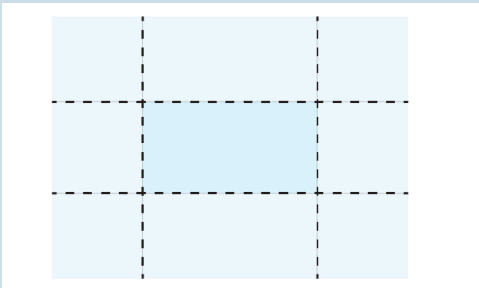
And then we can proceed from here, ending up at the same result for x.

Problem

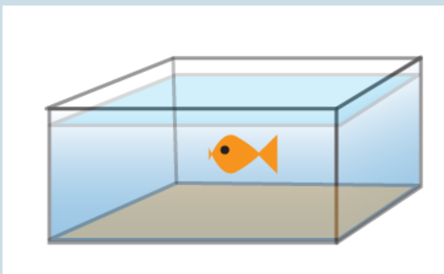
Let's have a look at a classic routine problem to demonstrate how this works. You'll probably remember that volume is calculated by multiplying length, width and height.

Fish tank factory

A small factory manufactures glass fish tanks. Each tank is made from a sheet of glass which is cut into sections. The factory always cuts squares from the edge, as it reuses these glass squares in other products that it manufactures. This is shown below.



The edges are cut at a 45-degree angle and joined together flush using silicone. The left over squares are then removed.



The sheet of glass that it is cut from is 1.4 meters long and 1.2 meter wide.

- The smallest tank that they manufacture is 30cm wide and 50 cm long. What is the maximum height of the fish tank, measured from the bottom to the top of the outside of the glass?
- If a tank must be at least 20 cm deep, what are the dimensions of the longest and widest fish tank the factory could manufacture? Would it be the same tank?
- Can you find an algebraic expression for the length of any rectangular tank manufactured using the same method and materials? What about its width or height?

This problem is loosely based on a problem from Stewart (2018).

A note on symbols for multiplication and division

It can take students a little bit of getting used to, but the $\times$ symbol is rarely used when doing algebra. This is to keep things concise and avoid confusion with letters (how could you tell what 4xx means?). If we are going to multiply numbers, like 4×3 then we could either include the multiplication symbol or use brackets. If we use brackets, we assume that a quantity outside a bracket is multiplied by a quantity within a bracket – eg both $4 \times 3 = 12$ and $4(3) = 12$ represent the same operation.

It's also unusual to see the symbol for division $\div$ in an algebraic expression. We usually use fractions such as $\frac{3}{5}$ and if we're trying to set things out on the same line we can also just use $3/5$. It's also common to see these written as decimals such as 0.6 in certain contexts or situations.

Implied brackets

We need to be careful with 'implied' or invisible brackets. When we have an expression like $\frac{7x+2}{3-x}$ this means that we have a number 'x' which we multiply by 7, add 2, and then we take the result and divide it by three minus the same number 'x.' So there are implied brackets both above and below the separating line (the vinculum): brackets around the $7x+2$ and around the $3-x$.

Common words, symbols and expressions

Here are some common ways to represent mathematical operations.

Common words	Common symbols or expressions
Addition, sum, add, total, plus	$x+y$
Subtraction, subtract, minus, less, difference, decrease, take away, deduct	$x-y$
Multiplication, multiply, product, by, times, lots of	$xy, x(y), x \times y, x \cdot y$
Division, divide, quotient, goes into	$\frac{x}{y}, x \div y, x/y$

Painters and shopkeepers

Head back to Chapter 4 to take a look at the problems.

- For the *Painters* problem, are you able to determine expressions and come up with a general solution for when Alma paints a house in A hours and Marvin paints a house in B hours?
- How can the *Shopkeeper* problem be set up algebraically?

Quiz

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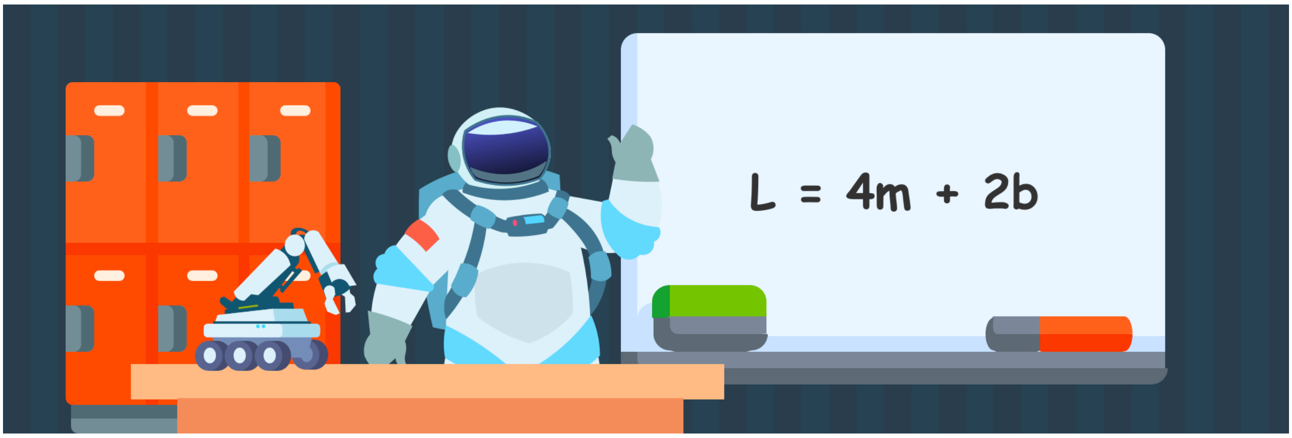
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=22#h5p-39

References

Stewart (2018). *Single variable calculus : concepts and contexts* (Fourth enhanced edition.). Cengage.

7. Properties of addition and multiplication



Algebra can be very powerful, but you need to play by the rules.

The following table lists a number of important properties of addition and multiplication that we can rely on in algebra. It should be remembered we can take these properties as granted only where a, b, and c are real numbers, i.e. any quantity that can be represented on the number line.

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Transcript

There are algebras for sets, vectors, matrices and other mathematical "objects", however these rules might not apply (since we will only be dealing with real numbers in this book, you can assume that each of these properties holds). Most of these you will already be familiar with, but giving them a name can help us think about them.

Algebraic expression	Name of the property (or rule)
$(a+b)+c=a+(b+c)$	associative law for addition
$(ab)c=a(bc)$	associative law for multiplication
$(a+b)c=ac+bc$	(right) distributive law
$c(a+b)=ca+cb$	(left) distributive law
$a+b=b+a$	commutative law of addition
$ab=ba$	commutative law for multiplication
$a+0=a$	additive identity: the effect of adding zero
$a\times 1=a$	multiplicative identity: the effect of multiplying by one.

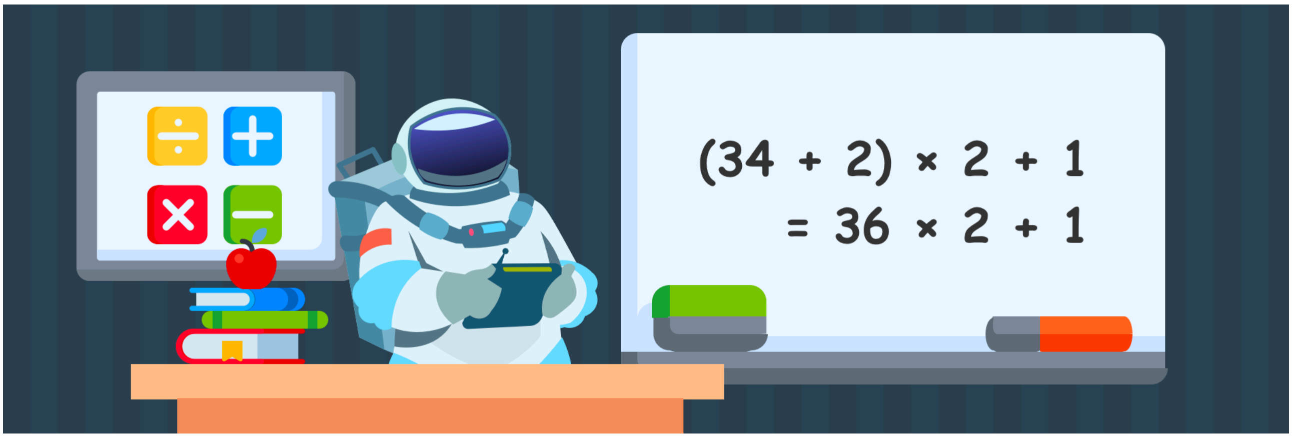
Quick check

Get a feel for each of the properties in the table above by substituting the following sets of numbers for (a, b, c), respectively.

- $2, 4, 6$
- $-3, -8, 2$
- $0, -1, 3$

As well as these properties, we also need to keep in mind the order of operations in which in which we perform each step when evaluating an expression. For example, with the associative law we just looked at, brackets were used to ensure that the 2 left or 2 right numbers are combined together first. Rather than put brackets everywhere, a number of conventions have been reached that dictate which steps of an equation we complete first. This is not because any of the operations are more important than any others, it is simply so that there is only one correct way of reading an equation and thus we can avoid ambiguity.

8. Order of operations



The order of operations when working with numbers is sometimes taught using the BODMAS, BEDMAS or BOMDAS.

These acrostics help us remember what to do an in what order.

Equally ordered operations

Acrostics like BEDMAS, BODMAS or BOMDAS are sometimes avoided by teachers today because they can raise confusion for equally ordered operations.

How should we approach a string of operations such as $6 \div 3 \times 2 \div 8 \times 6$?

We should simply do them in the order they are presented (the 'answer' to the sum above is, then, 3). The same goes for addition and subtraction.

Let's look through each of these operations. Download a copy of the Order of operations if it helps.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=197#oembed-1

Transcript

When we evaluate any expression, we should proceed in the following order:

Brackets

Brackets – evaluate anything in brackets (using the order of operations if there are complicated expressions within the brackets) and then it's easiest to replace the bracketed part with the result when we rewrite the expression.

For example,

$$\begin{aligned}&(34+2)\times 2+1\\ &=36\times 2+1\end{aligned}$$

Powers/indices

Even though powers can sometimes be thought of as repeated multiplication, we apply this operation before multiplying or adding. If some value is raised to a power which is an expression, the power should be evaluated first. So,

$$\begin{aligned}&4\times 3^{2+1}\\ &=4\times 3^3\\ &=4\times 27\end{aligned}$$

Multiplication/Division

Multiplications and divisions are also performed before addition and subtraction, completed in the order they are given.

So you should treat

$$3+4\times 6-1\div 4$$

as

$$3+(4\times 6)-(1\div 4)$$

which becomes

$$3+24-0.25$$

Addition / Subtraction

Finally, we evaluate the addition and subtraction (in the order they are given). For example, suppose we wanted to evaluate

3 – (9 – 2) × 3^2+ 10 ÷ 5 × 2

We would first evaluate the **brackets**

$$3-7\times 3^2+10\div 5\times 2$$

Next, the **powers:**

$$3-7\times 9+10\div 5\times 2$$

Then, **multiplication/division**

$$\begin{aligned}&3-63+2\times 2\\ &=3-63+4\end{aligned}$$

Finally, **addition/subtraction**

$$\begin{aligned}&-60+4\\ &=-56\end{aligned}$$

Algebra

Be careful of implied brackets, especially when working with algebraic statements.

In the case of algebraic statements, we sometimes leave out some of the brackets, multiplication and division symbols. Instead of writing division symbols, division is usually indicated by writing the expressions using the vinculum (i.e. similar to fractions). Multiplication is assumed if there is a number or variable next to another variable or expression and sometimes brackets will be left out if we are performing divisions or using powers raised to an expression.

Here are some examples to illustrate what we mean

The expression below	Would usually be written as
$3 \times x \times y$	$3xy$
$4 \times x \div y + 1$	$\dfrac{4x}{y} + 1$
$(3 + x) \div (2 + x)$	$\dfrac{3+x}{2+x}$
$y \times \dfrac{3+a}{2-x} + 4^{(x-1)}$	$\dfrac{y(3+a)}{2-x} + 4^{x-1}$

If these examples seem complex to you, please don't be intimidated. As you move through your journey with mathematical reasoning, you may or may not decide to employ such complex algebra. It's in your power to decide if and when you're ready for it.

These are just the conventions that are usually used. In some cases, we may decide to leave the brackets or symbols in if we are worried about the expressions being misinterpreted.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
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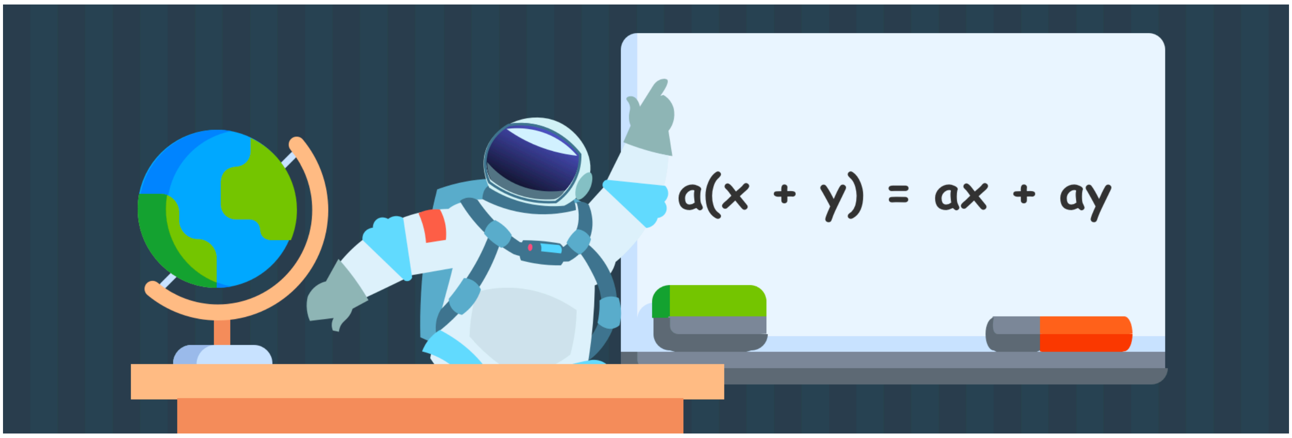
9. Describing relationships with variables



When it comes to problem solving, being able to express a relationship mathematically is a powerful thing.

However, the most difficult thing is to convert the information you have at hand to an algebraic expression. The key thing is to remember that the symbols (usually letters of the alphabet) represent quantities and behave the same way as numbers do.

For example, if we are traveling at $100km/h$, the distance we cover is $100km$ for each hour that passes. After 1 hour, we've travelled 100km, after 2 hours, we've travelled 200km and so on. So overall, if we were travelling at a steady speed, we could write $100h$ to represent the number of kilometres past, where h is the number of hours. We could also use $100t$, where t represents the number of hours instead of h – what we actually call the variable won't change the mathematics.

If we have more than one variable, we need to use a different symbol for it. For example, there is a classic problem about looking at a number of cows and chickens and counting the legs. For each cow, there are 4 legs and for each chicken, there are 2 legs.

It wouldn't work to represent this as $4L + 2L$ because that would mean for each L, we would have 4 and 2 of something. So we need to think carefully about how the equations work.

If there were 3 cows and 5 chickens, the number of legs would be $(3 \times 4) + (5 \times 2) = 22$, i.e. there are 3 cows, so we multiply 3 by 4 legs, and there are 5 chickens, so we multiply 5 by 2 legs. If there were a different number of cows and different number of chickens, the 3 and the 5 would change, but the number of legs per animal would stay the same.

So let's use an m to represent the number of cows (because they say moo) and b to represent the number of chickens (since they have beaks). This means the number of legs will be $(m \times 4) + (b \times 2)$.

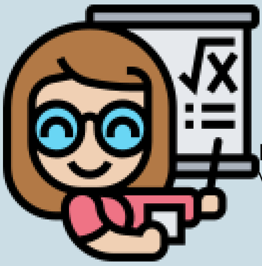
Since we're using algebra and we want to avoid confusing multiplication symbols for x's, we rewrite this without them, and we also put the numbers before the variables by convention. So we will have $4m + 2b$, and we could go a step further by saying that this is equal to the number of legs (L).

This would give us our final expression

$$L = 4m + 2b$$

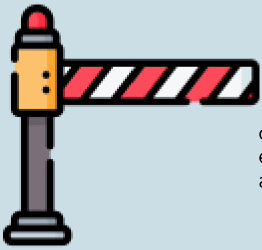
It's usually a good idea at this point to ensure you can state an equation in plain language in a simple sentence, just to make sure everything makes sense. In the case of the equation above, we could say "the number of all the animals' legs put together is equal to four times the number of cows plus two times the number of chickens in the paddock." It doesn't matter if the sentence sounds a little odd as long as it remains logical.

Algebraic expressions



Jean charges $40 for the first hour of tutoring and then $25 for each additional hour in given week. Create a model she can use to work out her income each week.

Kim can pay $10 for an eco-cup and then save 0.30c each time she buys a standard coffee. Write a model that can show her total savings based on the number of coffees she buys using the eco-cup.



The distance between Chengdu and Didong is 250km but it is necessary to stop on the way for a total of 30 minutes to get through toll-gates etc. Write an expression that can calculate the total time depending on how fast you travel on average.

There are a number of kangaroos, emus and dingoes in a paddock. Write an expression for the number of legs you can see in total based on the number of each animal.

You may have found that it is not always easy to write expressions from the information given in a situation. Sometimes this is the hardest part of problem solving – identifying which are important for your expression, what their coefficients (the number multiplying them) should be and how to combine them.

Common expressions

Some examples of common words, how they are used, and the algebraic expressions they might lead to are shown in the following table.

Word	Operation	Example	As an equation
Sum	Addition	The sum of Adam and Brenda's earning is $500	$a + b = \$500$ where a and b are Adam and Brenda's individual earnings
Difference	Subtraction	The difference between Adam and Brenda's age was 7 years	$a - b = 7$ where Adam is 7 years older than Brenda
Product	Multiplication	The expected loss is the product of the average payout and the probability of failure	$l = af$ where l is the expected loss, a is the average payout and f is the probability of failure
Times	Multiplication	She earns three times my salary	$h = 3m$ where h is her salary, and m is my salary
Less than	Subtraction	He predicted it would take 5 years less than expected.	$Y = E - 5$ where Y is the numbers of years he predicted it would take, and E was the expected time
Total	Addition	The total time taken to watch the three movies was 9 hours.	$t = m_1 + m_2 + m_3 = 9$ where t is the total time and m_1, m_2 and m_3 are the running times of the 3 movies respectively
More than	Addition	This drink costs $20 more than the cheapest one.	$d = c + 20$ where d is the cost of this drink and c is the cost of the cheapest one
Times more than	Percentage / multiplication	This drink costs 2 times more than the cheapest one.	$d = 2c$ where D is the cost of this drink, C is the cost of the cheapest drink – not that the drink costs 2 times as much would mean double, but 2 times more implies it's 2 times the amount added on to the price of the cheapest drink.

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10. Using expressions to solve problems

Mathematical models can be very useful.

If we need to solve a given problem a repeated number of times, if we need to get at information that is not directly accessible, and also if we need to justify our methods, we can often turn to mathematical models.

See if you can use what you've learned over the past few pages to help Mae out.

Badge machine

Mae loves making badges to give away at her band's gigs.

She can send away online, where it will cost $2.50 per badge plus $7 postage, or she could look to hire a badge machine for $300 and pay 20 cents per badge for the parts.

Which option is better?

Once you've spent some time on the problem, you can take a look at one possible solution in the video below.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=1236#oembed-1

[Transcript]

Rearranging

This application of inverse operations to both sides of the equation, which was so central to this method of helping Mae with her problem, is at the heart of *rearranging* to solve equations. We first encountered this on the previous page. Remember, we're not technically *moving* things to the other side, but it seems like we are because when we use an inverse operation it will cancel out to the identity on one side, but appear on the other side. We just need to make sure that we are careful, that we adhere to the order of operations, and remember the properties of arithmetic.

Let's practice rearranging equations.

Check your understanding

For each of the following, find the value of the variable.

- $4x + 1 = 91$
- $3a - 21 = 29$
- $\frac{4}{5}x = 20 + 2$

In each of these cases, we had a linear expression on one side and a value on the other. In some cases, we may want to find out when two linear expressions are equal to one another (as was the case in the badges question).

11. Working with fractions

Many of us are out of practice with using fractions.

For many people, working with fractions goes into the category of 'things I used to know how to do when I was 11 years old.' We're often taught fractions by being asked to memorise a series of rules, so it's understandable if you can't remember every one of them. If you're already confident with addition, subtraction and multiplication using fractions feel free to skip this chapter.

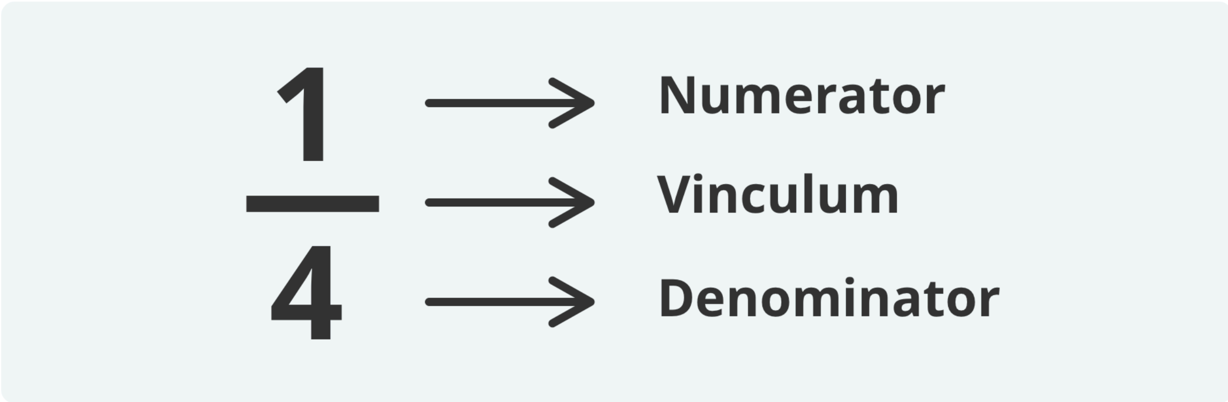
Although fractions can seem less intuitive than other ways of representing the same numbers (which is easier to get a sense of: $\frac{1163}{125}$ or 9.304?), they can be very useful. If you find it confusing to work with fractions, this in turn will make it hard to work with algebra – so it's worth spending some time to get the hang of them.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=220#oembed-1

Transcript

What are fractions?

If you associate fractions with equally sized slices of cake, you're probably not alone. You are probably familiar with terms like numerator and denominator. The vinculum is the little line that separates the top of the fraction from the bottom.



Fractions as parts of a single item

Most of us are taught, accurately, that fractions are equal parts of a whole. If you take a further step back, a fraction is 'simply' a division. In other words, one quarter is equal to one divided by four.

$$1 \div 4 = \frac{1}{4}$$

This holds true no matter what the numbers in the numerator or denominator are. If the number on top is bigger than the number on the bottom then you have more than one whole thing. That is, if you had $\frac{3}{2}$ slices of pizza, you would have three halves of a pizza or one and a half pizzas.

Fractions represent divisions, but it's also important to be aware that fractions are numbers. They can be placed on a number line, like any other number.

The division view of fractions

Another way to think about fractions is 'the division view.' In this way of understanding fractions, we start with whole items in the numerator and divide these whole items by the number in the denominator.

In this view, $\frac{3}{4}$ would mean we took three whole things and then divided all of them by four.

Common ways of representing quotients

A quotient is just another word for 'division' – so fractions, decimals, ratios and percentages are all quotients.

In most cases, it's fine to convert between fractions, decimals and percentages. Just keep in mind they all represent the same number.

$$\frac{1}{5} = 1 : 5 = 0.2 = 20\%$$

One fifth, 1:5, 0.2 and 20% are all equivalent figures and the way you choose to express them really comes down to the context you're in.

If you are converting, note that in many cases decimals are imperfect approximations of fractions. For example, if you put $1 \div 3$ into a calculator, it will give you an imperfect approximation (0.3333...) whereas $\frac{1}{3}$ is precise.

Addition and subtraction

As students get older, they're introduced to some more complex problems. These are the ones that can confuse many adults. Let's look at perhaps the quintessential fractional problem that we encounter in our youths.

Have a go at the problem below. Don't spend more than 15 minutes on it, though.

Pizza slices

A whole pizza is sitting on the table.

You take $\frac{1}{3}$ of it, and your younger sister takes $\frac{1}{4}$ of it.

How much pizza is left?

This is an example of a problem that is routine for someone who is already familiar with the method, but can be quite tricky if we've forgotten how to do it.

You may have some intuition about this to begin with which we could use to set some limits – perhaps you might 'guess' that there's somewhere between one half and one quarter of the pizza remaining.

If you struggled with this problem we recommend that you have a look online for some methods on adding and subtracting fractions – there are many good resources out there including Adding Fractions and Subtracting Fractions.

Multiplication

The next problem is another potentially routine one that essentially asks us to convert one fraction ($\frac{km}{km/h}$) into another. This can be a tricky problem, and it might help if you know how to multiply and divide fractions. Have a go.

How long to the turn-off?

You're driving down a highway.

At the moment you pass a street sign telling you that your turn-off is 15 km away, you are travelling at 100 km per hour.

If you keep travelling at the same speed, how many minutes will it take you to reach the turn-off?

One solution is presented below.

Entry

Let's think about the entry phase for this problem. There are many ways that we could approach it.

As an example, I might draw the highway, figure out how far you would travel every minute, and then add them up until I get to 15 km (the picture might end up looking like a number line).

The main thing to contend with here – and this will probably be the key problem to overcome in any method – is one of conversion. How would I figure out how far I travel each minute, or how long it takes to travel a kilometre? In other words, how do I get from kilometres per hour to kilometres (or just metres) per minute?

Attack

I'm going to start by simplifying the problem to something I already know the answer to – that way I can easily test to see if my methodology works. Instead of travelling 15 kilometres, I'll figure out how long it will take to travel 10 km at a steady speed of 100 km/h. The answer to this question is that it will take 6 minutes – here's how I know:

- If I travelled one hour at 100 km/h, I would travel 100 kilometres. Similarly if I travelled half an hour, I would have travelled half the distance or 50 kilometres. Therefore, if I travel one tenth of an hour, I will travel one tenth of one hundred kilometres – which is 10 km.
- One tenth of an hour is $\frac{60\ minutes}{10} = 6\ minutes$. So it seems logical to say it will take 6 minutes to travel 10 km.

So far so good – and to be honest at this point I can easily answer the question. If it takes 10 minutes to travel 6 kilometres, then it should take half that time to travel half that distance at the same speed. This means it would take 3 minutes to travel 5 kilometres. I want to travel 15 kilometres, so it should take 6+3=9 minutes to get there.

Generalising

That's good, I have an answer that I'm fairly confident about. But that's less interesting to me at this point than the main challenge – can I come up with a method that will work no matter what distance is being travelled?

Let's see if I can use fractions and algebra to get to the same result. To recap on my strategy, I want to

1. Find out what fraction of an hour it would take to travel a certain distance at a steady speed. That is, $\frac{distance}{speed}$.
2. I then need to convert this fraction of an hour into minutes. 1 hour is 60 minutes, so I just need to multiply the fraction from step one by 60.

To set this out algebraically, I could use the letter t to represent time and write the expression something like this:

$$t = \frac{distance}{speed} \times 60\ min$$

I am confident about my answers for the time it would take to travel 10 and 15 kilometres, so I'll use these figures to test the equation out.

Let's start with 10 kilometres – I'm confident the answer is 6 minutes.

\begin{align} t &= \frac{distance}{speed} \times 60 \ min \\ t &= \frac{10}{100} \times 60 \ min \\ t &= \frac{1}{10} \times 60 \ min \\ t &= \frac{60 \ min}{10} \\ t &= 6 \ min \end{align}

Looks good. Let's try 15 kilometres. It should take 9 minutes to travel 15 kilometres at 100 km/h.

\begin{align} t &= \frac{distance}{speed} \times 60 \ min \\ t &= \frac{15}{100} \times 60 \ min \\ t &= \frac{3}{20} \times 60 \ min \\ t &= \frac{3(60 \ min)}{20} \\ t &= \frac{180 \ min}{20} \\ t & = 9 \ min \end{align}

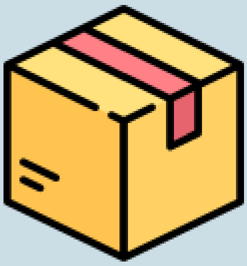
And, just to be sure, let's try to figure this method to figure out how long it takes to travel 150 kilometres – at 100 kh/h, I'm expecting the result to be 90 minutes (or 1.5 hours).

$$t = \frac{distance}{speed} \times 60\ min$$
$$t = \frac{150}{100} \times 60\ min$$
$$t = \frac{3}{2} \times 60\ min$$
$$t = \frac{3(60min)}{2}$$
$$t = 90\ min$$

Putting it all together

There are many real world problems where it can help to incorporate the addition/subtraction and multiplication/division of fractions. Have a go at the one below.

Flat packs



You're working at a factory warehouse.

You receive products in parts, which are then placed into flat packs and delivered to customers who assemble them.

A new product comes in three parts, A, B and C. All three parts need to be delivered to the customer who ordered the item.

The cardboard shipping boxes you use can each hold up to 10 kilograms of weight, which is enough for 3 lots of part A, or 4 lots of part B, or 6 lots of part C. If you receive an order for 10 items, what is the least number of boxes needed to ship them?

PART III

ALGORITHMS

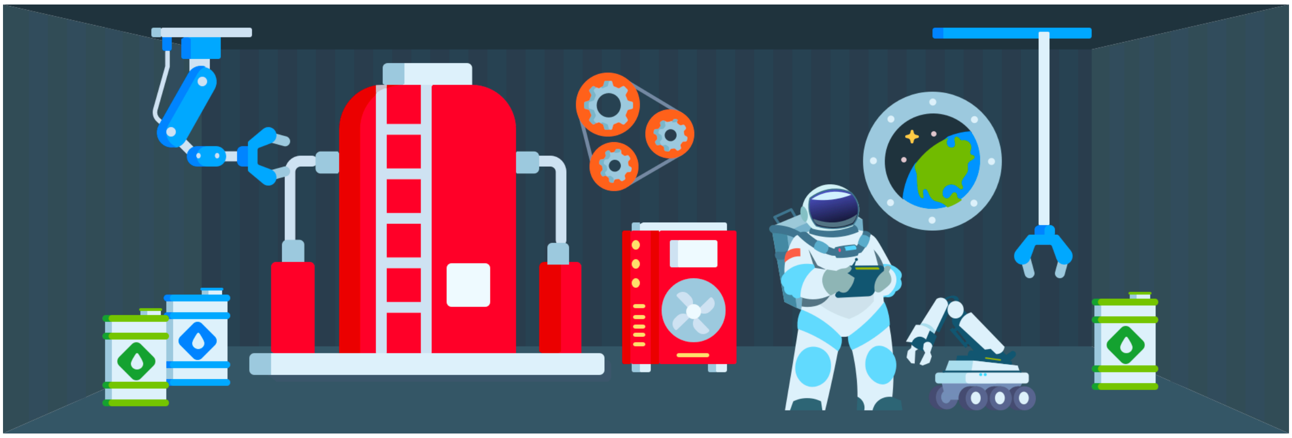
For this topic, we'll be looking at programming and how it can be used to aid mathematical understanding and problem solving. Being able to program gives us access to potential solutions and patterns at a far greater scale than doing things by hand, however it itself is an art that requires as much creativity and insight as problem solving does. Writing algorithms requires similar skills to problem-solving – although planning becomes especially important. Your understanding of algebra and order of operations will also be very important in problems requiring the coding of calculations.

Learning outcomes

By the end of the section, you should be able to

- Describe the characteristics of an algorithm
- Use the programming application Scratch to create algorithms in order to solve problems

12. What is an algorithm?



An algorithm is just a list of instructions for a computer.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=24#oembed-1

Transcript

As the video says, what makes them different from instructions from Ikea is that we can use them to help solve general problems (like 'what are the prime numbers under a million?') as well as specific ones. Take a look at this video that gives a good definition.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=24#oembed-2

Stop and think

Suppose you have the set of numbers: 3, 6, 10, 20, 7, 49, 2, 5, 20; what would be a set of instructions to determine the highest number? Which operations would the computer need to understand and what else would be required?

Have a look at the video below for one solution once you've spent a few minutes thinking about it.

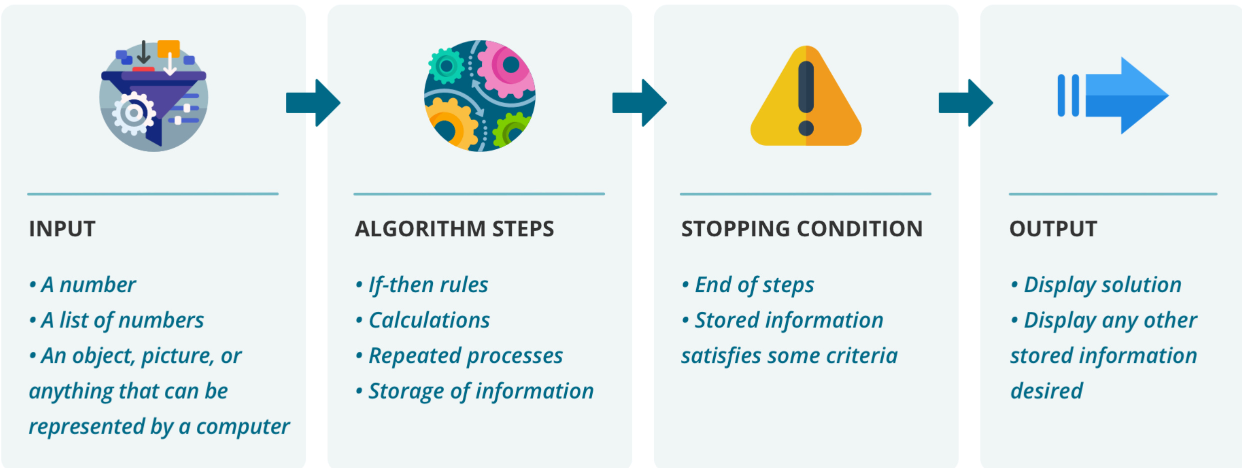
One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=24#oembed-3

Transcript

Input through to output

If the algorithm we have is designed to solve a problem, then often this solution will depend on the particular case or situation, and the algorithm will take this information into account. We can refer to this broadly as the **input** to the algorithm. So in the case of determining primeness, we would need to know which number the algorithm will be checking, while determining the largest number will depend on a list of numbers, and counting the number of people in the room will depend on the room we're looking at.

We also need to know what the solution is, and so usually this is the output. Like the input, the output could be one number or a set of numbers, or something even more general. Of course, if we think of the sequential process a computer might go through to give us this output, we usually need a stopping condition. In some cases, the stopping condition will just be when the algorithm finishes the list of instructions, however sometimes the actual number of steps the algorithm takes may be unknown.



Download accessible version of algorithm diagram

The Collatz Conjecture

The Collatz conjecture is another great example of a simply stated problem that is very difficult to solve. Let's suppose we have a positive integer to start with. Then, if the number is even, we divide it by 2, otherwise, we multiply it by 3 and add 1. We keep doing this, and the conjecture states that no matter what number we start with, we will end up with 1.

7 is not even, so we multiply by 3 which gives us 21 and add 1. So now we have 22.

22 is even, so we divide it by 2, to give 11.

11 is not even so we multiply it by 3 to get 33, and add 1. We now have 34.

Does it feel like this will end up approaching 1? Use the Scratch program below to explore the conjecture – when you enter a number, it will give you the sequence of values until it reaches 1. It will also plot the points on a graph (although it's only set up to plot numbers under 150 or so).

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=24#h5p-2

The Scratch program implements the algorithm from the Collatz conjecture and allows us to specialise and look for patterns without having to do all the calculations. What are some smaller conjectures you can make based on the numbers you've tried? This is considered such an interesting problem, archetypical to the field of computer science, that it adorns the exterior of the Warsaw University Library (among texts in different languages, science and statistics formulae and sheet music).

Your first algorithm

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=24#oembed-4

Can you remember how to tell if a number is prime or not? Have a look at Simon's video above, and this method if you've forgotten.

See if you can write clear "algorithmic" instructions to check whether an input, **n** is prime. You don't need to write the specific code, but make sure you have a logical order of steps and include a consideration about which steps would require special operations that would need to be understood by a computer.

For example, to check divisibility, we can quickly check divisibility ourselves by using a calculator: if **n** divided by **3** gives a **whole number result**, then we know **n** is divisible by **3** – if implementing this idea on a computer, we'd need some way for the computer to know that **n divided by 3** is **a whole number**, or just some operation that is able to say that **n is divisible by 3** is a **true** or **false** statement.

See image rights and reuse information

13. Scratch

Scratch will help us become familiar with the structure and capabilities of algorithms.

Scratch is a program developed by the Lifelong Kindergarten Group at the MIT Media Lab (at the Massachusetts Institute of Technology). It is **fun** and **free** and **easy** to get started.

Software setup

Go to Scratch using your internet browser. You can always undertake projects here as long as you have an internet connection, or you can also download an offline version. Once you have everything working, click on the “Create” button.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=274#h5p-3

Scratch basics

We’ll now use Scratch to introduce some of the key components of programming. In addition to the following instructional videos, it’s also a great idea to do some exploring yourself of what each of the different types of programming blocks do. The video will talk you through the basics.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=274#oembed-1

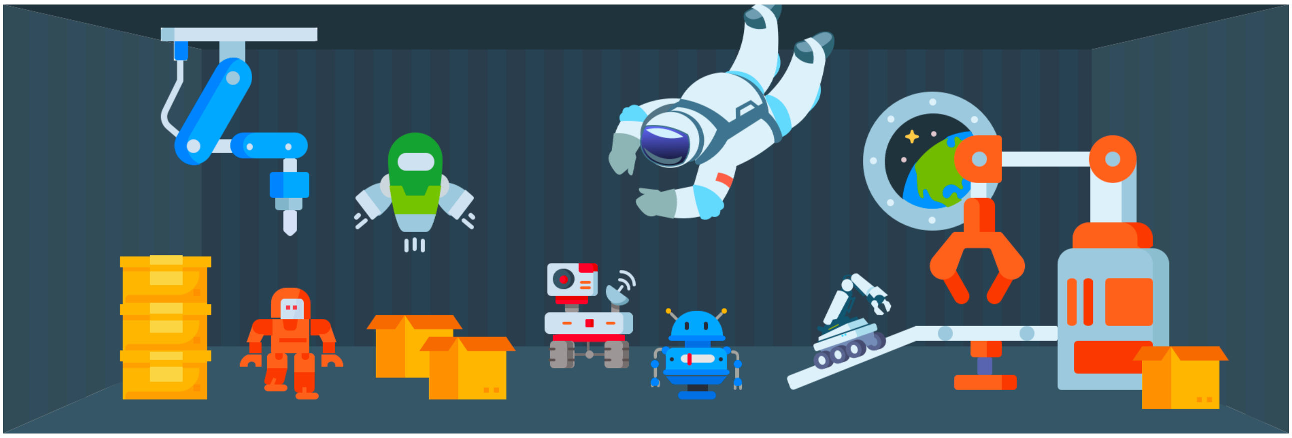
[Transcript]

Play around with Scratch

Have a play around with the scripts we've looked at as well as some of the other ones available to see what they do.

See [image rights and reuse information]

14. Conditional instructions (if-then rules)



If x then y

We've seen that algorithms often will use these logical rules with the structure below:

If	Then
P	Q

In these cases, P is some proposition or statement that can be evaluated as TRUE or FALSE and Q is the instructions to take. For example, with a primes algorithm, we might have the instruction:

If	Then
a is not divisible by b	move on to the next divisor.

The statement 'a is not divisible by b' will either be true or false. If the statement 'a is not divisible by b' is FALSE, then 'a is divisible by b' is TRUE. Double negatives like this can be useful, but make sure you keep track of them so that you don't get confused later on.

If-then

For example, suppose we are checking if the number 251 is prime.

1. First, we check whether 251 is not divisible by 2. Clearly this is TRUE (as $\frac{250}{2} = 125$ with a remainder of 1. Therefore 251 is truly not divisible by two).
2. Since the condition 'a is not divisible by b' has been met, the algorithm will move onto the 'then' part which will be the next number to check (3).

The problem is that we need to tell it how to do this as well, which we'll look at when we come to variables.

If-then-else

If the statement is **not** true, the action after the "then" won't be executed and the algorithm will just act on the next line of script. In some cases, we might have two potential actions depending on whether the first statement is true or false. For example, with the primes, we might have the following:

If	**Then**	**Else**
a is divisible by b	STOP. The number is not prime.	Move to the next divisor

In this case, we didn't use the negative form of the statement P, but we could also have done it the other way around.

If	**Then**	**Else**
a is not divisible by b	Move to the next divisor	STOP. The number is not prime.

Let's have a look at some examples of how to use the if-then rules in Scratch.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=276#oembed-1

Transcript

Draw an octogon

Can you create a program in Scratch that:

1. make the cat trace out an octogon (you'll need to use pen blocks)
2. asks the user how many sides of a polygon they'd like drawn, and then draws it
3. Asks for an input, and, if it's divisible by 5, it does so. If not, it doubles the number.

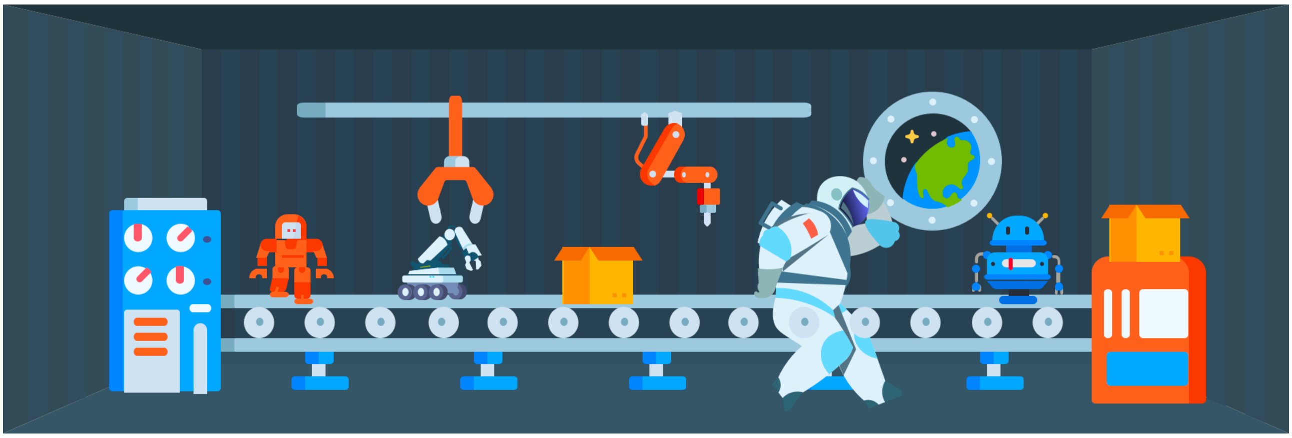
This is a short quiz that will help you get to know some of the terminology we will use when talking about Scratch and coding as well as some of the features of Scratch.

An interactive H5P element has been excluded from this version of the text. You can view it online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=276#h5p-4

How are you enjoying Scratch?

Remember that to improve your coding, the more you practice and explore, the better you'll get at debugging and ensuring that your code does what you want it to do. A web search on how to do something will often be fruitful too (one of the pieces of advice in the What to do if you get stuck section of the chapter on attack).

15. Loops

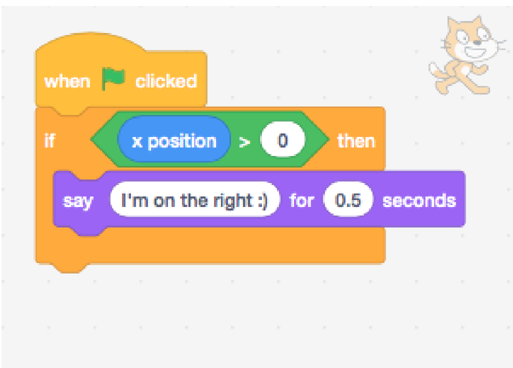


Sometimes the steps of our algorithm need to be repeated until a condition is met.

For example, when we look for primes, we'll keep on checking different numbers for primeness, but the process each time is the same. There are essentially three types of loops we might introduce into an algorithm:

- Repeat a process a given number of times
- Repeat a process indefinitely (or until the process is stopped manually)
- Repeat a process until a condition is met

For example, previously we created a program that determined whether or not the scratch cat was on the right. This only worked when we placed the cat and then executed the code, however if we want the program to keep on checking as we move the cat around, we can put in a loop that will keep on checking until we end the program.



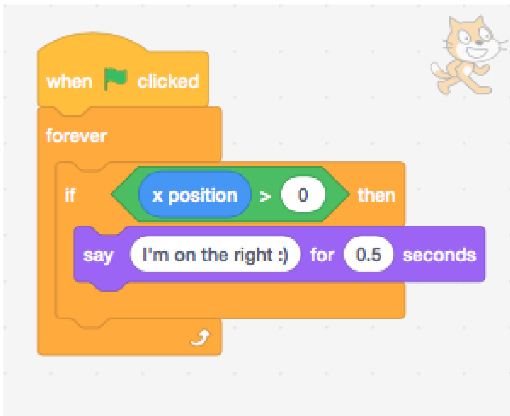


Image: screenshots from Scratch

Let's have a look at some other examples in Scratch

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=283#oembed-1

Transcript

Loops involving if-then statement

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=283#oembed-2

Transcript

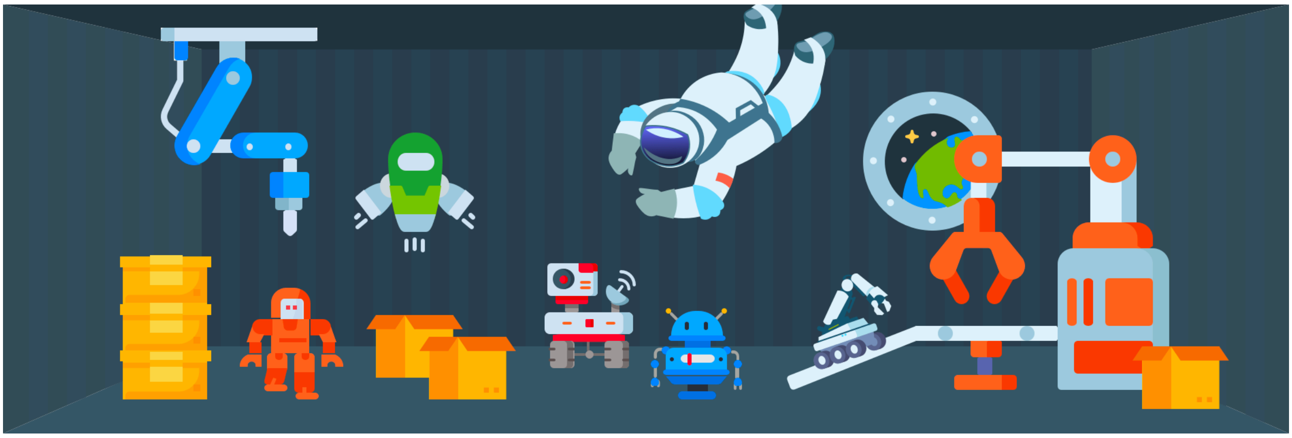
Coding

Create a program in Scratch that

1. includes user input and a loop.
2. includes a loop that uses an if-then statement.

See image rights and reuse information

16. Assigning values to variables and solving problems with Scratch



Sometimes we need variables.

We already saw when we looked at if-then rules that sometimes we want the algorithm to do things depending on inputs or on the values of some variables. In many cases we need to introduce new variables, or keep on updating existing variables in order to solve problems. For example, if we want to check if a number is prime, we will have a loop that checks the divisibility of a number at each stage. But, then we need to update the number we're checking. Whether we are introducing variables, or updating existing ones, this process is referred to as 'variable assignment'.

Different programming languages will have different notation for assignment. For the algorithm that counted people in a room, we had an initial assignment $N \leftarrow$ which was read as 'let N equal 0.' Then, after counting each person there was an updated assignment: $N \leftarrow N + 1$

Or, set N to N + 1. This kind of grammar differs to how we often think or work in mathematics, because we are not usually working in a dynamic environment, and so $n = n + 1$ doesn't make sense.

Let's have a look at some simple loops that involve assigning and updating variables.

Assigning variables in Scratch

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=286#oembed-1

[Transcript]

The counting cat

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=286#oembed-2

[Transcript]

Guessing game

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=286#oembed-3

[Transcript]

Keeping track

When you start making games or algorithms, one of the most difficult things is keeping track of all the variables and all the possibilities. You have to account for each path the algorithm might take, otherwise if a case isn't accounted for, the code might crash or just not run as you expect.

Coding

- Try to design your own simple game. Start with the idea, then map it out as a process and make sure you know how to implement each step (and that it won't take too long) and then have a go at playing it yourself (or show someone else). If you are interested in making small interactive games in Scratch, you can also incorporate additional sprites (like the cat). You can set these sprites up to move themselves and use events keys like "when spacebar clicked" to control your character.
- Try to design your own simple algorithm that solves a problem. This could be a simple calculation that is based on multiple user inputs, or it might be something more difficult.

Debugging

Can you fix this Scratch program so that it works properly? You'll need to click the 'see inside' button.

Other block-code programming languages

There's more out there for you (or your students) than just Scratch. There are a number of other simple programming languages that work similarly (or the same as) scratch. In particular, you can program Jimu robots and Sphero robots using similar commands. There's a version of Scratch on the iPad called "Scratch Junior" that has a simplified set of commands and there's also Logo, which is a programming language that can be used for drawing shapes.

You may wish to explore any of these.

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PART IV

INTERPRETING VISUAL INFORMATION

This section is about interpreting graphs.

In this section we'll look at how spatial understanding applies to interpreting information presented in 2-dimensional graphs, including looking at gradients. This component is based on the teaching resource "Shell Centre – The Language of Functions and Graphs", which, although dating back to the 80s, is fantastic for building intuition about graphs.

Learning outcomes

By the end of the section you should be able to:

- Interpret visual information accurately
- Sketch graphs

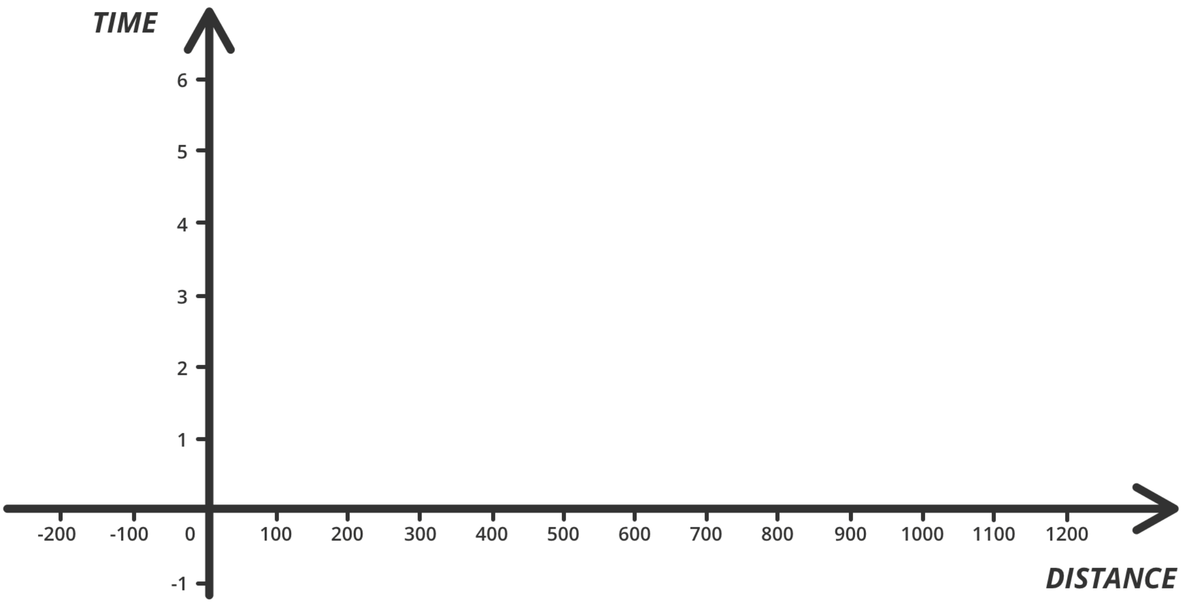
17. Interpreting graphs from points

The Cartesian Plane is a really useful way of visualising the relationship between two variables.

In an earlier chapter of this book (working with fractions), we looked at a basic problem which asks us to determine how long it would take to travel a specified distance at a steady speed (the question was something like 'how long would it take to travel 15 km if your speed was a steady 100 km/h').

One way to think about this question is as a relationship between two **variables**: distance and time.

One way to visualise (or graph) this relationship would be to use a Cartesian plane.

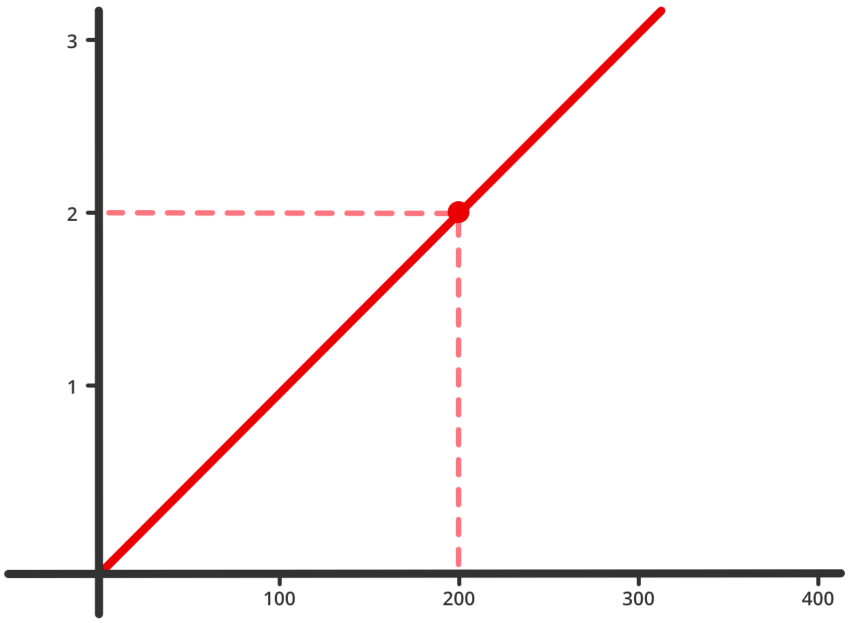


The intersecting lines are conventionally called axes (singular: axis). Each axis represents one of the variables. Time is represented by the vertical axis and the horizontal axis represents distance travelled.

Conventionally, the horizontal axis is called the x axis and represents the dependent variable and the vertical axis is called the y axis and represents the independent variable.

If you have trouble identifying the dependent and independent variable, don't worry about it too much at this stage.

As you can see below, this would be a useful tool for solving problems which asked us about distance and time at a fixed speed. The dot here shows us that it would take 2 hours (y axis) to travel 200 km (x axis).



And, in fact, it could even be used to solve problems about distance and time when the speed also varies. The graph below shows a car coming up to its cruising speed.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=292#h5p-40

Why graphs matter

These kinds of representations are useful for being able to interpret relationships in the data or differences between points that might not be so intuitively apparent in the numeric relationships. Such plots are hence incredibly useful for communicating a great deal of information, however sometimes our intuitive interpretation can be thrown off if we are not used to dealing with the quantities represented on either axis.

Let's start with a few small problems.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=292#h5p-5

How did you go? This video recaps some considerations for these problems.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=292#oembed-1

Transcript

Acknowledgement

Many of the problems shown throughout this chapter have been adapted from The Language of Functions and Graphs (1985) by Shell Center for Mathematical Education (this version modified by Deakin University and licensed under CC BY NC 4.0 with permission from Shell Centre for Mathematical Education)

See image rights and reuse information

18. Are graphs just pictures?

We now consider some situations involving the representation of changes in speed and distance over time.

Having a grasp of these relationships is essential to the topics of calculus and kinematics that students study in the later years of high school, however the intuition should be developed in students much earlier. The key trickiness in each of the following situations is the impression that the visual information provides us with, and how this can confuse us when trying to transform that information into the relationship as represented on a graph.

Speed over time

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=301#h5p-6

Download accessible text version

Once you've arrived at a tentative conclusion, you can see a worked solution using Scratch (YouTube).

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=301#h5p-7

Download accessible text version

Modelling gravity

Build the algorithm yourself from the video above (or see the program's project page and click on the “see inside” button), then explore how changing the initial values of each of the variables affects how the cat jumps.

Acknowledgement

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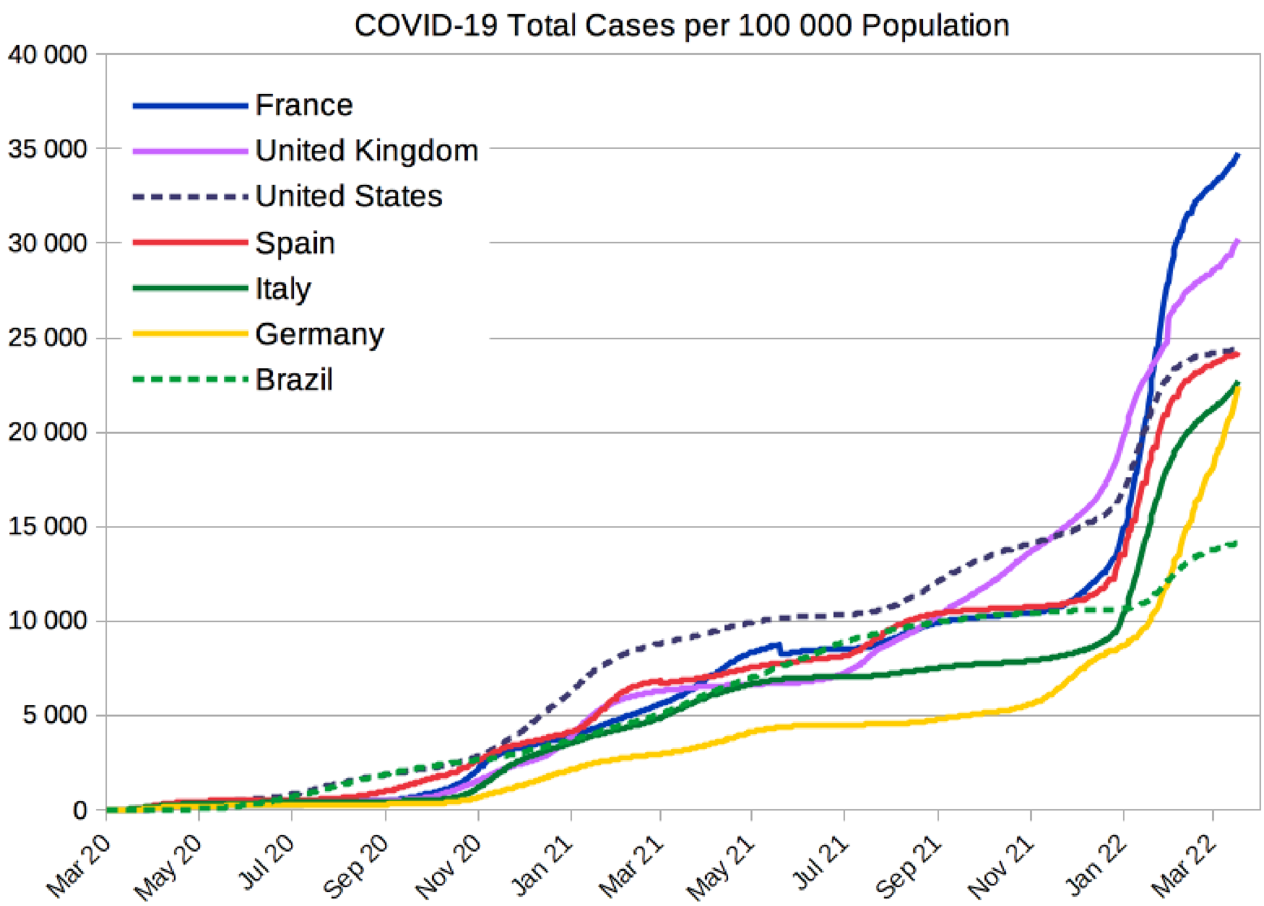
19. Sketching graphs from words

Transforming words to visuals representations can help build intuition.

One way to build intuition about how to read graphs and interpret the visual information presented is to practice transforming relationships, that are expressed in words, into anticipated visual representations.

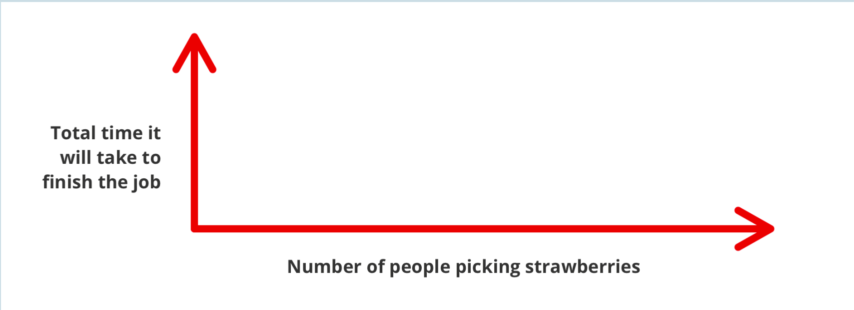


[Traut](), [CC BY-SA 4.0](), via Wikimedia Commons

Here we have a graph of Covid-19 cases per 100,000 people between March 2020 and March 2022 across several nations. You can see how, even with this quite complex graph, just how difficult it would be to convey this same information meaningfully using only words or a data table. Sketching helps!

Let's have a go at sketching our own.

Picking strawberries



Imagine you have a field of strawberries ready for harvest.

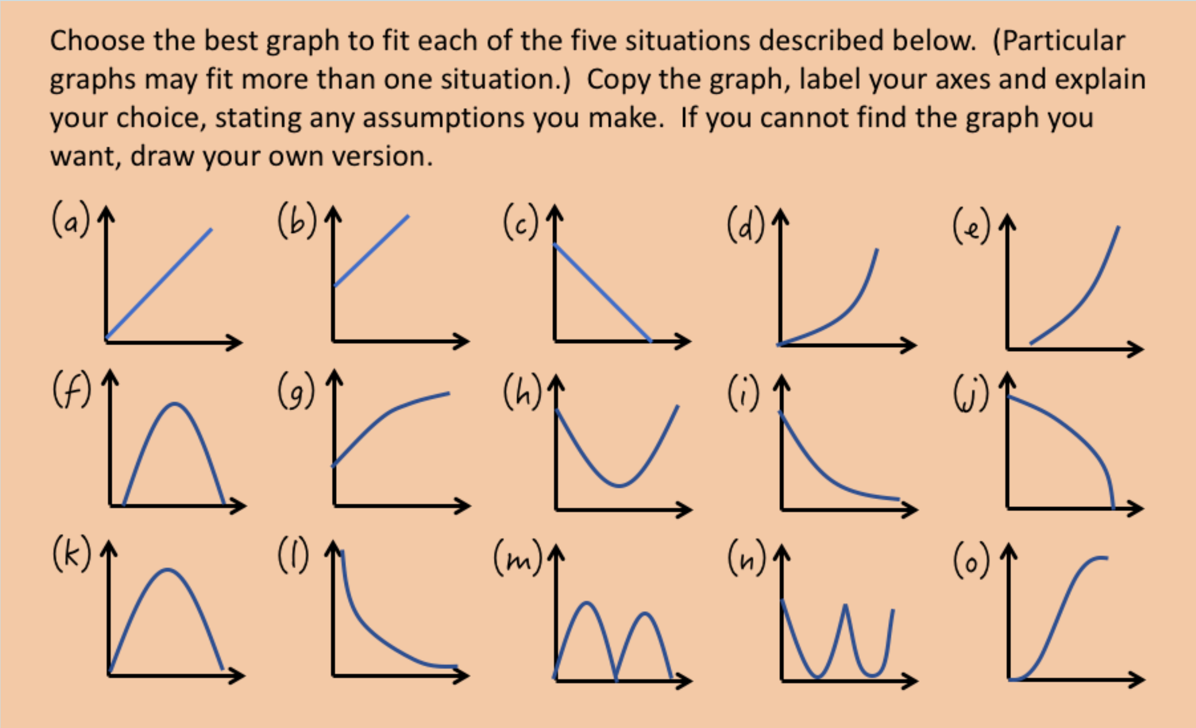
Create a graph of how long it will take to finish harvesting the entire field, depending on how many people you have helping you pick the strawberries.

Tip: Consider using 'Total time it will take to finish the job' as the vertical axis and 'Number of people picking strawberries' on the horizontal axis.

Once you've had a go, you can look at [the following video]() (YouTube).

Problem Solving

The following problems ask you to anticipate how the relationships might appear on a scatterplot from the worded descriptions given. You'll need to refer to the image below as you go.



An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=304#h5p-8

Statistics and society

Visual information is often used to report on sociological trends and issues. These can be powerful tools for investigation, comparison and discovery.

Data exploration

For example, take a look at the online Google Visualisation application. You can use the left hand menu to change the two variables. You could, for example, select:

- On the x-axis: proportion of seats held by women in national parliaments (this is down the bottom under Social Development)

- On the y-axis: primary education, pupils (% female). This is in the Education category.

Take a look at the video if you get stuck at this point.

You can then use the slider to see how the scatterplot changes over time. Try choosing Australia and one other country (you can select them in the check box on the left to help you see) and compare them in two different years. For example:

In 1995, Australia had a higher percentage of female pupils but a lower percentage of women in parliament, then in 2005 they still had a higher percentage of female pupils, but now had a higher percentage of women in parliament.

Acknowledgement

The problems presented under the headings Picking strawberries and Problem Solving have been adapted from The Language of Functions and Graphs (1985) by Shell Center for Mathematical Education (this version modified by Deakin University and licensed under CC BY NC 4.0 with permission from Shell Centre for Mathematical Education).

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20. Sketching graphs from pictures

We now consider transforming visual information about distances between points.

In the following exercises, you'll be looking at how the movement of particles relates to their respective distances to fixed points.

Paths and particles

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=311#h5p-9

Some of these problems are quite challenging. The following Scratch code allows you to investigate some of the different paths traced by particles and how this changes the distance to A and B. Visit the project page and click "see inside".

The video below gives an overview of how the code can be used.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=311#oembed-1

Transcript

Problem solving exploration using Scratch

Use the Scratch program and change the placement of the objects to replicate the scenarios from the paths and pathways activities. Are there any conjectures you can make about the types of paths that will be drawn?

Acknowledgement

The problems presented under the heading Paths and particles have been adapted from The Language of Functions and Graphs (1985) by Shell Center for Mathematical Education (this version modified by Deakin University and licensed under CC BY NC 4.0 with permission from Shell Centre for Mathematical Education)

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21. Looking at gradients

The term gradient refers to the rate at which something is changing.

The gradient of an actual slope refers to how steep it is, which in turn is just the rate that the height changes with respect to movements along the slope. It might be easier to think of steps on a staircase. If the height of the steps is large relative to the width/depth of the step, then it will be steep and hard to climb. If the height of the step is small respect to the width/depth of the step, it should be easier to climb and descend the stairs. Incidentally, international building code regulations state that heights should be between 10.2cm and 17.8 cm, while the depth/width (referred to as 'the tread') should be at least 27.9 cm.

The concept of the gradient is often taught in the context of relationships between distance, velocity and acceleration. If your distance is changing at a high rate with respect to time, this means your velocity is high, while if your velocity is changing at a high rate with respect to time, it means your acceleration is high.

Let's makes some predictions

See if you can predict how the gradients will change in the following situations.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=313#h5p-10

Confirm your conjectures

Once you've worked through these, take a look at the video to see one way you could confirm your conjectures in Scratch.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=313#oembed-1

Transcript

You can visit the project page or to further investigate how the height of the changes with time, depending on the shape of the bottle.

Acknowledgement

The problems presented under the heading Let's make some predictions have been adapted from The Language of Functions and Graphs (1985) by Shell Center for Mathematical Education (this version modified by Deakin University and licensed under CC BY NC 4.0 with permission from Shell Centre for Mathematical Education).

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PART V
SEQUENCES AND PROGRESSIONS

In this section we'll take a look at sequences and progressions.

We begin to come to terms with some of the more sophisticated aspects of sequences the moment we progress from counting by 1s to counting by 2s, 5s or 10s. We have already seen a number of sequences when looking for patterns in problem solving, and in Topic 3 we looked at the language around sequential information (e.g. plots with respect to time). Here we focus on two special types of sequences, 'arithmetic progressions' and 'geometric progressions', which help us to understand general trends of growth or decay – linear and exponential respectively.

Learning outcomes

- Solve problems using arithmetic and geometric progressions.

22. Patterns and sequences

In this section, we'll take a look at sequences and progressions.

Number sequences can be thought of as numbers arranged according to some defined order. The one we're probably most familiar with is what most people would simply call counting: 1, 2, 3, 4, 5… and so on. We begin to come to terms with some of the more sophisticated aspects of sequences the moment we progress from counting by 1s to counting by 2s, 5s or 10s. We can also count by less than whole numbers: 0.25, 0.5, 0.75, 1… and we might even see sequences that seem to alternate: 2, 5, 3, 6, 4, 7,… or don't even follow a pattern as far as we can tell: 11, 6, 3, 5, 4,…

We have already seen a number of sequences when looking for patterns in problem solving, and in Topic 3 we looked at the language around sequential information (e.g. plots with respect to time). Here we focus on two special types of sequences, 'arithmetic progressions' and 'geometric progressions', which help us to understand general trends of growth or decay – linear and exponential respectively.

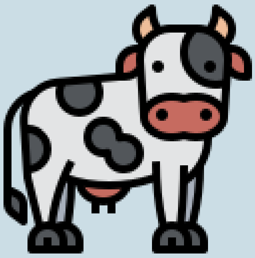
One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=318#oembed-1

[Transcript]

Let's get started with the following problems. Spend 10-15 minutes on one or both of the following questions. Be sure to annotate your work to make note of where you have specialised (is it random, systematic, artful?), generalised, any conjectures and justification arguments you make.

Sequences and progressions exercises

Cows



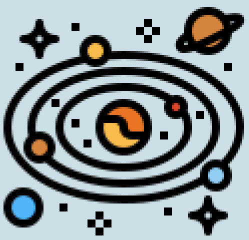
Suppose a cow gives birth to another female when it turns 2 and then another every year on its birthday. Each of its offspring do the same (with none of these cows giving birth to males). If this continues, and neither the first cow nor any of those born afterward die, how many cows will there be after 12 years?
(Adapted from Dudeney, 1967)

Planets

Between Mars and Jupiter lies a belt of asteroids. These are, perhaps, the remains of an old planet which disintegrated many years ago. We'll call this planet Planet $\mathscr{X}$.

The table compares the distances of some planets from the sun with that of our earth. For example, Saturn is 10 times as far away from the Sun as the Earth. Scientists usually write this as 10 A.U. (Astronomical Units).

Can you spot any pattern in the sequence of approximate relative distances? Can you use this pattern to predict missing figures? How far do you think Planet $\mathscr{X}$ was from the sun? Check your completed table with information available online. Where does the pattern seem to break down?



Planet	**Approximate Distance from the Sun**
Mercury	?
Venus	0.7
Earth	1
Mars	1.6
Planet $\mathscr{X}$	?
Jupiter	5.2
Saturn	10
Uranus	19.6
Neptune	?
Pluto	?

Progressions

Rather than sequences in general, we will focus on progressions, defined as 'a sequence of terms, where there is a *constant relationship* between any pair of sequential terms.'

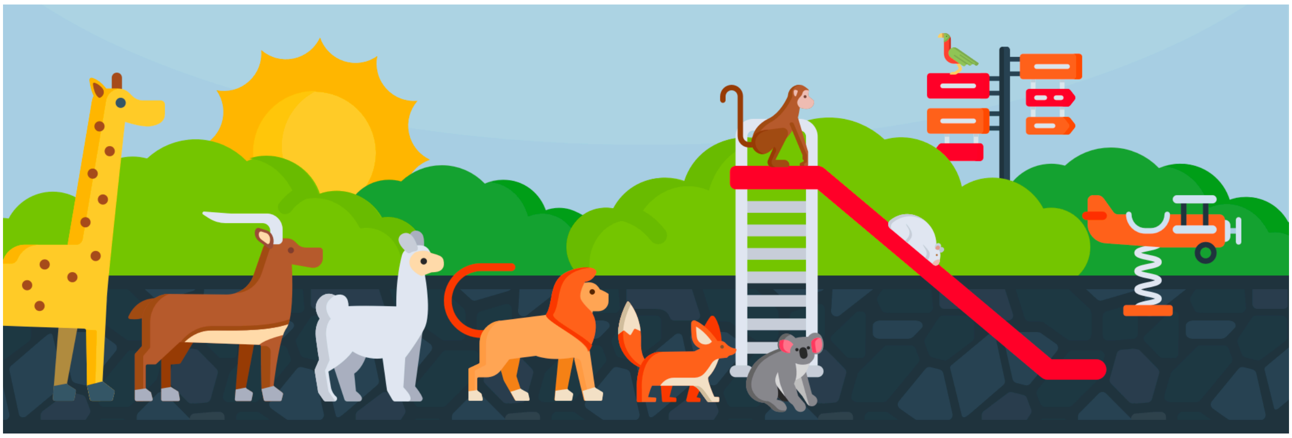
The two types of progressions we'll look at are **arithmetic** progressions, e.g. 2, 4, 6, 8, ..., where the constant relationship is the addition of 2, and **geometric** progressions, e.g. 2,4,8,16,32,..., where the constant relationship is the multiplication of 2.

Fibonacci sequences, (this is a hint for the Cows example — as for Bees in earlier topics) are not "progressions" because, although there is *a rule*, the relationship between each term is not constant. Similarly, the sequence of square numbers, 1, 4, 9, 16, 25, ..., is also *not* a progression because the relationship between two terms isn't constant here either. We will focus on some results pertaining to the arithmetic and geometric progressions, however you may also be interested to search for information on harmonic progressions.

References

Dudeney, H (1967), *536 Puzzles and Curious Problems,* Souvenir Press.

23. Arithmetic progressions



Arithmetic progressions are sequences that increase or decrease by the same amount in each step.

The simplest of which is the sequence that starts at 1 and increases by 1s, 1,2,3,4,5,.... However we can start from anywhere, e.g. from 100, and we can count by any number.

Definition: An arithmetic progression is a sequence of numbers where each new term is determined by adding the same constant to the preceding term in the sequence.

For example, the sequence 2, 5, 8, 11, 14, ... is an arithmetic progression starting from 2 with a common difference of 3.

We can express the sequence of terms in arithmetic algebraically in terms of the starting number (a) and the common difference (d).

$$a, a+d, a+2d, a+3d, \ldots, a+(n-1)d$$

Hence, the initial term of an arithmetic progression is a and the common difference of successive terms is d, then the n-th term (t_n) of the sequence is given by $t_n = a+(n-1)d$.

Why is it $(n-1)$?

- If $n=1$ then $(n-1)=0$, and $t_n = a+(n-1)d = a+0 = a$
- The second term will have $n=2$ and hence be $a+(2-1)d = a+d$

We could write our sequence out as $t_n = a+nd$, but we would just have to remember that now n = 1 means the *next* term after the first, and there would be an offset. So what we refer to naturally as the 10^{th} term in the sequence would actually correspond with n = 9.

Progression example

Let's look at the example of 2, 5, 8, 11, 14, …. The sequence starts at 2, and so $a = 2$. We can also see the constant difference of 3 between each two terms, so we have $d = 3$.

Term	When…	Then we have…
1st term	$n = 1$	$2 + (1 - 1)3 = 2 + 0 = 2$
2nd term	$n = 2$	$2 + (2 - 1)3 = 2 + 3 = 5$
3rd term	$n = 3$	$2 + (3 - 1)3 = 2 + 6 = 8$
4th term	$n = 4$	2 + (4-1)3 = 2 + 9 = 11

And so on. Don't forget that you always do the multiplication before adding (and always do the subtraction inside the brackets before multiplying).

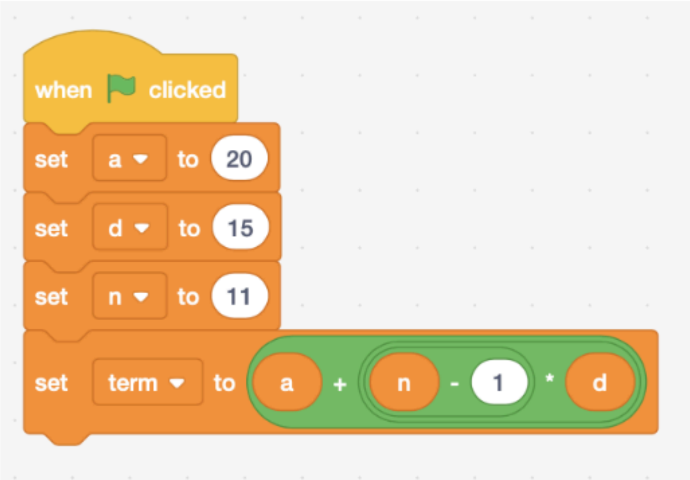
Quick check

You buy a shrub that is 20cm high. Each year it grows 15cm. Draw a plot of the shrub's height corresponding with the number of years since it was purchased.

Have a go at drawing the plot, then take a look at the video which gives a quick overview of what we've studied so far.

Scratch to the rescue

Scratch and mathematical programming software can really help us to understand relationships such as those that arise when we study sequences. Firstly, we can use Scratch as a kind of elaborate calculator. For example, we can code the n-th term of an arithmetic sequence as something like the following (pay close attention to how the correct order of operations is ensured with the combination of embedded green operation blocks). However we could also do away with the n-th term formula and still quite easily calculate it using a loop (see code on the right).



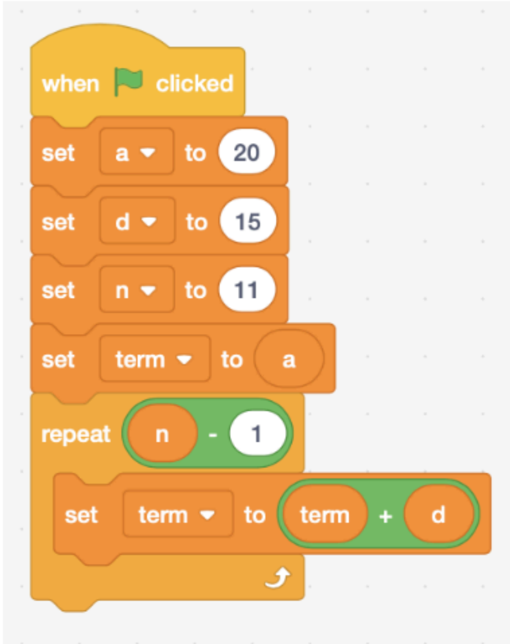


Image: screenshot from [Scratch]

The following activity presents a program that also plots the sequence of values generated by an arithmetic progression. You can [access the project page] if you want to see the code.

Scratch

The following scratch program lets us plot an arithmetic progression (and work out each term in the process) based on the value of **a**, **d** and **n**. Explore what happens with different values of **a**, **d** and **n**. Note that **a** and **d** can be negative and/or decimal numbers, but **n** refers to how many terms – so should only be a positive whole number.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=325#h5p-12

Excel

Excel is pretty handy for doing some calculations and charts too. You can create a formula by starting with an "=" sign, then use the asterisk "*" for multiplication, "+" for addition, "-" for subtraction, and "/" for division. You can reference a cell by its column and row labels, e.g. "A5".

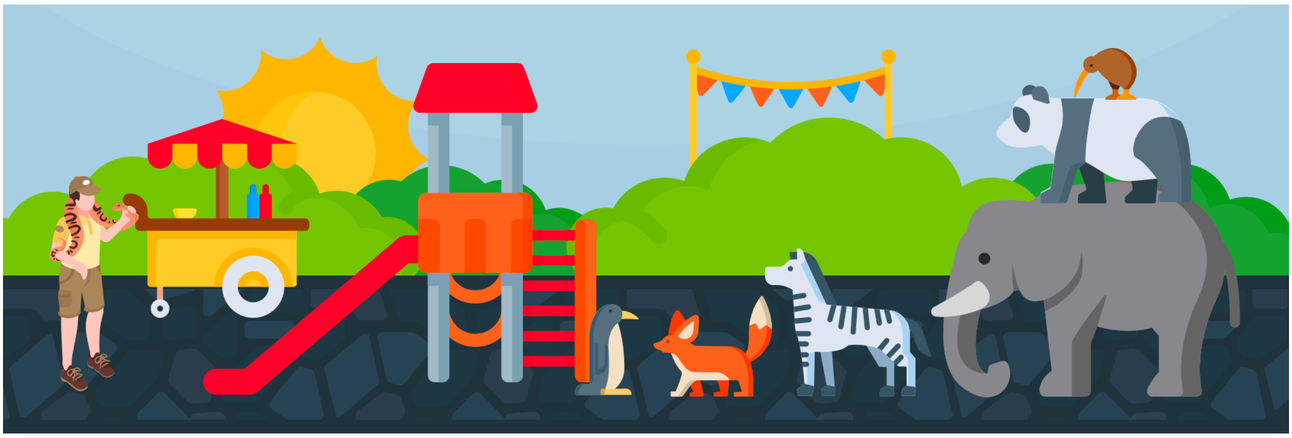
The chart linked below calculates the "t_n" term from the values of a (cell B3), d (cell C3), and n (B6, B7 etc).

You can double-click on the values below "a" and "d" to edit them and see how the values change. You can also double-click on the cells below "t_n" to see the formula used.

Progressions Template

One of the useful things in Excel is the ability to click on a cell and drag down a square that appears in the bottom corner to "fill-down", which will copies the pattern of the calculation with respect to the variables. In order to stop the a value from being drawn from the next cell down, you can use dollar signs, like "B3", which means all the cells filled when you drag down will keep using that value.

24. Sums of progressions



Sometimes we are much more interested in the sum of a sequence than just in knowing the n-th term.

Can you remember the handshake problem from the very beginning of the book?

Let's begin by specialising and work our way up a little bit. First let's reframe the scenario a little, assuming that everyone in the room walks in one by one, shakes hands with everyone else in the room, and then a new person arrives.

When two people are in the room, there is one hand shake (running total: 1). As a third person enters, they need to shake hands with two people (running total: 3). As a fourth person comes in, they must shake hands with three people (running total: 6).

In other words, as each new person enters, they need to shake hands with the total number of people in the room excluding themselves, and these new handshakes need to be added to the total.

So the number of *added* handshakes each time is an arithmetic progression (0, 1, 2, 3, 4...).

n	1	2	3	4	5
Added handshakes	0	1	2	3	4
Running total	0	1	3	6	10

Those of you with a keen eye will have also figured the pattern. To arrive at the running total of handshakes, you only need to add up together the preceding added handshakes. For example, when 5 people are in the room, this requires a running total of 10 handshakes. This is also the sum of all the added handshakes to this point (0+1+2+3+4=10).

Added handshakes

Let's try to write this relationship out in mathematical language. We'll start by defining the variables – these are the same as we used in the last chapter.

- n refers to the total number of people in the room. So if $n = 6$, then we have 6 guests.
- a is the total number of handshakes we start with, in this case 0
- d represents the common difference between them. This is the arithmetic progression which grows

by 1 for each person who enters the room.

Although it would be a complex way to count, we could model the number of *added* handshakes using the same formula we used for progressions in the previous chapter: $a + (n - 1)d$.

$$0 + (1 - 1) \times 1 = 0$$
$$0 + (2 - 1) \times 1 = 1$$
$$0 + (3 - 1) \times 1 = 2$$

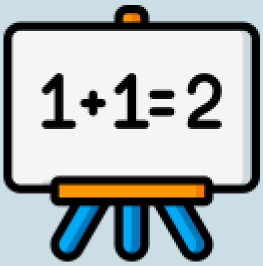
and so on...

We could simplify this further, by removing the parts that won't affect the end result (that is, the $0+$ and the $\times 1$). This would leave us with simply $n - 1$.

$$1 - 1 = 0$$
$$2 - 1 = 1$$
$$3 - 1 = 2$$

and so on...

Investigating sums



Can you work out what the sum of any 3 consecutive numbers will be? Some numbers can be expressed as the sum of a string of consecutive positive numbers. Exactly which numbers have this property? For example, observe that:

- $9 = 2 + 3 + 4$
- $11 = 5 + 6$
- $18 = 3 + 4 + 5 + 6$

Annotate your findings and see if you can come up with a few conjectures about **which** numbers can be expressed as a sum of consecutive numbers and which can't (i.e., we already know that 9, 11 and 18 definitely can be, but what about 2?).

Running total

This is good, but to get to the running total of handshakes we need to go one step further and add all the numbers in the progression together.

For the handshake problem, the sum of 0,1,2,...,19 that leads to the total number of handshakes produces the same result as the following formula.

$$Shakes = \frac{n(n-1)}{2}$$

$$\frac{1(1-1)}{2} = \frac{0}{2} = 0$$
$$\frac{2(2-1)}{2} = \frac{2(1)}{2} = 1$$
$$\frac{3(3-1)}{2} = \frac{3(2)}{2} = 3$$

and so on...

We can explain this formula by saying that n people each shake hands with everyone except themselves $(n-1)$ and, to avoid double counting, we divide by 2.

Formula for adding the terms of an arithmetic progressing

If we were to write our sum out in terms of a and d we have

\underbrace{a}_{\textit{1st term}}+\underbrace{a+d}_{\textit{2nd term}}+\underbrace{a+2d}_{\textit{3rd term}}+ \cdots+\underbrace{a+(n-1)d}_{\textit{nth term}}

This gives us an '**a**' for every one of the **n** terms (or ***n*** x ***a*** altogether) and then **d** will be multiplied by the sequence $(1+2+3+4+\ldots+n-1)$.

Which, as we have actually already established, adds up to $na + d\left(\frac{n(n-1)}{2}\right)$ or $an + \frac{n(n-1)d}{2}$. We could even express this over a common fraction, which would give us this monster:

$$\frac{2an + dn(n-1)}{2}$$

You can see it at work in scratch.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=337#h5p-14

Check your understanding

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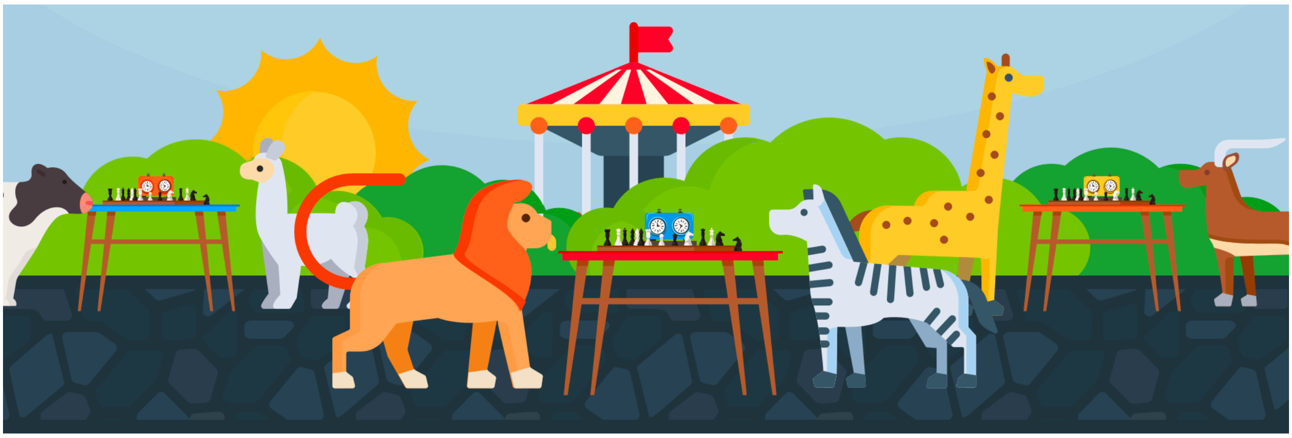
25. Geometric progressions



The rate at which quantities grow when they are continuously doubling is fairly well recognized.

Although, it's not something we always intuitively understand.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=365#oembed-1

[Transcript](#)

In gambling, the method of doubling a bet every time you lose until a win is obtained (and hence returning the value of the original bet) is referred to as a **martingale** betting system, and is all very well and good if you can back your bet, but the problem is that runs of losses are not all that uncommon, and so starting with a $10 bet, losing 5 times in a row requires you to risk $320 on your next bet.

Population growth

If a population grows by 10% of its current size each month, how long will it take to double its size?

If a population shrinks by 10% of its current size each month, how long will it take to halve its size?

Such progressions are also commonly observed in population growth. Human population growth is not usually that fast, but the growth of bacteria is often expressed in terms of how quickly they double. For example, on a hot day, bacteria in food can double every 20 minutes, so that a small amount of 100 bacteria on a piece of food at 8am in the morning could grow to over 26 million by 2pm. We also might hear about online content and data increasing in similar ways.

This is often referred to as **"exponential" growth**, and where there is a constant factor between each term in the sequence, we can refer to it as a geometric progression. Here is a graph of an exponential curve.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=365#h5p-15

Writing geometric sequences algebraically

It might help to have a quick way to figure out the total if we started with the number 2 and doubled it 31 times, without having to go all the way through the sequence. Expressing the sequence algebraically can help with this, meaning we can quickly calculate the total no matter what number we start with or how many times it has been doubled.

Let's use a to refer to the first term again, as we have done all through this chapter. Next, we need to consider the ratio. We'll call this r. The first number needs to be multiplied by something, which is then multiplied by itself, and multiplied by itself again, and so on.

So for the first three terms in our sequence we would have something that looks like this

$a, a \times r, a \times r \times r...$ or a, ar, ar^2

The nth term is then given by $t_n = r^{n-1}$. There's that $n - 1$ term again – we explain why we do this in the chapter on Arithmetic Progressions.

Let's see this formula at work. In this case it's tripling the number 2.

We have $a = 2, r = 3$ and the n^{th} term is ar^{n-1}

- When n = 1 (1st term): $2 \times 3^{1-1} = 2 \times 3^0 = 2 \times 1 = 2$ – any number raised to the power of zero is equal to one, as we'll explore in a later chapter.
- When n = 2 (2nd term): $2 \times 3^{2-1} = 2 \times 3^1 = 2 \times 3 = 6$
- When n = 3 (3rd term): $2 \times 3^{3-1} = 2 \times 3^2 = 2 \times 9 = 18$
- When $n = 4$ (4th term): $2 \times 3^{4-1} = 2 \times 3^3 = 2 \times 27 = 54$

...and so on.

Fractions

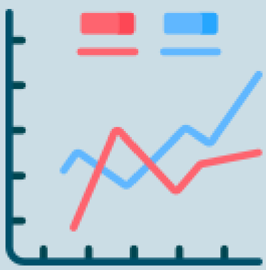
However we have some different sorts of patterns when the ratio r is a fraction. For example, if $r = \frac{1}{2}$ what will happen when we start from 1?

We'll have $1, ½, ¼, 1/8, 1/16, 1/32, 1/64,$ and so. Hence the values are getting **smaller** each time. This wasn't the case with arithmetic progressions, an arithmetic progression would just be going up with a smaller increase than usual, e.g., $1, 1.5, 2, 2.5, 3, 3.5...$

This is because the effect of multiplying by a number between 0 and 1 is to make a number smaller. This behaviour for geometric progressions is sometimes thought of as modelling half-life or decay. The following Scratch code can help you plot geometric progressions for certain values (depending on r, they could very quickly become too large!). You can also see the code on the project page.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=365#h5p-16

Building intuition with progressions



Graph at least five terms from each of the following progressions – you can use the Scratch program above, try an online graphing tool, or graph them yourself. You can even use an excel spreadsheet.

Then write a sentence summarising the behaviour of the graph. Try to think of an example of what might make a graph that looks like this (eg, the temperature of a cake over time as it comes out of an oven).

If you're struggling to understand the progressions below, take a look at the Scratch tutorial.

Arithmetic progressions

- $a > 0, d > 0$
- $a < 0, d > 0$
- $a > 0, d < 0$
- $a < 0, d < 0$

Geometric progressions

- a > 0, r > 1
- a > 0, 0 < r < 1
- a > 0, r < -1
- a > 0, -1 < r < 0
- $a < 0, r > 1$
- $a < 0, 0 < r < 1$

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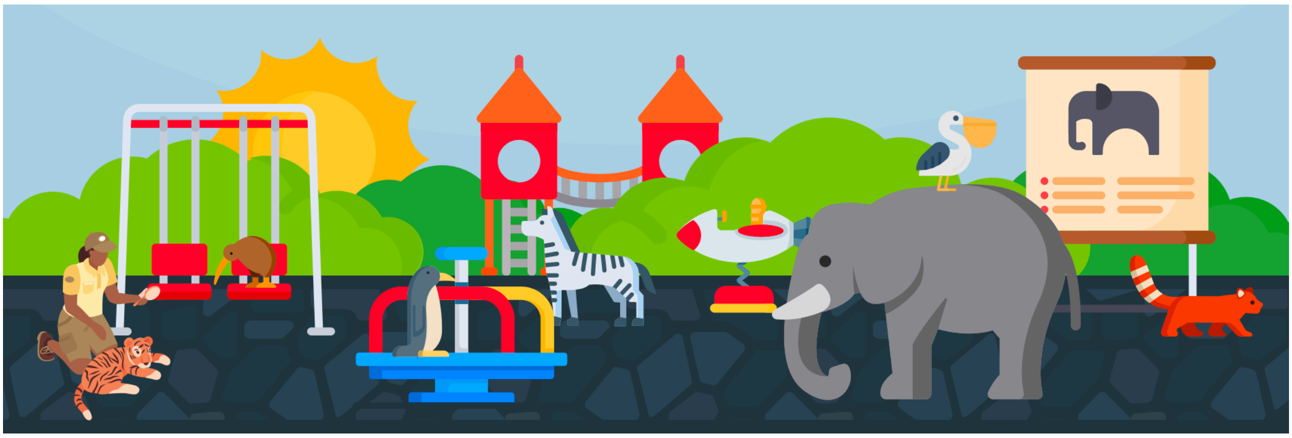
26. Summing geometric progressions



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Transcript

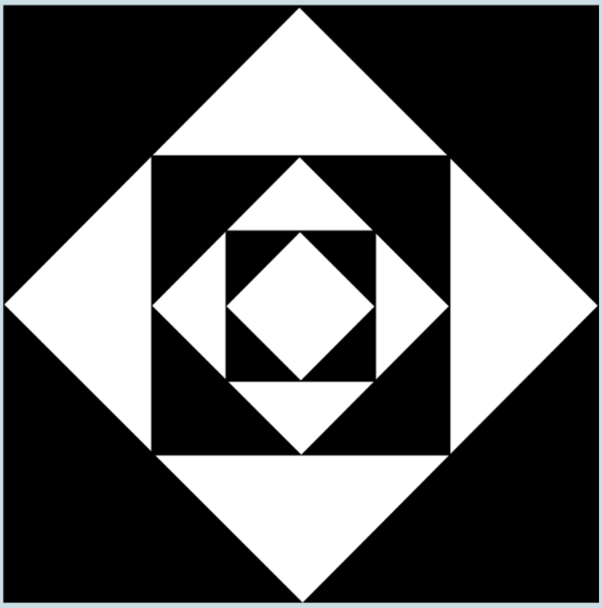
The algebra of how sums of geometric progressions mightn't be everyone's thing.

Although the derivation of the formulas is difficult, the most important thing is to be able to identify sequences that arise as either being **arithmetic** or **geometric** (or *neither*) and then being able to correctly apply the formulas if the sum or n^{th} term is required. This Scratch program will help you check your working.

An interactive H5P element has been excluded from this version of the text. You can view it online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=393#h5p-17

Geometric progressions

What fraction of the square below is shaded black? What if the pattern continues forever with smaller and smaller squares being drawn inside?



Hints

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https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=393#h5p-35

Still stuck? More hints (warning, spoilers!)

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=393#h5p-36

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PART VI

MEASUREMENT

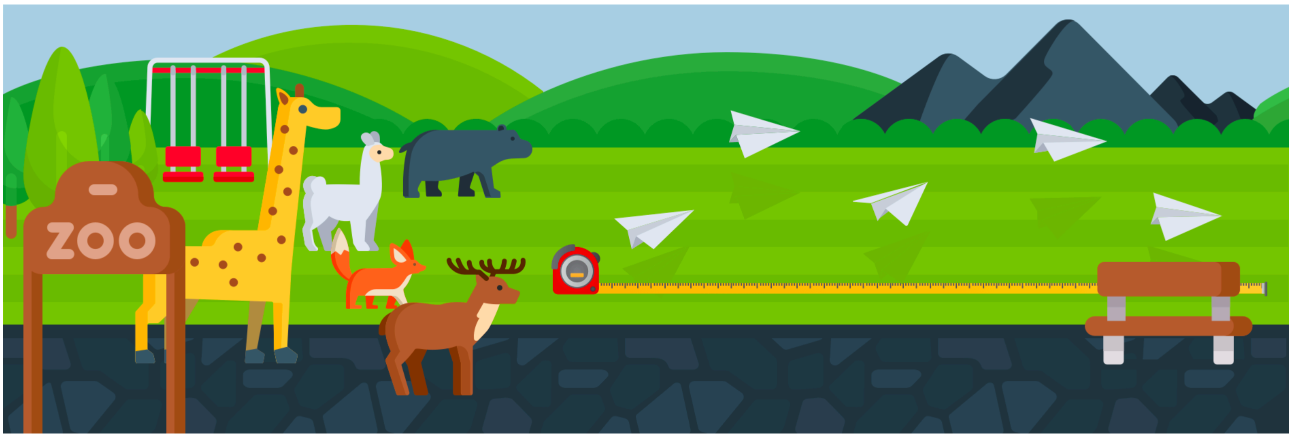
In this section we're looking at something you're probably all familiar with: measurement.

In the first topics we looked at problem solving and programming – those topics will be relevant throughout the entire book (and we'll keep coming back to them). However in this section we get to something a little more familiar as a mathematics topic: Measurement.

Learning outcomes

- Describe the metric system
- Convert units of measurement from one system to another (eg, metric and imperial systems)
- Express very large or very small numbers using scientific notation
- Convert numbers from the decimal system into other bases (eg, binary numbers).

27. Measurement



In this Chapter, we're going to start with something very familiar: measurement.

If the previous section on sequences and progressions can be considered 'advanced counting,' this chapter will help you (and your students) come to grips with measurement at a deeper level, and use it to solve all kinds of problems.

In particular, in this chapter you'll learn to

- Describe the metric system
- Convert units of measurement from one system to another (eg, metric and imperial systems)
- Express very large or very small numbers using scientific notation
- Convert numbers from the decimal system into other bases (eg, binary numbers).

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=398#oembed-1

Transcript

What does it mean to measure something?

Any measurement of an object or quantity comprises three components:

- A property (what it is actually being measured)
- A unit (a benchmark appropriate to the property)
- A number or value

For example, if you say you are 18 years old, the property is the age (or time), the unit is years (but you could also use months or moon cycles) and 18 is the value.

Having an intuitive notion of different benchmarks or units is a big part of being able to work comfortably with measurements. Through experience, we will encounter a number of benchmarks. Some become official units while others just serve to help our intuition.

Benchmarks and quantities

A quantity is anything that can be measured or counted. When we measure a quantity, we use a unit and a value. When using numbers for measuring physical quantities, a benchmark is a readily available reference to help visualise a physical quantity, e.g. 1 litre milk carton, a bucket of water, the length of a football ground, an Olympic size swimming pool.

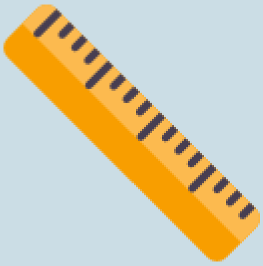
It is important that children develop a strong sense and intuitive feel for how big or small certain benchmarks are. For example, how big is a metre? How heavy is a kilogram? How much is 1 litre?

Stop and think

What are some other benchmarks? Can you give an example of a benchmark you use for all of the different properties: height, length, weight, mass, quantity, time, energy?

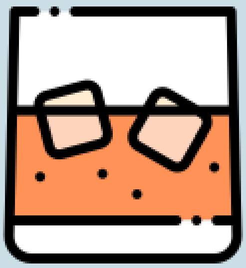
Now, let's get started on the mathematics of conversion.

Conversion



1 inch is equal to 2.54cm. How long is a 12 inch ruler in cm? How many inches is a 30cm ruler?

Ava thought she had bought 3.5m of cloth from the market. When she got home, she found that the stall holder had used a ruler that was 4cm short of 1m. What was the real length, in metres, of Ava's cloth?



Whisky is usually bottled at about 40% alcohol content. Cask strength whisky is usually bottled at about 60% and allows tasters to add water to bring out the flavour.

For a 30ml dram of cask strength whisky, how much water could be added until it was equivalent in percentage to the standard bottling?

How many standard drinks will it be if a 30ml dram at 40% is one standard drink?

Stuck?

Have you tried some strategies from our problem solving topics?

- Try specialising with a few examples
- Is there a general rule you can use? Can you introduce algebraic notation?
- Extend! For example, with the ruler problem could you generalise it to a ruler that is x cm shorter or longer than the original?

Having benchmarks can help guide our reasoning when we do these conversions. However we can also look more closely at the mechanics of what is going on and develop some algebraic methods and tools.

Coding

Can you define a function in Scratch that converts inches to cm?

The following video provides some examples of how you could use scratch to explore or extend the above problems.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=398#oembed-2

Transcript

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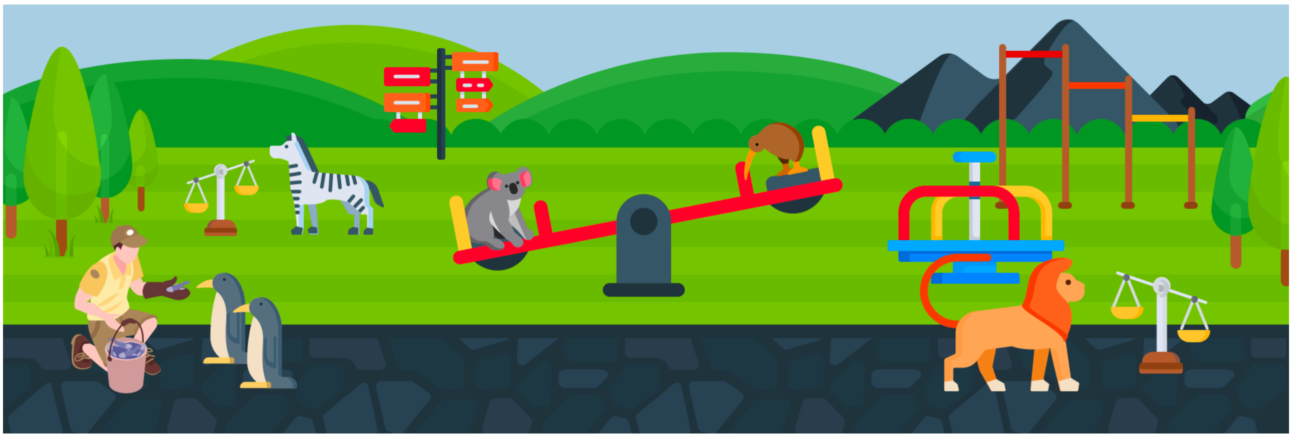
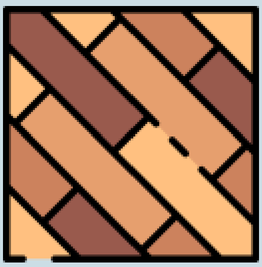
28. The algebra of ratio



To convert a measurement, we usually translate it from one unit of measurement to another using a ratio.

Most of us are used to performing this kind of operation with a calculator. However, to be more confident in our results – especially for more complex ratios like the whisky example in the previous chapter – it might help us to first conceptualise what we're trying to achieve using algebra. Recall that algebra is very useful for finding an unknown quantity in relation to something that we do know (like the length of 25cm in inches).

Metric and Imperial



Graeme and Jaitesh are working together to make a deck. Jaitesh is more comfortable with inches, whereas Graeme went through school using centimetres. Both need to be able to make quick conversions from the other's measurements.

Can you help by giving them a method they could use to quickly make the conversion if:

1. Graeme needs to figure out what 84 cm is in inches
2. Jaitesh needs to convert 16.5 inches into centimetres
3. Could the method be extended to work no matter what the original measurement is and whether it was measured in inches or centimetres?

If you got stuck on part 3 of the question above, the video might help.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=405#oembed-1

Transcript

Ratios and conversion

Using ratios can be an algebraic strategy for converting units from one form to another. While this might seem over the top for simple conversions, it can make problem solving a lot easier if something more complex is occurring.

Let's begin with a simpler conversion first just to demonstrate the concept.

Racing

If there are 1.6 kilometres in a mile, how many miles are there in a 5 kilometre race?

This, of course, can be answered more elegantly without algebra – $\frac{5}{1.6} = 3.125$ miles. However, let's look at it algebraically anyway, so that we can see how ratios work.

Kilometres to miles

Let's start with the ratio of miles to kilometres, which is approximately

$$\frac{miles}{kilometres} = \frac{1}{1.6}$$

Next, we'll remind ourselves what it is we don't know – as always with algebra, it's important to identify this clearly. What we don't know is 'how many miles are there in 5 kilometres?'

Using the same ratio (miles on top), we could set this out like this:

$$\frac{miles}{kilometres} = \frac{x}{5}$$

So, what next? All we need to do is to join the two together in a way that preserves the original ratio of $1:1.6$.

If we place this ratio on the left hand side of our equation, this will effectively mean that whatever we find on the right hand side will have to also preserve this ratio.

Let's see how it works (remembering we have miles on top and kilometres in the denominator):

$$\frac{1}{1.6} = \frac{x}{5}$$

Multiply both sides by 5, the 5 under the x cancels.

$$5 \times \frac{1}{1.6} = x$$

By the rules of multiplying fractions, this becomes

$$\frac{5}{1.6} = x$$

Swap x to the left hand side (because it feels neater) and solve

$x = 3.125$ miles

Over to you

Now it's time to have a go at a more complex conversion problem. If you're really getting stuck, it's simple to look up the conversion from kilometres per hour to any other unit of time and distance online – but try to have a go for yourself first.

Cheetah

Cheetahs can run up to 130 kilometres per hour. Cheetahs love to hunt Springbok, which can reach a speed of 88 kilometres per hour.

A Springbok is being chased by a Cheetah. The Cheetah is 50 meters due West from the Springbok. However, there is a safe place for the Springbok to hide only 200 metres to the East.

If both animals are moving due East at full speed in a straight line, will the Springbok escape the Cheetah?

Extension

Assuming there is nowhere for the Springbok to hide, how long will it take for the Cheetah to catch it? Answer to the nearest second.

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29. The metric system

Many ancient measurements were loosely based on the human body.

Cubits, hand spans and feet are well known examples. The problem is that they're not accurate for most people.

Online research

- What exactly is the metric system? Could you explain to a child what is so special about it?
- Where does the metre originally come from (Wikipedia can help with this)?
- If it's so great, why does the USA still use imperial units ?
- How would you build a measurement system from scratch?

The metric system

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=419#oembed-1

Transcript

The key to the usefulness of the metric system is that it is based on the decimal system, which makes conversion between units particularly convenient. For example 1 kilometre is 1000 m, 1 metre is 100 centimetres and so 1 kilometre is $1000 \times 100cm = 10,000cm$. We will focus on the following:

Prefix	Symbol	Meaning	Length	Mass	Capacity
mega	M	million	Mm	Mg	Ml
kilo	k	thousand	km	kg	kl
hecto	h	hundred	hm	hg	hl
deka	dk	ten	dkm	dkg	dkl
Benchmark unit		one	m	g	L
deci	d	tenth	dm	dg	dl
centi	c	hundredth	cm	cg	cl
milli	m	thousandth	mm	mg	ml

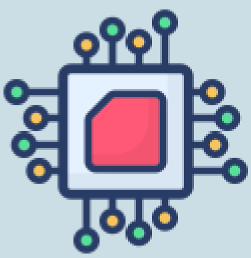
The prefixes meaning 1000 (kilo-), 100 (hecto-) , 10 (deka-) come from Greek, while 1/1000 (milli-), 1/100 (centi-) and 1/10 (deci-) come from Latin, however there are also prefixes that derive their names from Italian (pico- means a trillionth) and Danish (femto- and atto- are respectively a quadrillionth, quintillionth).

Online research

In the table on this page, we've shown units in the metric system for length, mass and capacity. There are many other things that we could measure that also work using the metric system. Spend up to 15 minutes finding out a little bit more about one of the units of measurements below that is unfamiliar to you.

- Newtons
- Joules
- Watts
- Hertz

How small are microchips?



We sometimes hear about measurements in nanometres.

A nanometre is a billionth of a metre (I.e., 1 000 000 000 nm = 1 m). Small microchips in computers can be small as 10 nm, and the average is 14nm.

What would these measurements be in millimetres?

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30. Scientific notation

Some numbers that apply to real phenomena are huge.

For example,

- Our solar system is approximately 11 800 000 000 km across.
- The distance light travels in one year is approximately 9 500 000 000 000 km.

Others are really small. For example, the time taken by light to travel one metre is roughly 0.000000003 seconds.

To write these out, we'd either need very long numbers with lots of zeros, or remember a bunch of different units or prefixes. Most people who are familiar with using long (very big or very small) numbers use scientific notation instead.

What is scientific notation?

Scientific notation is a way of writing large or very small numbers efficiently. After a bit of time, it gets a little more intuitive than counting zeroes, too.

The idea is simple enough: you start with a number between one and ten and then record the number of times it would need to be multiplied or divided by 10 to get to the actual figure.

Some examples of large numbers:

- $1.3 \times 10^3 = 1,300$
- $9 \times 10^8 = 900,000,000$ or 900 million.

For smaller numbers, we still use indices ('to the power of'), but recall that raising something to a negative power is effectively dividing it. So $2^{-1} = \frac{1}{2}$ and $2 \times 10^{-1} = 0.2$.

- $5.8 \times 10^{-2} = 0.058$
- $1.8 \times 10^{-}5 = 0.000018$

Definition of scientific notation

A number is in the form of scientific notation if it is written in the form of $a \times 10^b$, where:

- a is a number between 1 and 10, and
- b is an **integer**.

Non-standard notation

If the number **a** is not in the range 1 to 10, then it is not standard scientific notation. For example, if we want to write 0.5×10^4 in standard scientific notation, it should be rewritten as 5×10^3 (but both equal 5000).

We can see here that we're using index notation, so let's just quickly refresh our index laws.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=423#oembed-1

[Transcript]

A useful thing about scientific notation is that it allows us to easily compare the magnitude of numbers.

If you have 18942000000000000000000000 and 213940000000000000000 it's not so easy to see that the former is almost 100 000 times larger. However in scientific notation, we have 1.8942 x 10^{25} compared to 2.1394 x 10^{20} and we can see the difference in the power of 10 is 5, so almost 105 times difference. Sometimes this is referred to as order of magnitude, i.e. the two numbers differ by 5 in order of magnitude.

Real world examples

Now that we have a good understanding of scientific notation, we can consider a few interesting quantities:

- Approximately 10^{11} neurons in the human brain, 10^{14} synapses
- More than 1.241 x 10^{12} digits of *pi* known
- About 3 to 7 × 10^{22} stars in the observable universe, 10^{80} atoms
- A DNA molecule's width is about 2 x 10^{-7} cm, (hair is about 2×10^{-2})

The ones below are from Bill Bryson's "A short history of nearly everything", which relate some large quantities to a few benchmarks to help you build intuition about some impossible-to-fathom-ly large numbers.

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Scientific notation can make routine mathematics simpler

Scientific notation can also make it easier to perform some operations. To multiply and divide numbers in scientific notation, we just need to remember our order of operations and index rules.

Let's start with two routine problems to show you what we mean.

Indices

Suppose that $a = 1.4 \times 10^3$ and that $b = 3.4 \times 10^{-2}$

- What is ab?
- What is $\frac{a}{b}$?

Watch the video, then have a go for yourself, then look through the worked solution below (in that order).

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Transcript

Multiplying in scientific notation (worked solution)

Let's start with ab.

First of all, what is the question asking us? We need to recall that, by the conventions of algebra, writing two things next to each other means without any other symbols in between means that they're multiplied. For example, we can represent cows as having four legs by saying $legs = 4 \times cow$ or $l = 4c$.

So, $ab = a \times b$ or $(1.4 \times 10^3) \times (3.4 \times 10^{-2})$.

There are two 'tricks' we can use here to make this easier to calculate.

1. It doesn't matter what order you multiply them in: $5 \times 3 \times 2$, $3 \times 2 \times 5$ and $2 \times 5 \times 3$ all equal 30.
2. Because of the index laws, $a^b \times a^c = a^{b+c}$. In other words, $10^3 \times 10^{-2} = 10^{(3-2)} = 10^1 = 10$. To add a negative number, you subtract it. Eg $3 + (-2) = 3 - 2$.

Let's put these two little tricks into practice to solve the problem above:

$$\begin{aligned} ab &= (1.4 \times 10^3) \times (3.4 \times 10^{-2}) \\ &= (1.4 \times 3.4) \times (10^3 \times 10^{-2}) \\ &= 4.76 \times 10^{(3-2)} \\ &= 4.76 \times 10^1 \\ &= 4.76 \times 10 \\ &= 47.6 \end{aligned}$$

It's a good idea to check by writing it into your calculator as it originally appears to ensure the final numbers agree.

If you think about what this means, multiplying in scientific notation is made easier by the fact that you are working with base 10 (10 to the power of something), and that the power is always a single number.

All you need to do is

1. multiple the leading numbers together (1.4 and 3.4 in the example above),
2. add or subtract the powers of 10 ($10^{(3-2)}$ in the example above)
3. Multiply the results of steps one and two together (4.76×10).

Dividing in scientific notation

Now let's try $\frac{a}{b}$ or $\frac{1.4 \times 10^3}{3.4 \times 10^{-2}}$

Here are the two 'tricks' to keep in mind for this one, that are related to the tricks in the multiplication above:

1. Just as the laws of indices state that $a^b \times a^c = a^{b+c}$, it also states that $a^b \div a^c = a^{b-c}$. This makes sense because, for example,

$10^3 \times 10^2 = 1,000 \times 100 = 100,000 = 10^5$. Similarly, $\dfrac{10^5}{10^3} = \dfrac{100,000}{1,000} = 100 = 10^2$.

2. The equation can be split up into two fractions that are multiplied together, rather than one single equation.

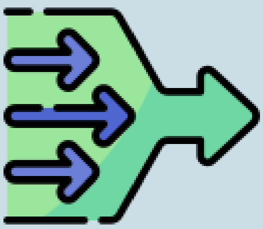
Let's see it in action:

\begin{align}\frac{a}{b} &= \frac{1.4 \times 10^3}{3.4 \times 10^{-2}} \\ &=\frac{1.4}{3.4} \times \frac{10^3}{10^{-2}} \\ &=\frac{1.4}{3.4} \times 10^{(3-(-2))} \\ &=\frac{1.4}{3.4} \times 10^{5} \\ & = 0.411764705 \times 100,000 \\ &≈ 41,176.5\end{align}

If you're not quite convinced about being able to separate the equation into the multiplication at the first step, remember that when multiplying fractions, we just multiply the top and the bottom, so if we were working in the other direction we would have

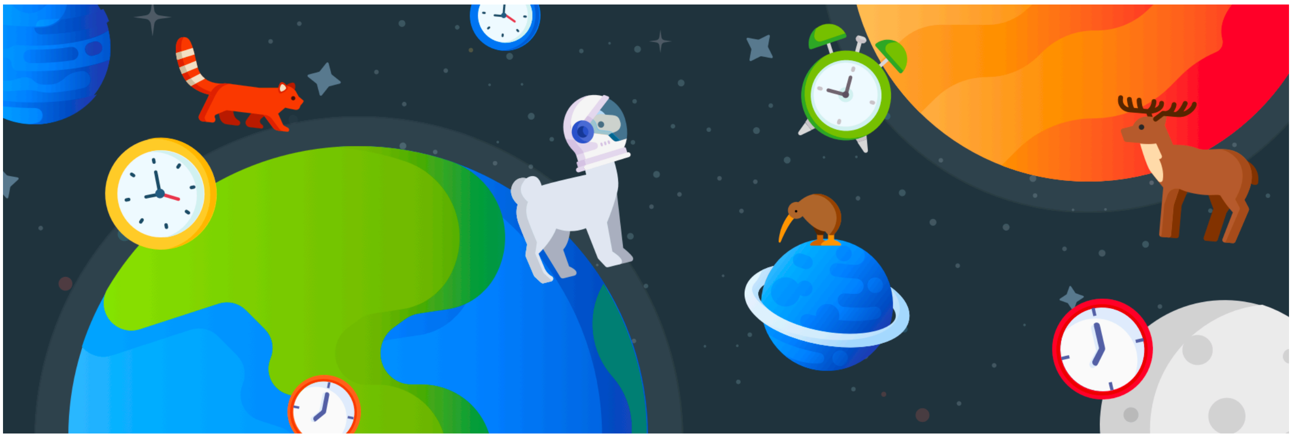
$$\frac{1.4}{3.4} \times \frac{10^3}{10^{-2}} = \frac{1.4 \times 10^3}{3.4 \times 10^{-2}}$$

The Feynman Technique for consolidating your understanding



Give an explanation of what scientific notation is in simple English. If there are things you're not sure about, do some additional research and rewrite your explanation. Include examples of numbers that are in scientific notation and not in scientific notation and give an example of where it might be useful to use it.

31. Base conversion



Let's return to the concept of our decimal place system.

Much of our intuition is built around this, and as we have seen, our measurement systems take advantage of its structure.

However considering bases other than 10 can give us a deeper understanding into how our standard place-value system works.

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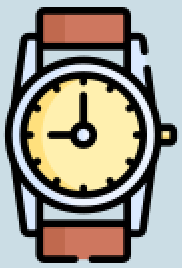
Transcript

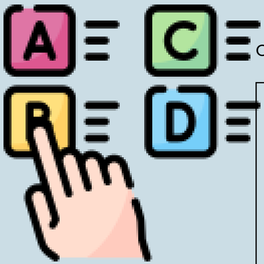
What's the time?



Suppose someone makes a fun watch.

The watch shows the hours and minutes in binary. So, for example, 1:18 in the afternoon, might be written as 1101:10010. What would the binary watch show at 7:43 in the evening? How many digits would we need to be able to show all times of day?

Quiz



Have a go at the following online quiz for converting between base 10 and binary or base 5 numbers.

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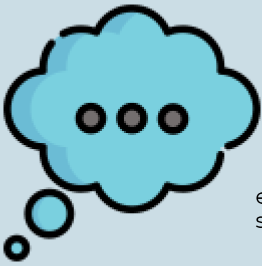
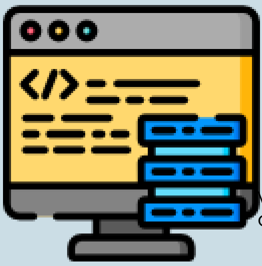
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=445#h5p-32



Stop and think

Recall that when we are adding base 10 numbers, we can add the numbers in each column and then use remainders to do the entire sum. Could you devise a similar system in a different base?



Coding

What are the steps for converting a given number in base 10 into base 5? Can you write out the steps clearly? Could you write Scratch code for this? (Hint: some key operators to use are the “floor” and “mod” blocks). See a worked example here.

Summarise

Write out your step-by-step process for converting from other bases (use your code from the Coding activity if you completed it). Read over your step-by-step process and consider whether it would make sense to someone who hasn’t studied this course, and refine the description so that it’s clear and without too much jargon or terminology. Check that the process works with an example, and make any adjustments if required.

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PART VII

GEOMETRY

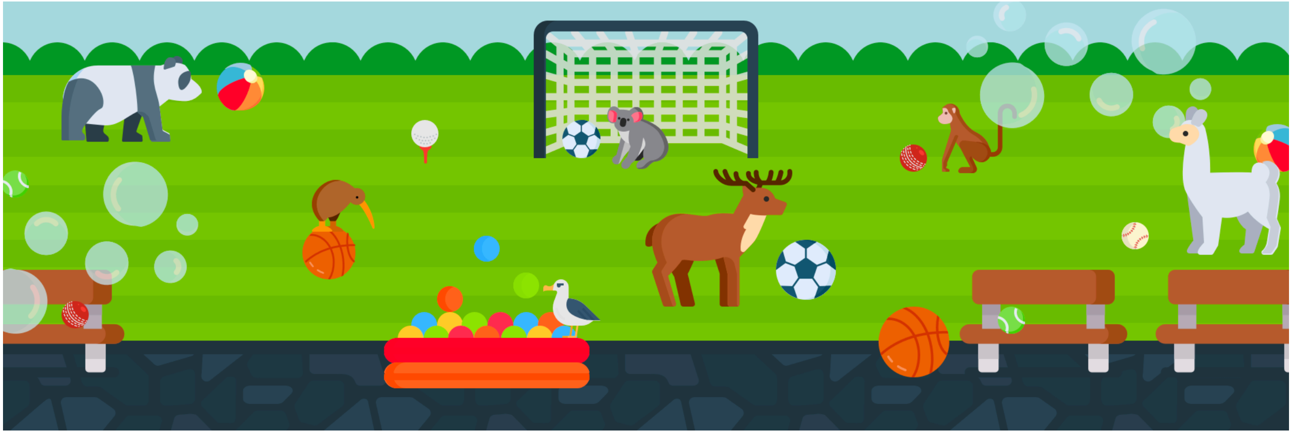
Geometry is an ancient discipline.

Here we begin exploring our understanding and that of the previous topic on measurement to study how geometric properties may relate to perimeter, area and volume.

Weekly learning outcomes

- Apply mathematical formula to solve routine problems regarding the perimeter, area and volume of triangles, circles, pyramids and spheres.
- Solve non-routine problems to develop methods for arriving at the volume of pyramids.

32. Introduction to Geometry



Geometry is an ancient discipline.

In this section, we're going to study how geometric properties may relate to perimeter, area and volume.

Geometry is one of the classic sub-disciplines of mathematics, and it underpins and intersects with more areas of mathematics than might initially be obvious.

In terms of mathematical history, you could get the impression that mathematics was *only* geometry a couple of thousand years ago in the times of Plato and Pythagoras, and further, that for anything to be true, it had to find an interpretation geometrically.

Many of the ancient problems and proofs involved being able to construct shapes and find relationships using only a **straight edge** and **compass**. Why? Because if we try to create, say, an isosceles triangle by *measuring* the side lengths, we will not be able to do this precisely and therefore such diagrams would not be able to prove anything as always true. We'll begin this topic with a return to this ancient art via the Euclidea problems.

Play a computer games

Spend about 10-15 minutes and try to complete a few of the Euclidea exercises in the **alpha** and/or **beta** sets of problems.

Obtaining all stars requires being able to obtain the constructions in as few moves as possible, where a move is counted either based on the legal tools (L) or based on using only straight edge and compass (E). Sometimes this is tricky, and it may take you days or several attempts to find the solution. It's not always intuitive, so although you should avoid looking up solutions online, also don't let yourself spend too long on any one problem if it's frustrating you!

You might like to try a YouTube search for solutions if you get stuck on problems 1.1 or 1.2.

To access Euclidea through your browser, go to the Euclidea website. You can also access Euclidea either by downloading the app on a smartphone, iPad or tablet via the Apple store or GooglePlay.

Note that by drawing the two circles which have a radius equal to the length of the line, their intersection gives the third vertex of the triangle, because it is clear that it is the same distance from both endpoints of the line.

Working with the geometric construction problems helps gain an idea about geometric relationships and there are some fascinating results, especially those needed to obtain the minimum E number of moves.

A reflection on getting stuck

We fully expect that you will all get stuck at some point using Euclidea. Take a few moments with a pen and paper, and write about any problem you found interesting, tricky or which particularly stumped you.

Here's an example Simon James wrote:

For the two Euclidea problems, 1.7 and 2.8, I spent days trying to nut out how to solve these in the minimum number of straight edge and compass moves (E). Then even once I'd worked it out (or in the case of 2.8, cheated by looking it up online), I'd forgotten how to complete them when I reattempted it some time later. This was a good reminder for me on the problem solving principles and of learning how to deal with being stuck. Of course I went through all the standard emotions of frustration, deciding that actually I'm not that good at geometry, giving an unhelpful excuse to myself that perhaps I'd never spent that much time on this type of problem. But the main things that I was reminded of after I did find the solution were:

1. I had discounted a number of pathways because of pre-conceptions about what information the new circle or line was giving me. I had figured that in 1.7, the first two circle moves couldn't be the way to go because all it gave me was the intersecting point at the bottom of the circle, which could more easily be obtained by a straight line. What should I have done?

I should have paid attention more broadly to what information was available, and I should probably have been more methodical in the way I trialled and eliminated steps.

2. Being positive about the experience of being stuck is very important. Some solutions are not intuitive and we can learn more by being stuck and discovering something new than by assuming that the failure is a reflection on our own ability. By learning about these solutions, I became much more open to potential solutions for other problems. I was reminded of the saying (I think it's in a recent Star Wars movie…) – failure is the greatest teacher.

Solutions of these two problems take less than a minute, despite the many hours and days the problems puzzled me for. Another good reminder that the concise solutions we often see in textbooks or even submit for an assignment are far from the reality of the real work involved.

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33. Triangles

We now turn from the proofs we obtain in geometric constructions to the geometry of length, area and volume.

Some key measurable properties of shapes we are often interested in are:
For 2-dimensional shapes

- Perimeter – the length around the outside of a shape
- Area – how many square units fit inside a shape

For 3-dimensional shapes

- Surface Area – how many square units could be used to wrap or cover the shape
- Volume – how many cubic units fit inside a shape

Recalling area and volume formulae

Are you already familiar with formulae for calculating these properties for some of the standard shapes? Which ones would you need to look up? Which rules could you explain and reason out to a child?

Triangles

Triangles will be our starting point for determining the some general rules for area and volume, as well as the properties of angles in other shapes.

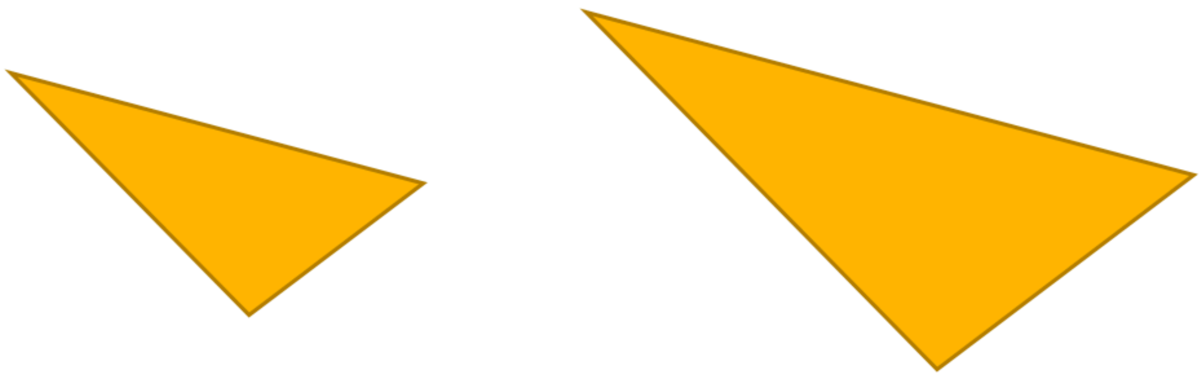
You probably already know that triangles can be classified in three ways:

- **Equilateral,** with sides of equal length, and equal angles

- **Isosceles,** two sides are the same, two angles are the same
- **Scalene,** different length sides and angles.

This kind of equivalency between angle property descriptions and side property descriptions does not happen with other shapes, e.g. with quadrilaterals, equal angles (squares, rectangles) does not imply equal side lengths (squares, rhombuses). If you spent a while playing Euclidea, you will have probably figured out that triangle play a key role in helping to establish the angle properties of other shapes.

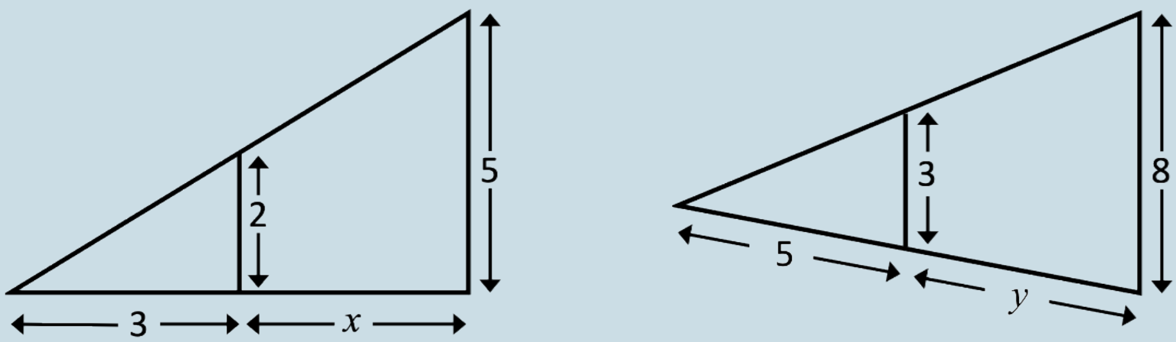
Many of the properties of triangles can be worked out from the way angles preserve ratio, or the concept of 'similar triangles'.



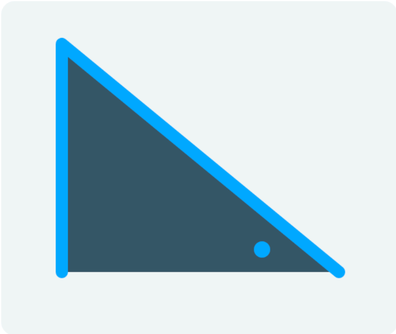
This is useful because it means a lot of what we can deduce about unknown lengths and, subsequently, areas will just amount to ratio problems.

Similar triangles

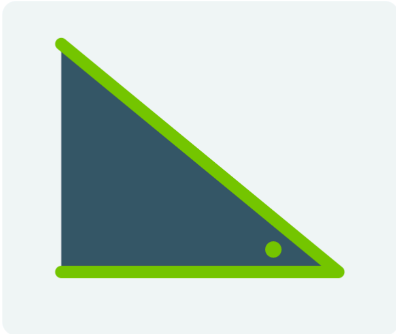
Use what you have already learnt about ratios to determine the unknown lengths x and y. You don't need the sine, cos or tan buttons on your calculator – you can get by with ratios.



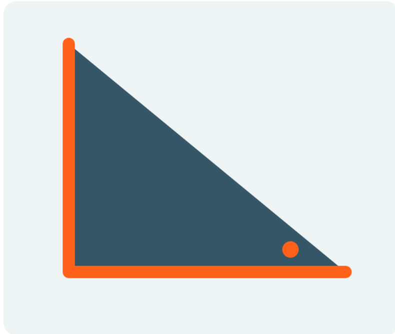
In fact, for right-angled triangles, some of the ratios between sides and how they relate to particular angles have special names and rules. These are the (fairly well known for at least 2,000 years) sine, cosine and tangent rules.



Sine *(opposite/hypotenuse)*



Cosine *(adjacent/hypotenuse)*



Tangent *(opposite/adjacent)*

These rules work because if we consider triangles with a right angle, the angle will automatically determine each of these ratios. In fact, sometimes the steepness of angles are described in terms of ratios, e.g. on the road when you see a sign that says 15% decline – it means that the road goes downwards 15cm for every metre you travel forward.

For now, just remember that an angle determines these ratios. We will now look at another well-known rule about right-angled triangles.

Pythagoras's Theorem

Pythagoras's theorem – named after Pythagoras but probably known about four centuries beforehand, is usually taught as an algebraic rule that relates the side lengths of right-angled tringles. You might remember this from school: if c is the length of the hypotenuse and a and b are the lengths of the remains sides, then $a^2 + b^2 = c^2$.

For example, if $a = 4$ and $b = 3$ then we have:

\begin{align} c^2 &= a^2 + b^2 \\ c^2&= 3^2+4^2 \\ c^2&= 9+16 \\ c^2&= 25 \\ \sqrt{c^2} &=\sqrt{25} \\ c &= 5\end{align}

If we didn't know that a, b and c were side-lengths, we should remember that c could also have been -5 in this case.

Online research

The sides 3,4 and 5 are all whole numbers, but this won't always happen. For example, if a=2, b=3, then c would be $c = \sqrt{2^2 + 3^2} = \sqrt{4+9} = \sqrt{13}$ which is an irrational number. We can only approximate this (try it on a calculator).

When a, b and c are all whole numbers, they are referred to as “Pythagorean triples”. Do a web search to find more sets of Pythagorean triples.

Proof of Pythagoras’ theorem

As a rule, Pythagoras’s theorem is certainly one of the most useful that is directly applicable in “real-life” situations that we come across throughout our schooling. Algebraically we can solve for the third side whenever we have the two others (we can practice this later) but Pythagoras’s theorem also gives a number of opportunities for beautiful geometric and other mathematical proofs, which (unfortunately) sometimes aren’t introduced when taught in school.

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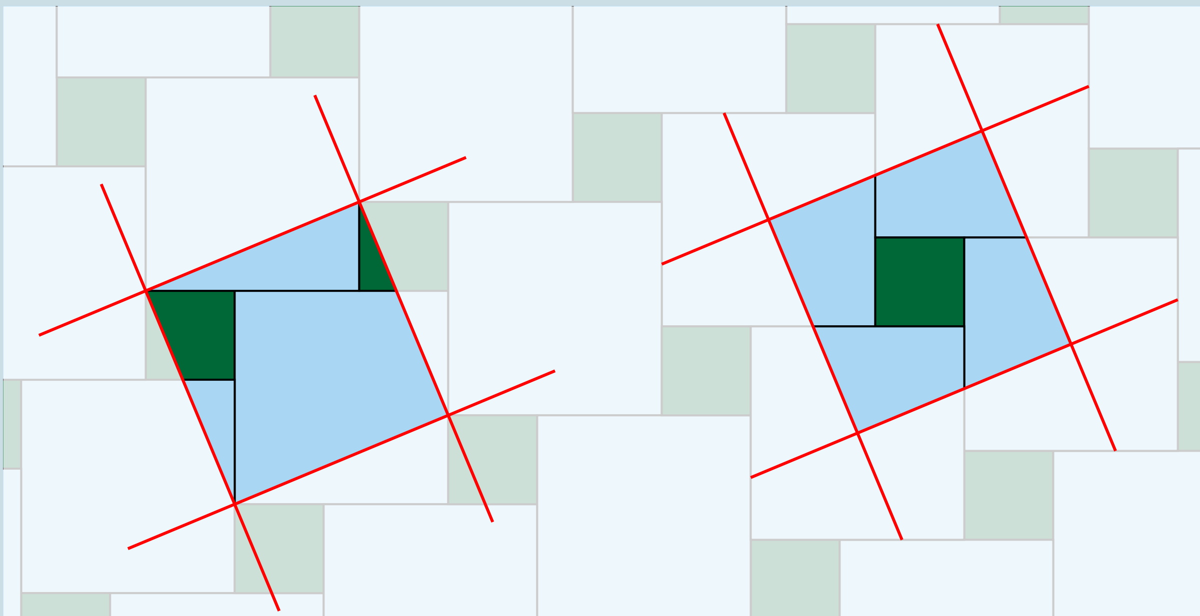
Transcript

Online research



Do some research on some of the other proofs for Pythagoras's theorem. Provide explanations of how the proofs work, including diagrams and algebra where appropriate. Another interesting one (given that you may have studied tessellations before) is one based on what's called a Pythagorean tiling (see above).

Try the quiz

An interactive H5P element has been excluded from this version of the text. You can view it online here:
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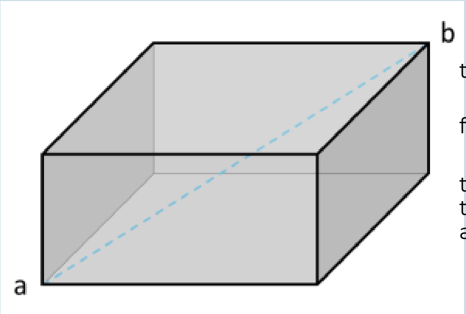
Pythagoras's theorem in 3 dimensions

Let's start with a problem.

Getting from a to b



1. Consider a rectangular polyhedron as shown. What will be the length from the vertex a to the opposite vertex b?
2. How could you determine the height of a square pyramid from the length of its edges?
3. Could you find a way to work out the distance between any two points along the edge of the polyhedron (tip: Do you recall the notion of Cartesian Coordinates? What about if we think about coordinates in 3D?)

Can you use Pythagoras's theorem to make a general rule?

Perimeter and Area for triangles

Using Pythagoras's theorem, we can now work out the perimeter of right-angled triangles provided we know two side-lengths. We are also aware of the trigonometric ratios. We can hence easily calculate perimeter in such cases, as well as area (since a right-angled triangle is just half a rectangle). However there is also a formula for calculating the area of any triangle as long as we know the three side lengths.

This is Heron's formula:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \dfrac{a+b+c}{2}$.

It might be hard to see where this formula comes from, and it looks tricky to memorise, but in the following video, we can show some steps for working out an equivalent formula using algebra and a little problem solving.

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Transcript

This example involves some tricky algebra – but it serves to show how we can often make deductions using things we already know in order to find unknowns. We know now that we can calculate (perhaps in a round about way), the area of any shape as long as we can break it down into triangles we know the lengths of. For polygons, this can usually be accomplished if we are given enough information, however it gets a little more difficult with circles.

Coding triangles

Can you create an Area function in Scratch that calculates the area based on the side-lengths a, b and c? Have a go and then watch the video showing how it could be done using Heron's formula.

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Transcript

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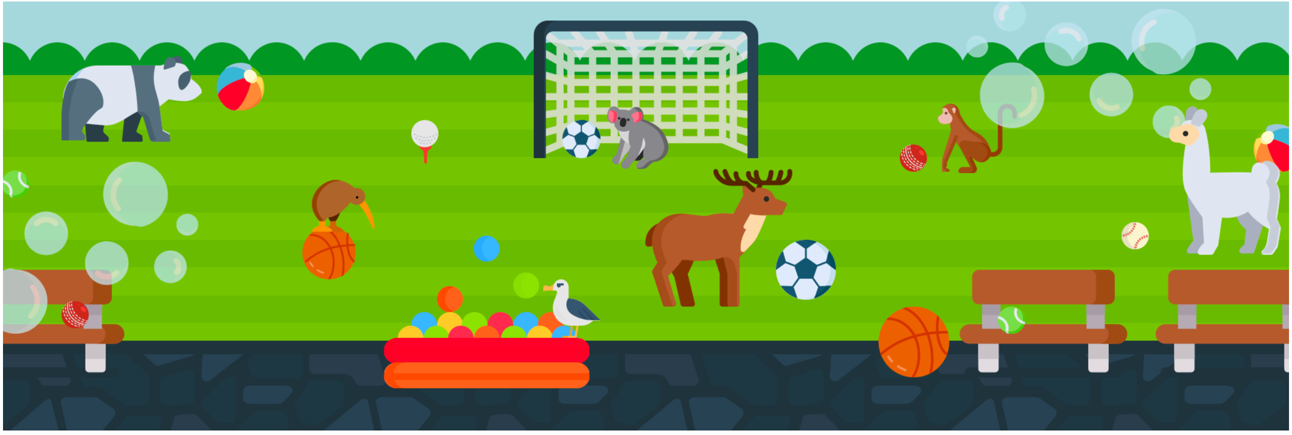
34. Circles



Mathematicians tried for centuries to figure out how to determine the length of a circle based on its radius or diameter.

Many methods for calculation were investigated, but a simple rule eluded them because the relationship needed to be described in terms of a new type of number, i.e. the irrational number π (pi). Today, we recognize π as the ratio of a circle's circumference to its diameter, equal to a value close to 3, but as you might recall, it's not a rational number (i.e. it can't be expressed as the ratio between two whole numbers). Pi was approximated to high degrees of accuracy, even thousands of years ago, and today we have the technology and computing power to express it to millions of decimal places of precision.

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Let's take a look at how we can use a similar technique in Scratch to approximate π.

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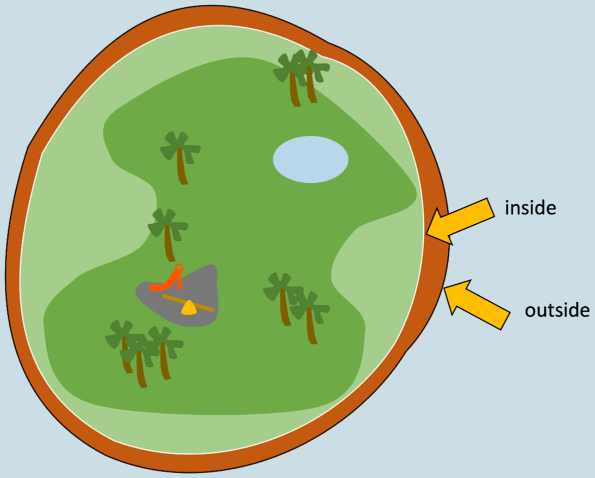
Transcript

Running track



There is a running track that goes around your local park that's meant to be 1km for one lap, however if we assume the track is roughly circular, you might be covering more or less than 1km depending on whether you're running on the inside or outside of the track. What would be the difference if the track width is about 1.5m?

Area of a circle

We might all now be well aware of the formula for determining the area of a circle: $A = \pi r^2$. However, it, too, can be a difficult thing to calculate with precision. To see how we can use π to approximate the area of a circle, it helps to think of the area of a circle in terms of triangles. The video below explains.

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Transcript

Quiz

The following quiz gives you practice for using the formulas related to the area of a circle.

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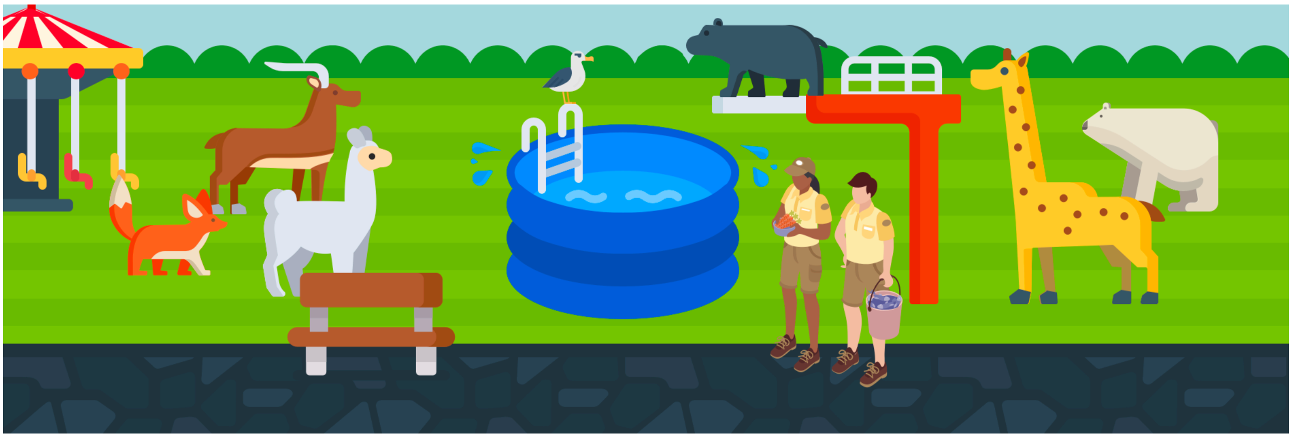
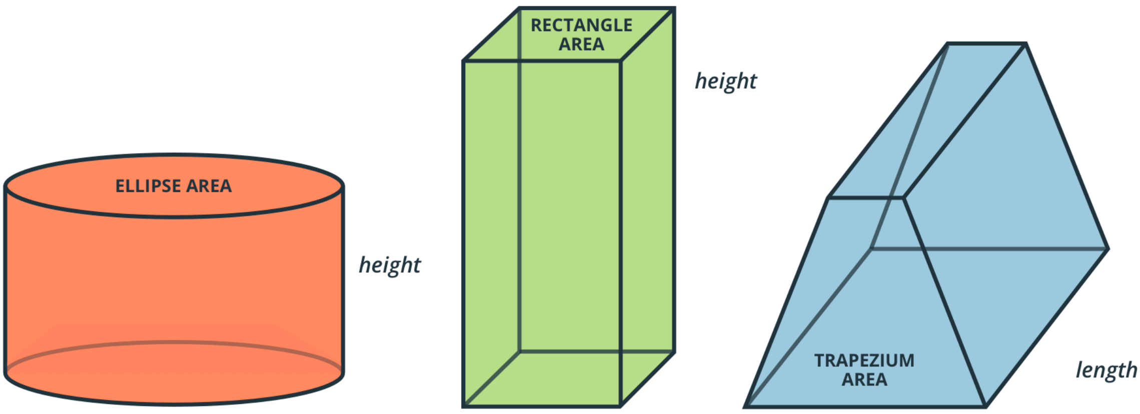
35. Volume



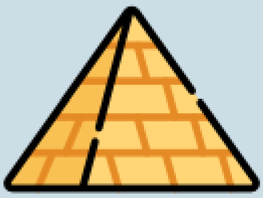
For volume, we are essentially extending the idea of area, except now we are considering the area for each step in height.

It might not seem obvious, but if the cross-sectional area is the same for any two shapes, then solids made by extending them upward will have the same volume too.



However there are some special shapes, too, such as cones and pyramids, which gradually get smaller as their size increases.

Determining the volume of a pyramid



Determining the volume of a pyramid

A stepped pyramid, with five layers, has a base length of 50 metres. Each layer decreases at a uniform rate, and the length of the top layer has a length of 5 metres..

Can you figure out a method for determining its volume?

- Hint: can you figure out the volume of one layer? Can you come to a reasonable conjecture about the length of each layer?

What if it had a straight, sloping edge rather than steps?

Once you've had a go at the activity above, take a look at the solution offered in the video below.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=484#oembed-1

Transcript

Quiz

Have a go at the following quiz which gives practice for working out volumes.

An interactive H5P element has been excluded from this version of the text. You can view it online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=484#h5p-34

Notes

Using the tool below, create a brief description of the formula for calculations that we have covered so far this week (plus the volume of a sphere).

Include an example with some reasoning about where the formula "comes from". If there are steps you're not sure of, check back over the week and update accordingly. Try to make your explanation as simple as possible.

If it's too tricky (explaining the area of a sphere can be pretty difficult), think of a way you might be able to remember it and differentiate it from different formulas.

Warning: the text input into the app below will not save if you close your browser or navigate away from the page. Please export your text to keep a copy.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=484#h5p-20

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PART VIII

GRAPHS AND NETWORKS

In this topic we look at some mathematical results pertaining to networks and graphs.

The term ‘graph’ here is a specialised term for a mathematical representation of relationships consisting of nodes and edges, rather than the way it is commonly used to just mean a diagram or a plot with axes. What we look at here also represents the starting point for topology, which can roughly be described as the mathematical study of whether or not things are ‘close’ or connected.

Learning outcomes

- Solve routine problems using Eular paths, Eular cycles, mimimum spanning trees and graph colouring.

36. Graphs and networks

In this topic we look at some mathematical results pertaining to networks and graphs.

The term 'graph' here is a specialised term for a mathematical representation of relationships consisting of nodes and edges, rather than the way it is commonly used to just mean a diagram or a plot with axes. What we look at here also represents the starting point for topology, which can roughly be described as the mathematical study of whether or not things are 'close' or connected.

Throughout the topic, you'll learn how to solve routine problems using Eular paths, Eular cycles, mimimum spanning trees and graph colouring.

Let's start with one of the classic problems in the discipline of topology: the Königsberg bridge.

The Königsberg bridge

This problem is a good reminded that sometimes it's impossible to achieve the goal of the problem. In these cases, it's left for us not to 'answer' the question, but instead to show that it's impossible.

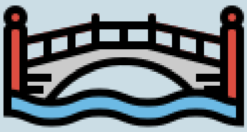
Let's take a look at the problem.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=32#oembed-1

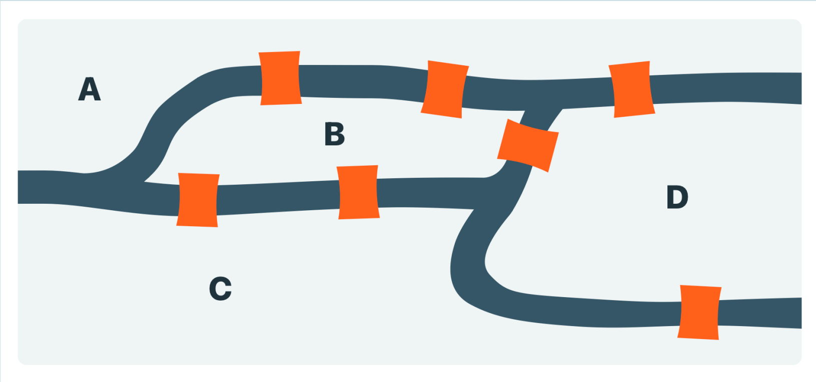
Transcript

Or if you'd prefer a written expression of the problem:

The Königsberg bridge



The image below represents a map of the bridges of the city of Königsberg in the 1800s.



How could someone walk through this city, crossing each bridge once and only once?

Remember the entry, attack and reflection phases as you work through the problem.

If you enjoy the problem, there are several [more topological problems](#) developed using Mathigon.

In case we hadn't made it clear enough already, there is no way to cross each bridge once and only once. However, *attempting* to solve the problem can still help us develop valuable insights and methodologies. The video below shows some potential methodologies for attempting to solve the problem.

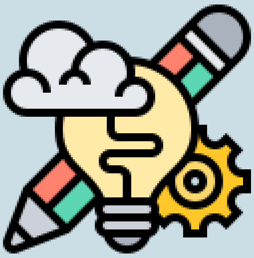
One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=32#oembed-2

Transcript

Recall that often in problem solving, the problem is not the real problem. A very limited number of people would actually care about whether or not it is possible to take a walk in Königsberg without recrossing a bridge, what we really care about is whether there are any general results that seem to apply in this framework. Which arrangements of bridges and regions will allow us to determine such a path? Is there a way we can tell quickly?

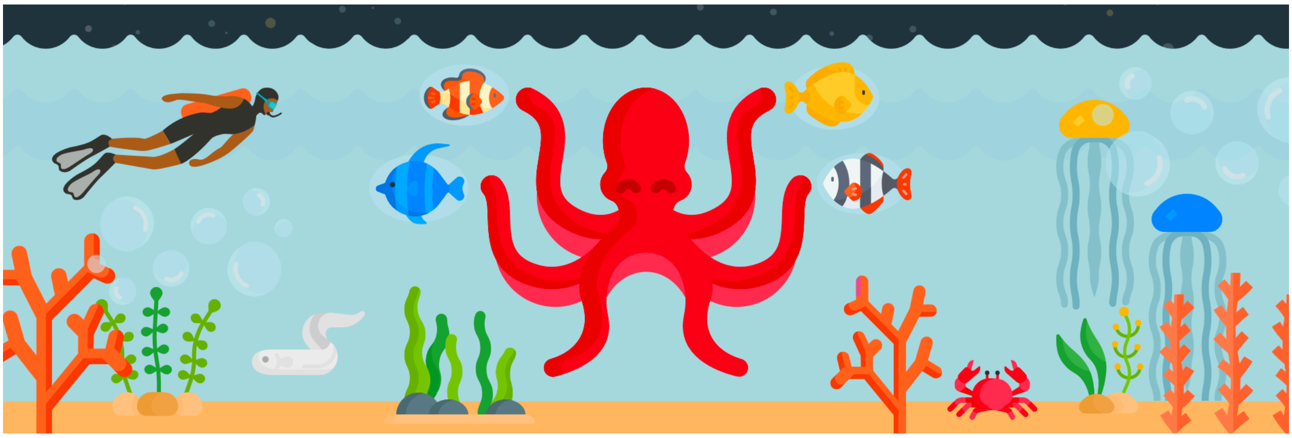
Let's practice the important process of generalising, to see if we can get a good feel for what's involved in problems of this kind.

Generalising from the Königsberg bridge problem



What are the important aspects of the Königsberg problem that make it impossible to find this kind of walk? Use some problem solving strategies to reflect: e.g. can you add a bridge that makes it possible to find the walk? Could you move a bridge that makes it possible to find a walk? Could you remove one bridge? How does the important information change with each setup?

37. Graph theory



Let's start with some terminology

The video will give you a good overview of some of the terminology we'll use over the week.

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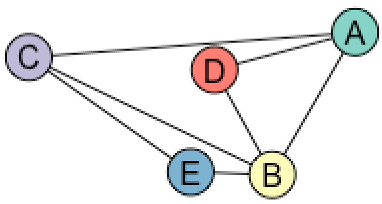
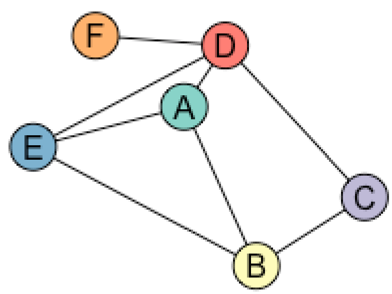
[Transcript]

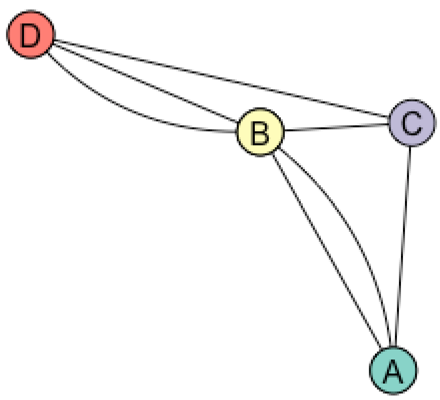
Let's start with a **graph.** A graph, in the context of 'Graph theory,' is a figure made up of points (referred to as **nodes** or **vertices**) connected by lines (**arcs** or **edges**).

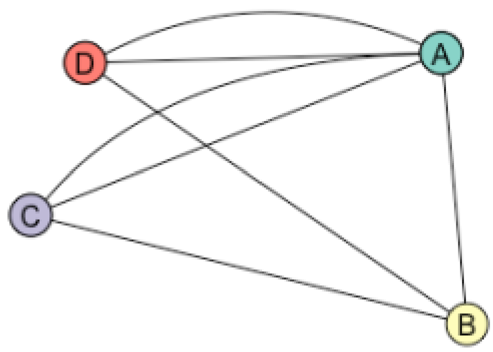




The left graph has 5 vertices, 7 edges and 4 'faces' (the regions enclosed by BCE, ABD, ACBD and the outside region counts too). The graph on the right has 6 vertices, 8 edges and 4 faces.

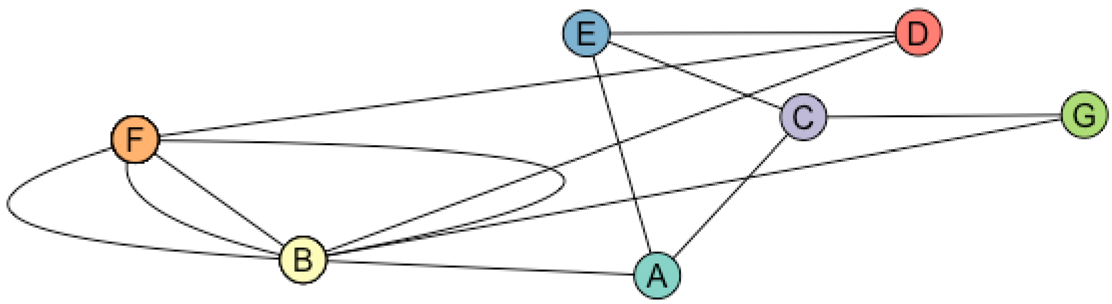
As mentioned in the introductory video, a key idea that differentiates graph theory from geometry is that often the actual position of a node, or the length of an arc is meaningless. The following graphs could both be used for representing the situation in the Königsberg bridge problem.





Even the labelling doesn't particularly matter – we just need to represent the middle region as a vertex, connecting twice to the vertices representing the upper and lower regions, then the remaining vertex (representing the right-most region) should connect once to each region.

The 'degree' is the number of edges going into or out of a vertex. In the graph below, the degree of vertex A is 3 and the degree of vertex B is 7.



We can also count faces – as long as it's a graph that can be layed out so that no edges are crossing over one another (if it's possible, such graphs are said to be '**planar**'.)

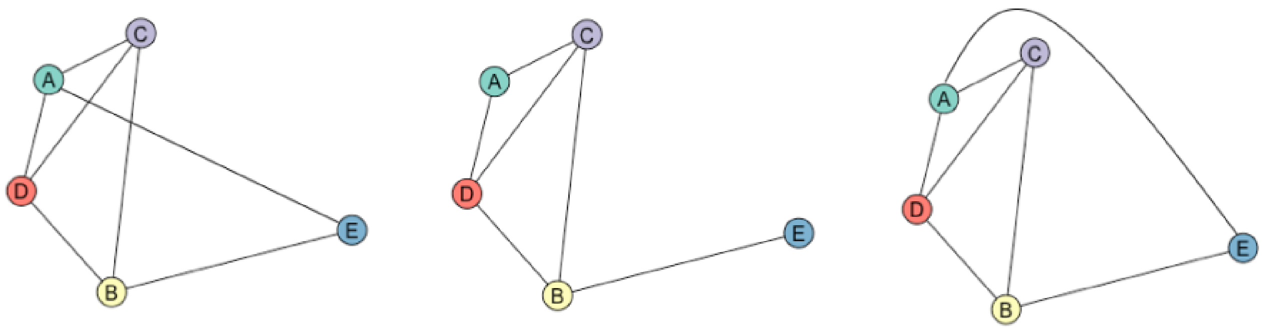


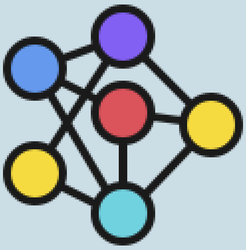
Figure 1 **Figure 2** **Figure 3**

It's hard to work out the number of faces from figure 1 because the line AE crosses some of the other edges. If we remove this line (figure 2) and redraw it so that it doesn't cross any edges (figure 3), we can then see that it has 4 faces – the regions enclosed by ACD, BCD, ACBE, and the region outside.

More important terms

- A **path** – a sequence of edges that may connect two vertices.
- A **cycle** (or sometimes '**circuit**') – is a path that starts and finishes from the same vertex, usually not allowing backtracking (e.g. a path ADA in the above 3 graphs would not usually be considered a cycle)
- **Connected** – a graph is connected if there is a path from any vertex to any other vertex in the graph
- **Isomorphic graphs** – two graphs are isomorphic if they have the same number of vertices and if the edges correspond exactly. Usually this is independent of the labelling – but a common way of describing isomorphisms is that one graph can be obtained from the other by a continuous "morphing" – moving of the vertices and/or bending of the arcs.

Planar graph



Can you draw a connected planar graph with 6 vertices and 8 edges?

How many faces does it have? Is it possible to draw it differently so that it has fewer or more faces?

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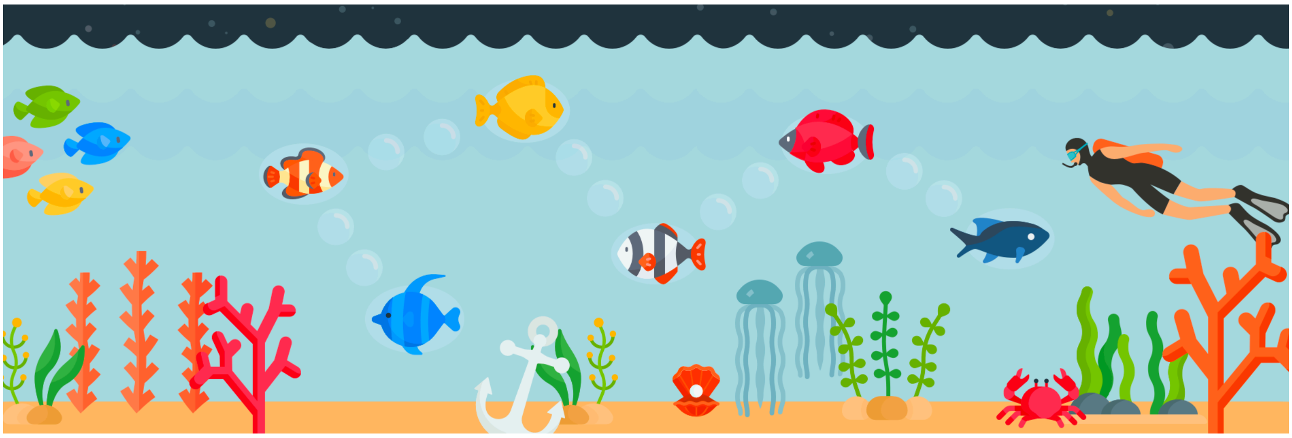
38. Euler paths and Euler cycles



In the normal definition of a path, there's no restriction on the number of times we can use an edge or how many times we can pass through a vertex.

The kind of path that arises from the restriction in the Königsberg bridge problem has been called "An Euler Path" (after Euler). So we can summarise the following definitions:

- An **Euler path** is a path where every edge of the graph is used exactly once.
- An **Euler cycle** (or sometimes Euler circuit) is an Euler Path that starts and finishes at the same vertex.

Euler paths and Euler circuits have no restriction on the number of times a node can be used, e.g. in the bridges problem it was no problem that we pass through the central region any number of times (in fact, we need to in order to cross all the bridges), however there is also a special path called a "Hamiltonian path" which instead has the restriction that we can pass through each vertex only once: for example, if you were to describe a sequence of trips around Australia where you visit each of the capital cities only once.

We'll now consider Euler's solution to the Königsberg bridge problem and the theorem that allows us to solve any similar problem involving graphs.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=512#oembed-1

Transcript

The following video gives some examples for finding Euler paths.

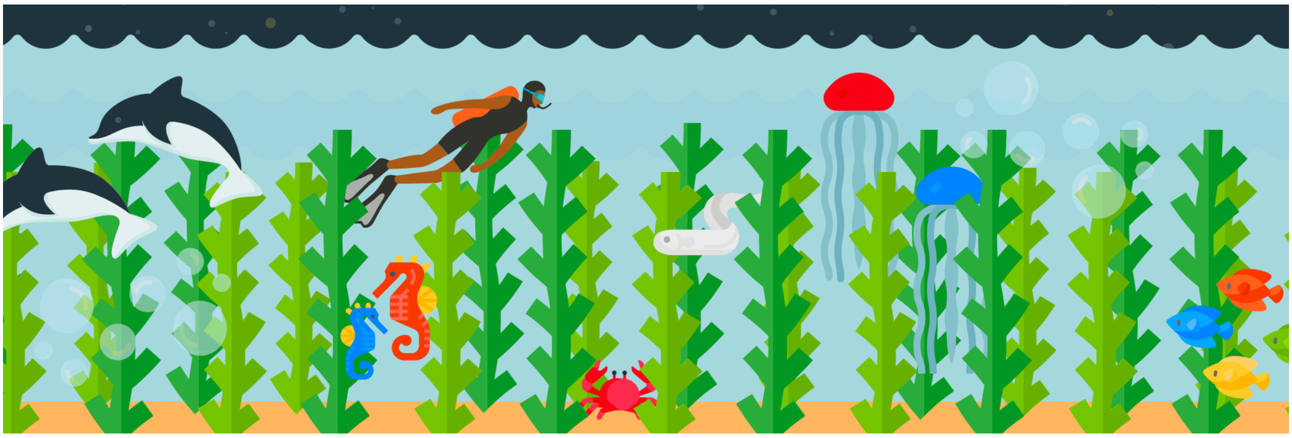
One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=512#oembed-2

[Transcript](#)

Explain it in your own words

Give a description of the Königsberg bridge problem (include a diagram) and your own explanation of why it's not solvable so that someone who'd never studied it before could understand. If there are things you don't remember, go back over the notes and consolidate your description, simplifying the language wherever possible.

39. Minimum spanning trees



The areas of Graph Theory and Topology are huge – mathematicians can spend their lives devoted to working on specific sets of problems.

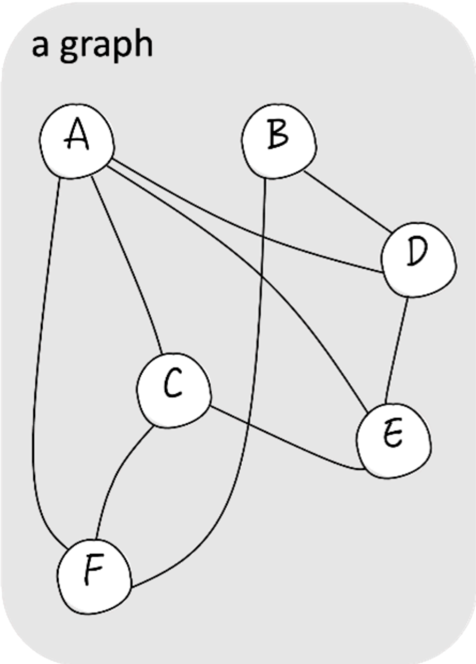
We'll now look at a couple of common problems, how to represent them with graphs, and how to solve them. The first is the problem of finding a minimum spanning tree.

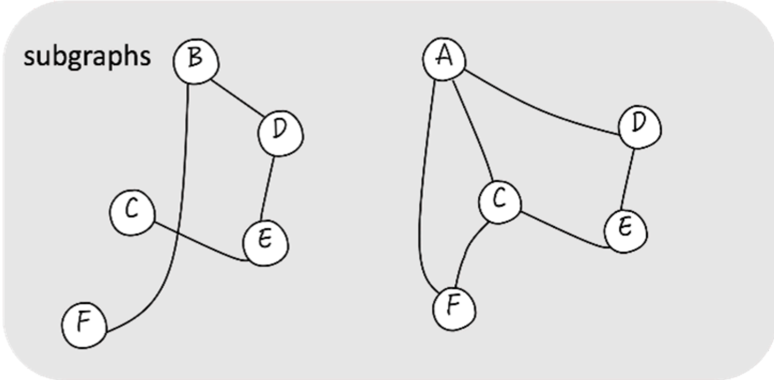
One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=516#oembed-1

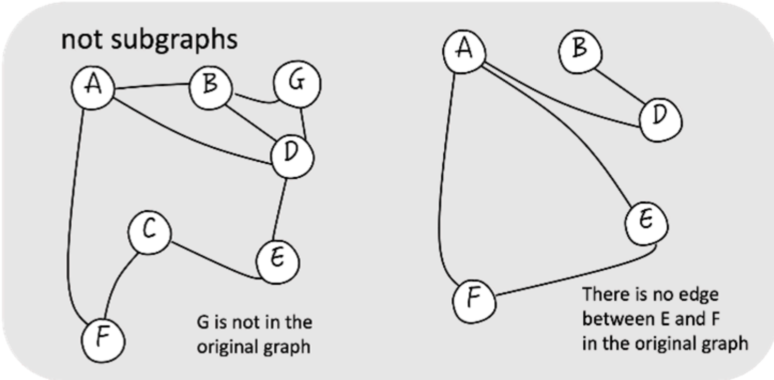
Transcript

A **tree** in graph theory is usually defined as a **connected** graph with **no cycles**. Recall that when we think of cycles, we don't usually allow back-tracking. We could also think of trees as connected graphs with only one face (the "outside" region). There's even the concept of a forest in graph theory – but we won't consider these.

A spanning tree is a special type of **subgraph**, which means that we need a graph to start with – we can't just say that a graph is a 'spanning tree' without referring to that original starting graph. Starting with any graph, a subgraph is a graph that can be constructed using any of its existing edges and vertices (see figure below). If we include an edge, we need to make sure that the vertices connected by that edge are included too.







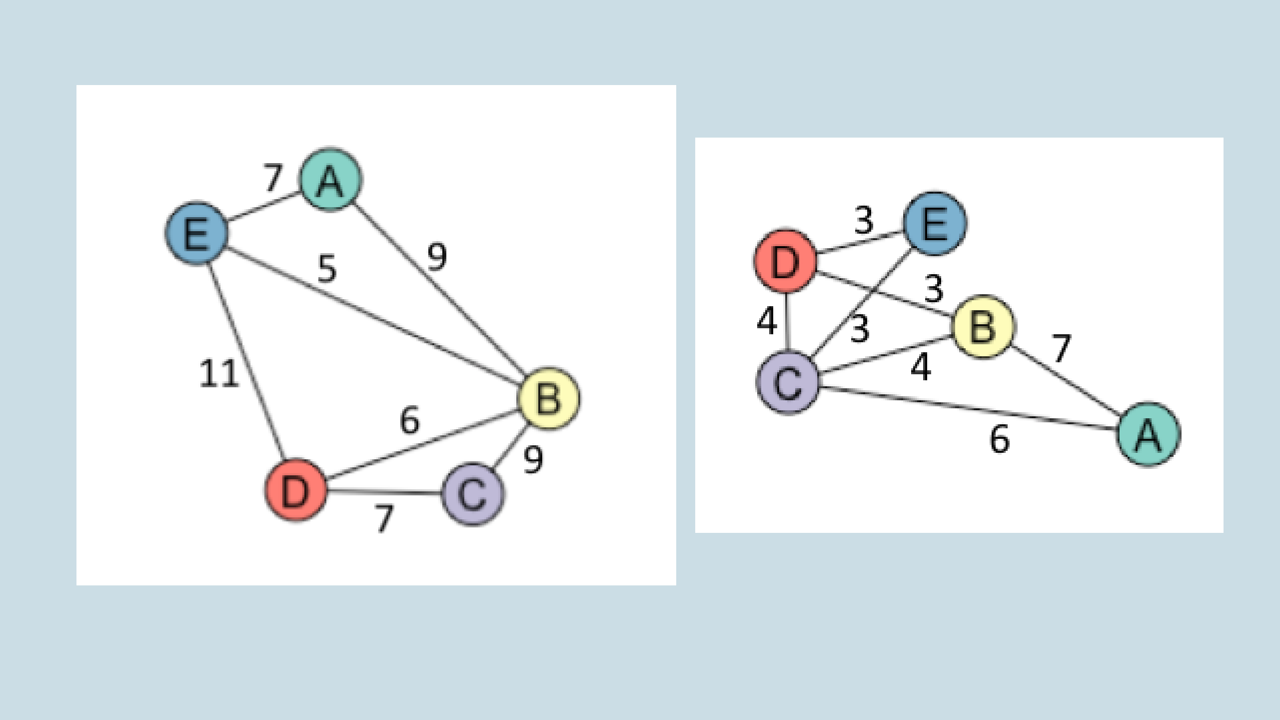
So now, we can define a **spanning tree** as a *subgraph* that includes *all vertices* of the original graph and is also a *tree*.

In applications like what was shown in the video, we are interested in finding the minimum spanning tree, where the aim is to find a spanning tree that minimises the sum of edge weights.

Minimum spanning trees

Can you determine the minimum spanning trees of these graphs?

Can you come up with a method of determining the minimum spanning tree of *any* graph, no matter the weighting or number of edges and vertices?



Useful algorithms

Once the number of edges and vertices grows large, it becomes harder to be sure that the spanning tree we find has the smallest weights possible. We need to approach the selection of the edges logically and methodically. Here are two potential algorithms that might help us achieve this. Both begin with a graph as an input, and output the minimum spanning tree of that graph.

Algorithm 1: Build from the least expensive	**Algorithm 2: Eliminate from most expensive.**
1. List all of the edges in order from least expensive to most expensive. 2. Consider each of the edges in order. For each edge, if using the edge does not form a cycle, use it, otherwise, discard it. After following this sequence you should be left with a spanning tree.	1. List all of the edges in order from most expensive to least expensive. 2. Proceeding through the ordered list, for each of the edges, if discarding the edge would not result in the graph becoming disconnected, discard it, otherwise, keep it. After following this sequence you should be left with a spanning tree.

Algorithm 1 is also known as "Kruskal's Algorithm" and was first published by Kruskal (1956). Algorithm 2 was also proposed by Kruskal in the same paper, but is more commonly referred to as the "Reverse-delete algorithm".

The video shows these algorithms in action.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=516#oembed-2

Transcript

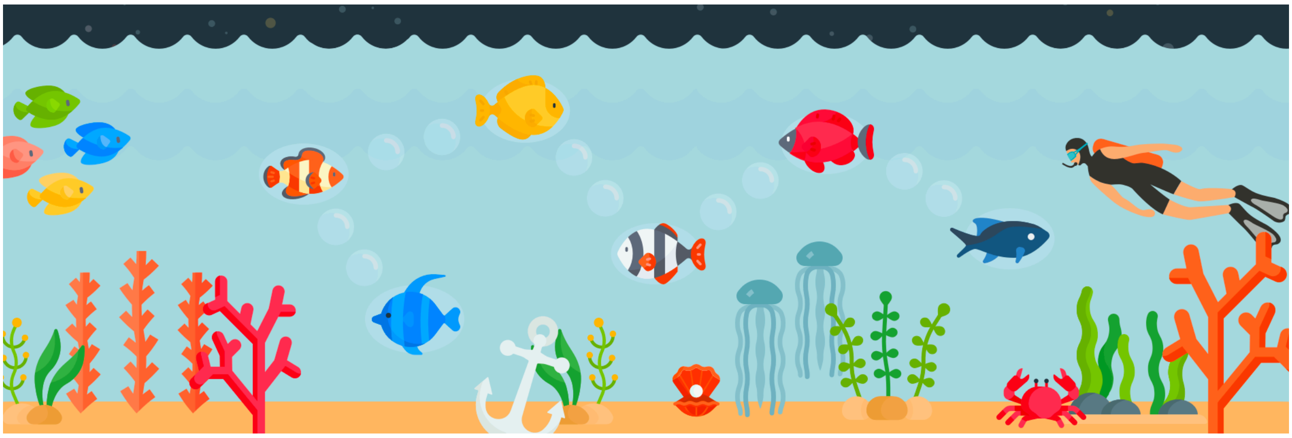
Generalise

How many edges will there be in a tree (with respect to the number of vertices?)

References

Kruskal, J. B. 1956. On the shortest spanning subtree of a graph and the traveling salesman problem. *Proceedings of the American Mathematical Society*. 7 (1): 48–50. doi:10.1090/S0002-9939-1956-0078686-7.

40. Graph colouring and chromatic numbers



If you had to give every vertex that shared a connection a different colour, how many colours would you need?

Another important problem in graph theory is that of finding the chromatic number of graph – or the minimum number of colours required to assign a colour to every vertex such that no adjacent vertices are the same color.

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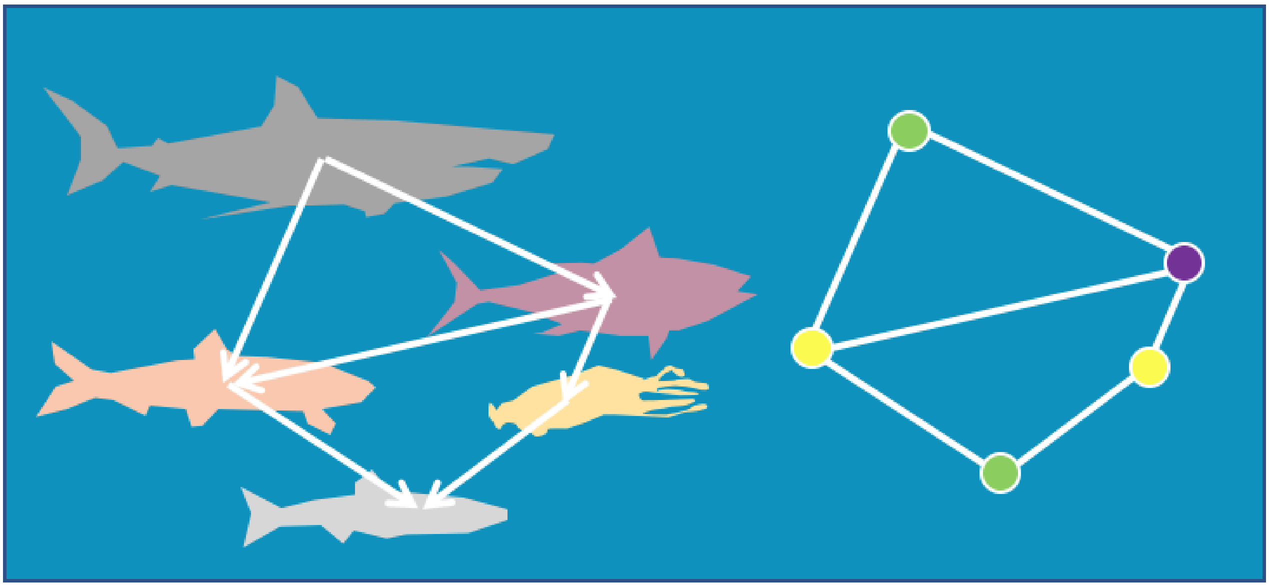
Transcript

Although the problem of the chromatic number arose from colouring maps, it can actually be applied to quite different problems and situations. Suppose we have a number of fish, and we have some knowledge about whether some of the fish eat the other ones, or shouldn't be kept in the same tank. Well the chromatic number tells us how many tanks we need so that we can keep the fish away from their predators, and the colouring tells us how to assign those fish to the different tanks.



However, it can be difficult to find an optimal colouring of a graph. Once a graph is large enough, we could proceed in colouring from vertex to vertex but not be quite sure whether we've done it in the best way or not. Fortunately, there's an algorithm that can give us a systematic way of proceeding. It isn't guaranteed to obtain the best colouring, but it is usually pretty good, and provides us with another example of where a solution method is described as an algorithm rather than a rule (Welsh and Powell, 1967).

The Welsh-Powell Algorithm

Input: A graph

Output: A colouring of the graph that should be close to optimal

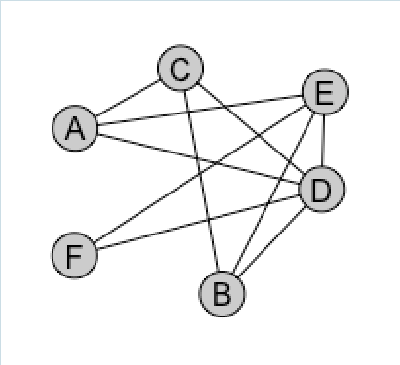
Step 1. Order the vertices of the graph according to their degree in decreasing order.

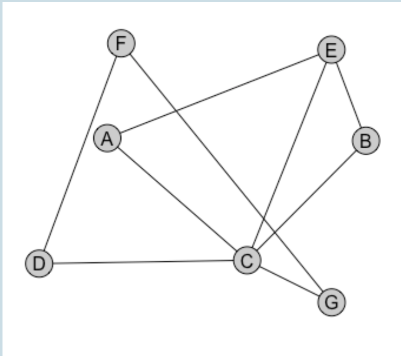
Step 2. For each colour in {Colour 1, Colour 2, Colour 3…}, proceed through each of the vertices in order, if the vertex hasn't been coloured yet and it is not adjacent to a vertex of the colour you're using, assign that colour, otherwise move onto the next vertex. Once you've run out of vertices, move to the next colour and start again.

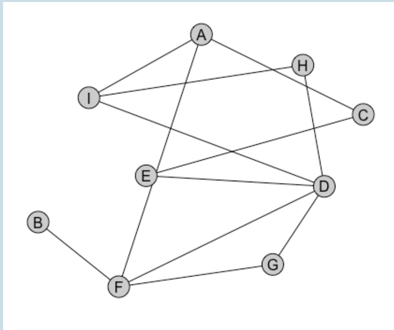
You should end up with a complete graph colouring. The number of colours you used should be the chromatic number.

Applying the algorithm

See if you can apply the Welsh-Powell algorithm for the following graphs to find an optimal colouring.







Watch this online video for an example of applying the Wesh-Powell algorithm, once you've had a go for yourself. Take a look at some more worked examples, too, if you're interested.

References

Welsh, D. J. A. and Powell, M. B. 1967. An upper bound for the chromatic number of a graph and its application to timetabling problems, *The Computer Journal*. 10 (1): 85–86. doi: 10.1093/comjnl/10.1.85

PART IX
PROBABILITY

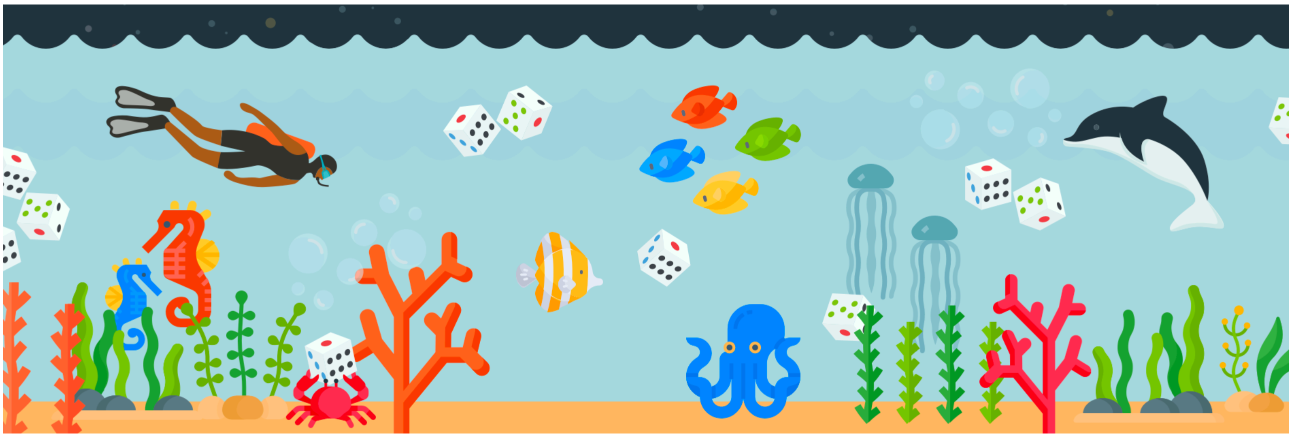
What methods do you use to calculate (or guesstimate) how probable something is to occur?

The more developed field of probability that underlies modern statistics and practical mathematical research traces its origins back to an apparent contradiction that arose in calculations about a dice game in 1654 that called on the expertise of two brilliant mathematicians, Pascal and Fermat. One of the curiosities of probability (that makes it particularly interesting) is the number of results that do seem to be contradictory or that are counter-intuitive. It's also what makes reasoning about probability important, because being naïve to some of these results makes us vulnerable – not only to things like gambling, but also to reasoning that is used by political figures about economic strategies, human influence on climate change, evidence inferring someone's guilt and so on. We will be focusing on some basic ways to construct probabilistic arguments and summaries.

Learning outcomes

- Describe and apply methods for developing an intuitive sense of the probability of events occurring
- Apply tree diagrams and Venn diagrams to describe the probability of events concurring
- Solve non-routine problems involving probabilities

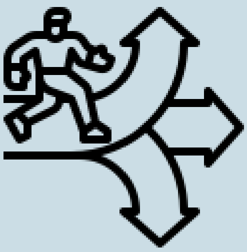
41. Estimating probability



When estimating the chance of something occurring, most of us make estimates based on our intuition and prior experience.

For example, when it's cold and cloudy we expect the chance of rain to be higher than when it's blue skies as far as we can see. So let's start by having a look at a few scenarios that effectively represent a guess.

Likelihoods



Have a think about what the probability is of the following events occurring. In your mind, rank the likelihood from 0 (not possible) to 10 (it will definitely happen). Try to think of the reason you'd pick that rating as you go.

- Your favourite sports team winning the championship next year
- Having twins
- Rolling 3 6s in a row on a dice
- That the USA will have a female president within the next 10 years

The language of probability

One or more interactive elements has been excluded from this version of the text. You can view *them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=573#oembed-1*

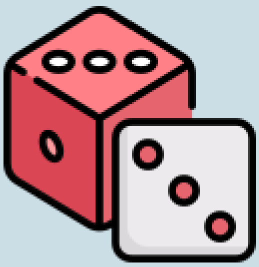
[Transcript](#)

It'll be easier to have a discussion about probability if you're familiar with a few common terms. The following terms were emphasised in the above video:

- **Event** – a description of something that can be said to happen or not – e.g. flipping a head on a coin, a train arriving on time. The flipping of the coin itself is not an event – unless we're concerned with the probability of someone actually deciding to flip a coin.
- **Trial** – usually used in simulation – an opportunity for an event to occur. If we're interested in the probability of flipping a head, the trial would be the flipping of the coin.
- **Sample space –** the set of potential events that can occur – e.g. if you flip a coin, the sample space would be {head,tail}, whereas if you roll a die, the sample space is {1,2,3,4,5,6}.
- It also discussed **gathering data**, generating data through real simulation or **computer simulation**, and **theoretical calculation**.

Doubles

Some of the questions below can be difficult. Don't spend more than half an hour all together on these.

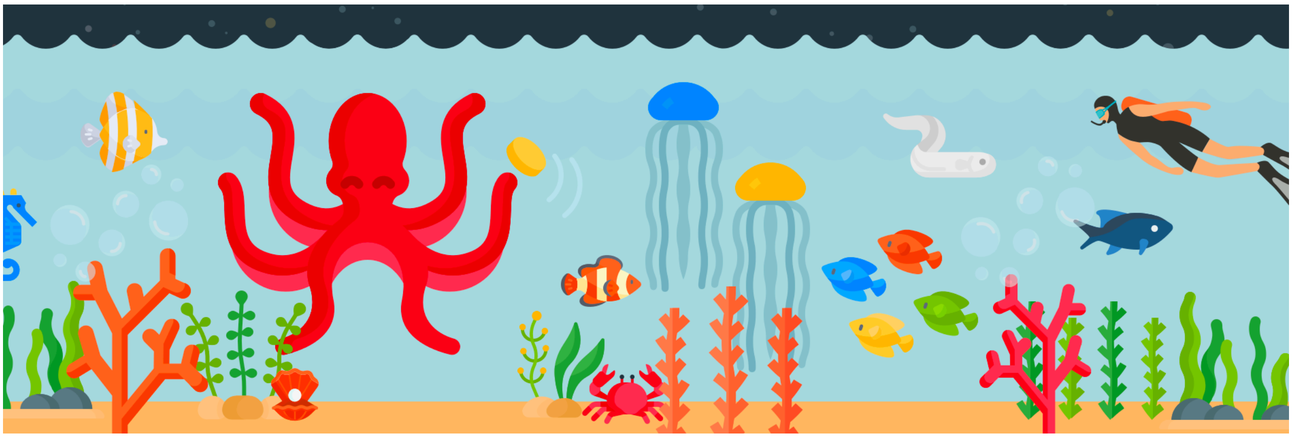


You have a pair of standard, six-sided dice.

- What is the probability of rolling a 3?
- What is the probability of rolling doubles (of any number)?
- What is the probability of rolling two doubles of any number in a row? (eg, double 1 and then double 5 would be acceptable).
- What is the probability of rolling two double 3s in a row?

See [image rights and reuse information](#)

42. Success fractions and sample space



Let's take a look at a few counter-intuitive examples

Consider the table of 1973 applicants for University of California, Berkely.

	Applicants	Admitted
Males	8442	3738
Females	4321	1494

To consider what this means, and how we might interpret it, we should first take a look at the concepts of success fractions.

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Transcript

Take a quick look again at the table above. Consider the 'probability' of being admitted based on being male, each application is like a 'trial' and each admission a 'success', so that we have a 44% chance of success for males, compared with 35% for women.

Of course, it feels like there is something fundamentally different when we think of events like tossing a coin, which seem to be governed by 'real' chance, to things like sports matches, job applications or meal preferences, where it's more the concept of frequency.

Actually, in the Berkely example, this difference can be attributed more to the preferences of females and males, because a higher proportion of males applied for courses that had higher probabilities of success. For example, suppose the numbers looked like this:

	Applicants for courses that are easy to get into	Successful applicants	Applicants for courses that are hard to get into	Successful applicants
Males	150	75 (50%)	50	10 (20%)
Females	50	25 (50%)	50	10 (20%)

If we take the success fraction of males and females as groups, this would result in $\frac{85}{200} = 42.5\%$ success for males and $\frac{35}{100} = 35\%$ success for females – even though they had the same success rates when broken down. This is called Simpson's paradox, and is an example of how tricky reasoning about correlation can be. These hidden breakdowns could also be going on in other probabilistic situations.

Sample space

When trying to determine theoretical probabilities, the key is often to determine the sample space – a set of possible outcomes that can occur. It's easiest when each of the outcomes is equally likely. For example, when rolling a die, the potential outcomes are that a 1, 2, 3, 4, 5, or 6 is rolled, and each of these possibilities is equally likely.

We can hence calculate the theoretical probability of an event based on this sample space. For example, consider the following examples.

Event	Outcomes resulting in success	Probability
Rolling a 2	2	$\frac{1}{6}$
Rolling an even number	2,4,6	$\frac{3}{6} = \frac{1}{2}$
Rolling a number greater than 3	4,5,6	$\frac{3}{6} = \frac{1}{2}$
Not rolling a 6	1,2,3,4,5	$\frac{5}{6}$

It's worth noting that different "events" can have the same probability of success.

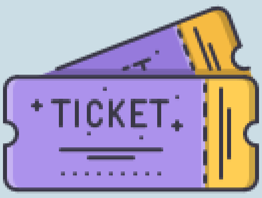
It becomes more difficult when we have multiple processes leading to an event. A simple example is when we extend the sample space associated with a single roll of a die to the rolling of 2 dice. We now need to think of all combinations that can occur with respect to both dice.

Let's be organized and represent this sample space using a table. The row indicates the number of the first die, the column indices the number of the second die.

	1	**2**	**3**	**4**	**5**	**6**
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

In this case there may be a number of ways of representing the 'sample space', depending on what we're ultimately interested in. If we are just interested in the possible outcomes of 2 dice, then the above table is sufficient for representing the sample space of the 36 equally likely outcomes. However if we're interested in the sum of the two dice, then the sample space could also be expressed as {2,3,4,5,6,7,8,9,10,11,12} , keeping in mind that in this case we'd need to be careful in making calculations because each of those sums is not equally likely. For example, looking at the table, there is only a $1/36$ chance of rolling a sum of 2 while there is a $6/36 = 1/6$ chance of rolling a 7.

The traveller and the ticket inspector



Suppose a forgetful man boards a train every day, but only remembers to validate his ticket 2/3 of the time. The train he boards has random ticket inspectors, who will board the train and check whether tickets are validated 1/6 of the time.

How could you represent the sample space of random events here? What are the different outcomes?

If we relate this to rolls of the dice, which numbers could we use to represent the forgetful man validating his ticket and which numbers (on the second die) could we use to represent the ticket inspectors boarding the train?

43. Tree diagrams and independent events

Tree diagrams come in handy when we've got more than a couple of events.

For describing sample spaces of 2 sequential events (like rolling 2 dice), we found a table useful, however once we are describing three or more sequential events, we might prefer to use tree diagrams.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=584#oembed-1

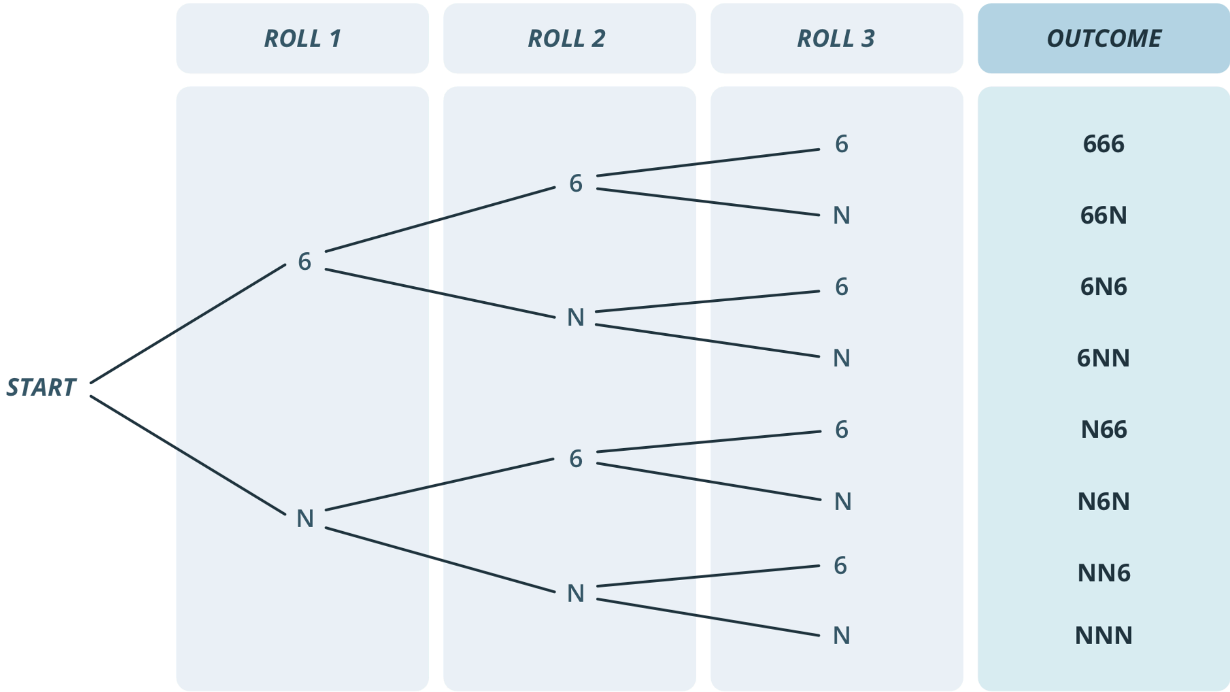
Transcript

The multiplication rule applies whenever we have sequential outcomes that are 'independent' of one another. Independence means that what happened previously has no impact on the probability, i.e. there is a 1/6 probability of rolling a 6 on the second die we roll, no matter what the roll was on the previous one. Similarly, when you flip a coin, it doesn't matter what happened previously (even if you have flipped 10 heads in a row), the probability of flipping a head on the next flip is still ½.

Multiplication rule for probabilities: $Pr(A) \times Pr(B)$

If we denote the probability of an event A occurring by Pr(A) and the probability of an event B occurring by Pr(B) then the probability of A and B occurring is $Pr(A) \times Pr(B)$

Let's have another look at the tree diagram for rolling a die 3 times.



In the video, we just worked out the probability of rolling NNN and subtracted this from 1 to work out the probability of at least one 6. However if we wanted to work out the probability of exactly one 6, then we'd need to consider the different paths that lead to this. We have 6NN, N6N or NN6.

For rolling 6NN we have the calculation:

$$\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{25}{216}$$

which will be the same as rolling a N6N or NN6 (since the order of multiplication doesn't change the result – but that means if we want to know the probability of rolling a single 6, we have to add these three probabilities together to get 75/216).

Free throws

Suppose a basketball player has a 5/6 chance of making each free throw. If each free throw is treated as an independent event (i.e., missing the first doesn't discourage her and make her more likely to miss her second), what would be the chance of her hitting 2 out of 3 free throws? Draw a tree diagram and calculate the total probability.

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44. Conditional probabilities

For an event to be called independent, the chance of one event happening must have no impact on the probability of another.

Similarly, a conditional event is one that depends on another event also happening. Sometimes these relationships will be quite straight forward (the chance of someone's eyes being brown bears no relationship to their scores on an IQ test). However, at other times, understanding whether or not something is independent can be a little trickier.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=588#oembed-1

Transcript

The result discussed in the video is so counter-intuitive, but ultimately it comes down to the rarity of the disease, and the framing of our thinking. In particular, the question that we needed to be asking ourselves was: Given I tested positive, what is the probability that I actually have this very, very, very rare condition? This is a key reason why, when we are taking a screening test, we're asked if there is a family history of the illness we're being screened for and often other questions that might make it more likely that a positive test result actually means we have the illness – it's important to avoid patients being anxious if there is a high probability of a false positive.

Causation, correlation and coincidence

One of the things that can be persistently tricky when using probability to investigate and analyse real world phenomena is the precise relationship between two events or variables. This often boils down to a question of whether something causes something else to happen or whether they just seem to go together (often because they have the same cause).

Applying direct heat onto ice will cause it to melt. However, there is only a **correlation** between a child's height and their performance on a standardised maths test intended for 12-year-olds. It is the child's age that causes both phenomena and so they are correlated. Occasionally, of course, two things that happen appear to be correlated but bear no real relationship to one-another.

In real world situations, it can be difficult to figure out what is a cause and what is a correlation. Take the example of life-expectancy. We know from many years of research that a person's genetics, income, level of education, 'health literacy,' job security, ethnic background and place of birth are all factors that impact on how long most of us have on this earth. None of these factors individually determine our life expectancy and clearly many of them are inter-related. Of course, it is also very important to keep in mind that there will always be individual variation – that is, some people can buck the trend and die much sooner or live much longer than expected. Many researchers treat these factors as a kind of group (in the context of medicine, you might have heard of the term 'risk factors').

At some point we may need to acknowledge that the question 'does this group of risk factors cause an increase or decline in life-expectancy or are they simply correlated' is difficult to answer with certainty. Although it should also be noted that there are some statistical inference methods and experimental methodologies aimed at doing just this.

Conditional probability

To determine conditional probabilities, it's usually best to think in terms of two-way tables. For example, what is the probability that a 6 has been rolled, given that the sum of two dice is 7 or greater? This is very different to asking what the probability of the dice summing to 7 or greater is, given that a 6 is rolled. We can organize the different outcomes into a table like this.

	6 rolled	**6 not rolled**
Sum greater than or equal to 7	(1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,1), (6,2), (6,3), (6,4), (6,5)	(2,5), (3,5), (4,5), (5,5), (3,4), (4,4), (5,4), (4,3), (5,3), (5,2)
Sum less than 7		(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (4,1), (4,2), (5,1)

There are 20 outcomes where the sum is greater than or equal to 7, and out of these, 10 involve a 6. So this means that the probability of a 6 being rolled, given that we know the sum is 7 or greater, is 0.5.

On the other hand, for independent events, one event should have no bearing on the other. So for example, the probability of rolling a 6 on the second die, given that the first roll was 6 is $1/6$, however the probability of rolling a 6 on the second die, given that the first roll was something other than 6 is also $1/6$. So sometimes we describe independence in terms of this conditional probability not being affected, especially when it is looking at events relating to real world phenomena or statistical data (e.g. the probability of someone having type O blood is independent of whether they are male or female, so we would say that blood type and gender are independent).

Independence based on conditional probability

If we denote the conditional probability of an event B, given that an event A has already occurred by $Pr(B|A)$ (read as the probability of B given A) the events A and B are considered independent if $Pr(B|A) = Pr(B)$.

If we have calculated our probabilities using experiments or from data, we could assume that two events might be independent if this formula holds approximately.

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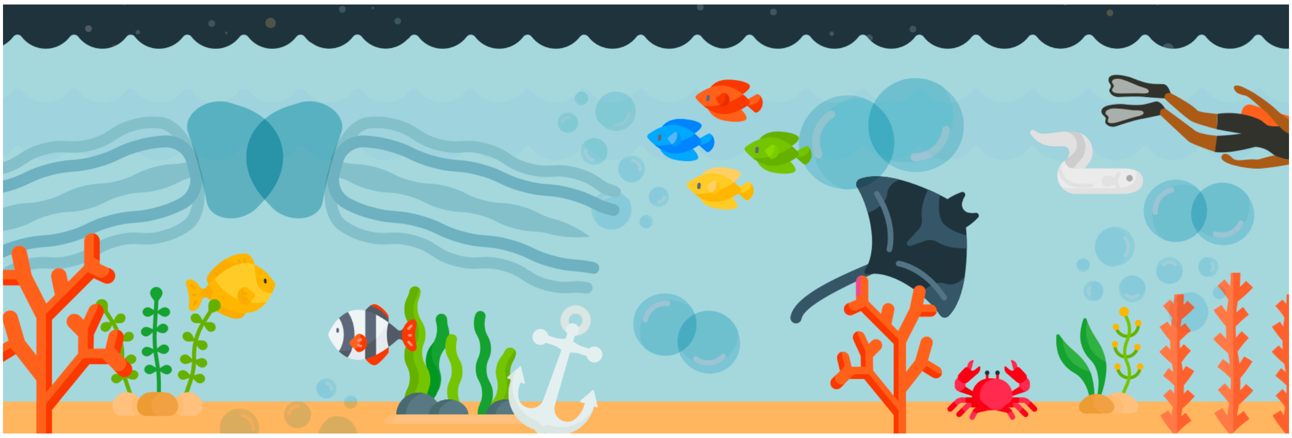
45. Venn diagrams



Venn diagrams help us understand the relationship between sets.

Venn diagrams are a nice and intuitive way of representing relationships between different sets. In probability, each circle will usually represent an event or set of outcomes.

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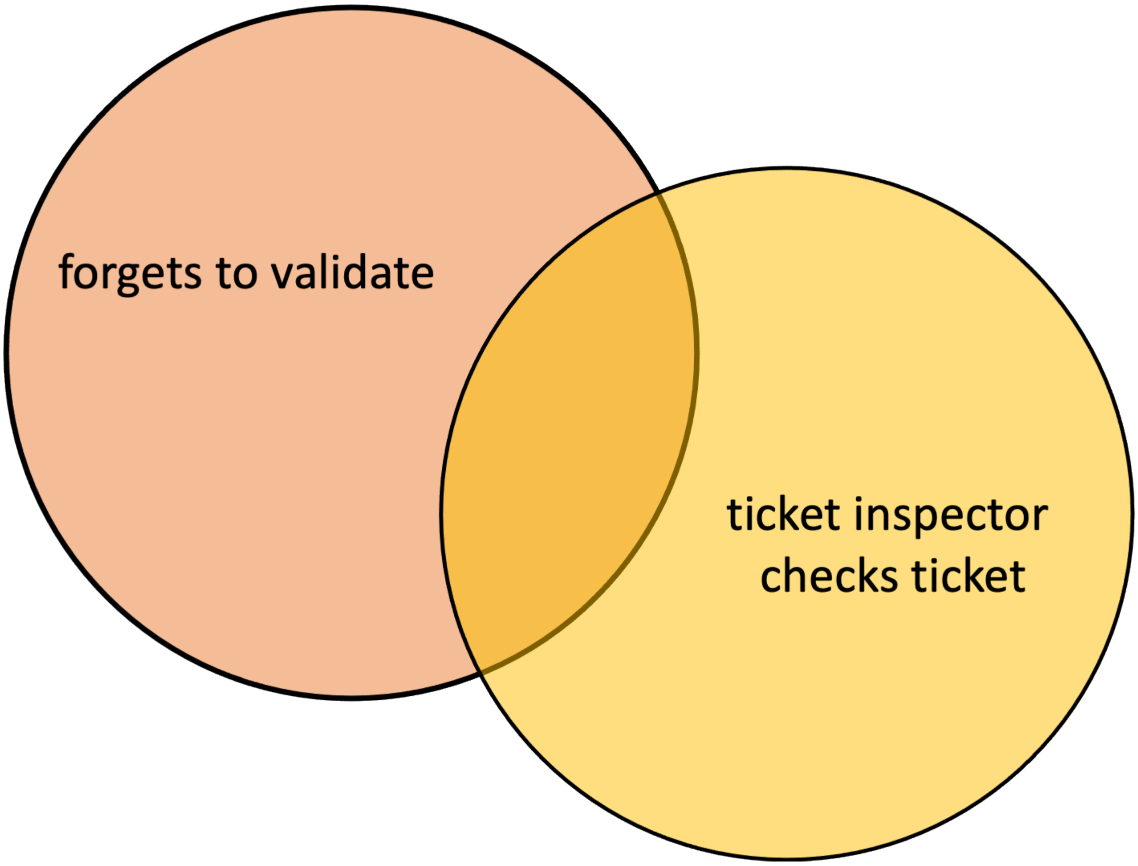
Transcript

Taking the example from the video, the set of odd outcomes if you roll a die is {1,3,5} (the curly brackets tells you it's a set). The set of outcomes greater than 2 is {3,4,5,6}. Using a Venn diagram, we could easily show the overlap between these two sets.

The overlap between the circles represents an outcome where both events occur, in this case, the overlapping region occurs when we roll a die and the number is both odd and greater then 2.

However sometimes the nature of the events and how they are represented using Venn diagrams can lead to misunderstandings.

Both circles in the above example relate to the same roll of a die. We might also use Venn diagrams to represent events that seem less related to one another. For example, the following might be used to represent the forgetful train passenger problem.



In this case, each circle represents the occurrence of an event. If both events occur (the overlapping region), then the forgetful man could be fined for not having a ticket. On the other hand, we should remember that the region outside both circles represents the man remembering to validate his ticket, and the ticket inspector not boarding the train.

Independent events

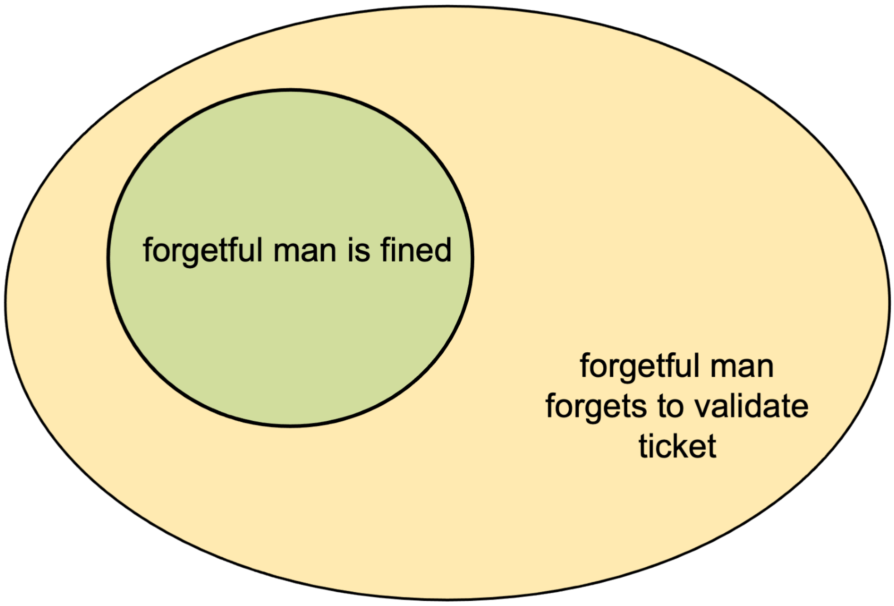
Recall that for independent events, the probability of one occurring doesn't have any impact on the likelihood of the other, and the probability of both occurring can be found by multiplying the probabilities of each. Venn diagrams can help us conceptualise these kinds of situations and we can use this rule to show or check whether the two events are independent.

In the dice example, the probability of the intersection, "both odd and greater than 2" is equal to ½ multiplied by 2/3, and so these two events can be considered independent, however the probability of "both odd and greater than 3" is not independent, because this would suggest that the chance of rolling a 5 is ½ (odd) multiplied by ½ (greater than 3) which is equal to ¼.

A common misconception arises when it comes to understanding how the concept of independence relates disjoint (or mutually exclusive) events or subsets.

In the case of disjoint subsets, if one occurs, then it is impossible for the other to also occur. If you roll an odd number, you won't be able to also roll a 4 (we are considering the same roll of the die). Disjoint events are hence never independent.

Similarly, two events where one is a subset of the other won't be independent either. The forgetful man forgetting to validate his ticket and the ticket inspectors boarding his train are two independent events, but the man being fined by the inspectors will be a subset of him forgetting to validate.



The probability that he will be fined is not independent of him forgetting to validate his ticket. If he forgets to validate, it has a direct effect on whether or not he will be fined (it increases the probability from 0 to 1/6). On the other hand, if he is fined, it means that the probability that he forgets to validate automatically goes to 1, since he can't be fined if this doesn't occur.

Are free throws really independent?

Are free throws really independent? Suppose we collect data from 100 instances where a player took two free throws with the following results.

	Misses Second	**Hits Second**
Misses First	15	20
Hits First	25	40

Draw a Venn Diagram with one circle for the first free throw and one circle for the second free throw.

Are your circles disjoint? Or is either a subset of the other?

What is the probability of hitting the first free throw?

What is the probability of hitting the second free throw?

What is the probability of hitting both free throws? Is it the same as multiplying the two probabilities together?

PART X
SIMULATION

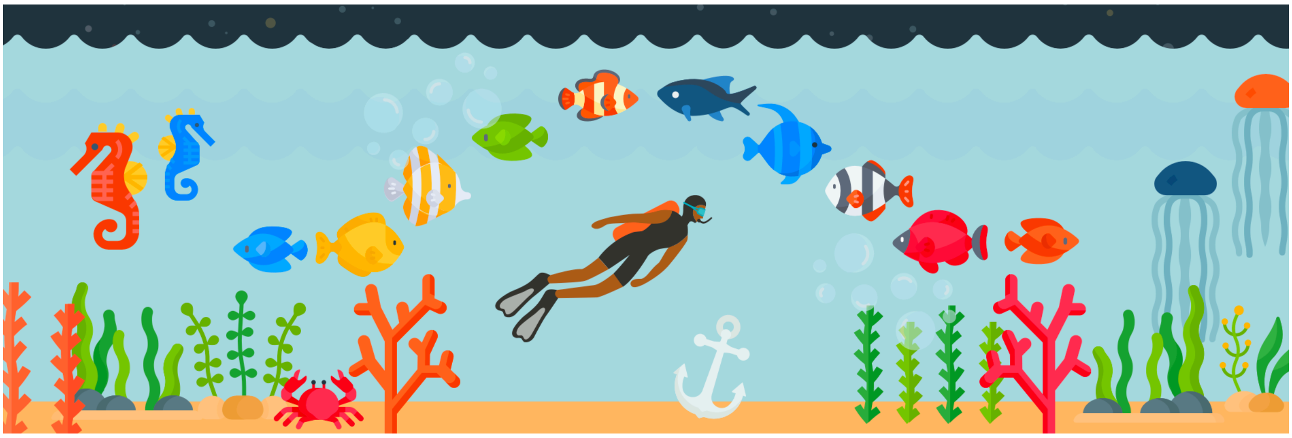
What is ‘normal’?

We now briefly look at the famous “Bell Curve” – the distribution that does very well in describing a number of natural phenomena and random processes. Understanding how this distribution works depends on a fundamental understanding of probability but also helps us understand some forms of randomness.

Learning outcomes

- Explore computer simulations
- Describe the normal distribution, standardisation and other distributions

46. Simulation



In this section we ask ‘what is normal?’

When we say ‘normal’ we’re talking strictly in the sense of numbers, not making a judgement about morals or values or any qualities whatever. Nonetheless, it’s important to understand what we *mean* by normal in mathematics so that the judgements we do make based on numbers are grounded in solid foundations.

Let’s start with a problem. Say you were pointed to a tree in early summer and asked to give your best estimate of how long a leaf is likely to be, if it were picked from the tree at random. Another way of asking that question might be ‘what is the normal or average length of a leaf’?

Leaves

Come up with a method for finding the average length of leaves on an individual, fully grown tree. Try to keep it plausible – that is, could you really do it within the space of an afternoon and without destroying the tree?

There is a huge array of different ways to do this. However you did it, it’s likely that your method involved measuring a sample of (or all of) the leaves, writing down how long these individual leaves were and figuring out from this information what the ‘average’ length of the leaves were. This would have involved some kind of calculation (to find the average leaf length, the calculation would be the sum of the length of all the leaves combined divided by the number of leaves in the sample, but there are other ways to do it too).

You might also have paid attention to the distribution of the leaves. This might help us understand how likely we are to find a leaf of an average length. What proportion of leaves were very short or very long compared to the rest of them? We might well discover that, while we’re more likely to find an average length leaf than any other, they’re still fairly rare. That leads us to another related concept: probability.

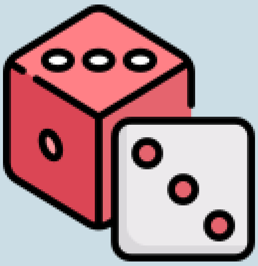
Probability

When the kind of distribution described above is graphed using the kinds of techniques we looked at in the section on interpreting visual information, it often creates the famous 'bell curve' shape. This distribution does very well in describing a number of natural phenomena and random processes. Understanding how this distribution works depends on a fundamental understanding of probability but also helps us understand some forms of randomness. For example, in schools, understanding how the normal distribution works is essential for interpreting student results and statistics from standardised tests.

We're feel comfortable and confident about asking the probability of a coin flip coming up heads or tails, or a rolling a 6 on a die. But in truth, most of our intuition starts to get a little bit shaky once we start adding a little complexity.

Games of change

What is the probability of a coin flipping sequence landing heads, then tails, then heads, then tails?



How likely are you to roll 3 sixes in a row on a six-sided die?

These problems aren't always easy, but thankfully, simulation can help. On the next page we'll look at the example of the Birthday Paradox to show you how.

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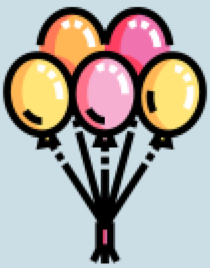
47. Computer simulation

Computers can help with more complex simulations.

While we can use dice, coins, counters in a jar, pieces of paper in a hat, or more sophisticated devices for running real simulations, these will be time consuming and impractical for obtaining accurate probability estimates. This is why research often involves computer simulation. By 'simulation' we mean imitating trials and events where the mathematical structure should be the same as in real life. We can use this to support theoretical calculations, to obtain results for probabilities where theoretical calculation is difficult (or impossible), or to predict probabilities when it would be time consuming to collect real data.

Let's consider the Birthday Paradox and write a program for modelling it. We'll start with the problem. Have a go at coming up with a method for answering it, but don't spend more than 15 minutes as we don't expect you to quickly find an accurate answer.

The Birthday Paradox



There are 23 people at a train platform, waiting for a train to come. What is the probability that at least 2 people in that group will share a birthday?

Before we get into modelling the probability with a computer, let's have a go at real simulation.

Simulation

Birth Month simulation – we'll do birth *month* because it means we don't need to find a deck of 365 cards!

1. Take a set of 12 playing cards (numbered 1 – 10, then include a jack and a queen for 11 and 12). We'll use these to represent the probability of being born in a particular month (and we're approximating these probabilities as 1/12 each, even though there might be fewer people born in February).
2. Draw 5 cards (replace the cards in the set of 12 each time you draw a card and reshuffle) and write down the sequence of 5 numbers. This represents one trial. If you drew the same number twice, write "success" next to the 5 numbers.
3. Repeat the trial 9 more times (10 in total).
4. How many times did you have a success? What's the experimental probability you obtained for any group of 5 people having two people born in the same month.

Now let's have a go at writing code to program the set of birth months.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=530#oembed-1

Transcript

In this case, Scratch allows us to simulate 100s of 1000s of trials in seconds, so we should be able to get a good estimate on the probability. You can take a look at the Scratch code and then 'look inside' to change the parameters and adapt it so that it estimates the probability of two people sharing a birthday if you have a group of:

- 10 people
- 20 people
- 30 people
- 23 people

You can increase the number of trials if you want to try and make your estimate more reliable. With enough trials, running the Scratch code each time would have yielded a slightly different result. How can we be sure that the simulation we ran is reliable? We can run multiple simulations – and in this case if the number of trials is high enough, the results might not vary much.

Calculating the exact probability

Although programming is often used to estimate probabilities that are too difficult (or even impossible) to calculate theoretically, algorithms that simulate probabilistic scenarios can also be used to validate or test a known result. You might already have some idea as to how you might calculate the theoretical probabilities for the birthday and birth month problems – the resource below gives an overview, as well as some more code that you could use to do it for you.

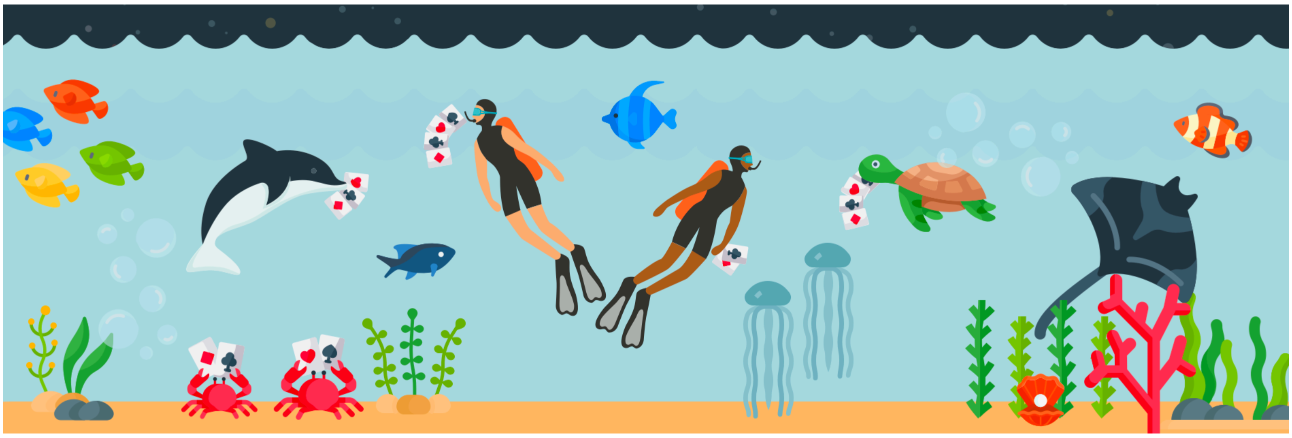
Take a look at Calculating the theoretic probability for the birthday paradox to understand the calculations behind the birthday paradox.

In the previous example, we knew how to model the probability of a person being born on any particular day. Can you imagine the probabilities that we'd need to take into account for the simulations described below?

Asteroid trackers from Spain have upgraded the chance that asteroid 1999 RQ36 could hit our planet, saying it now has a **one-in-a-thousand chance** of impacting the Earth in the year 2182. Previous estimates gave a **1 in 1,400** chance that this asteroid could strike Earth sometime between 2169 and 2199. Currently, however, NASA's Near Earth Object website gives between a **1 in 3,850** and a **1 in 3,570** chance that 1999 RQ35 could potentially impact Earth on Sept. 24, 2182. To make everyone breathe a little easier, that's a **99.97200000%** chance the asteroid will completely miss the Earth.

Nancy Atkinson, "Researchers Say Asteroid Has 1 in 1,000 Chance of Hitting Earth in 2182", Universe Today, 27 July 2010, http://www.universetoday.com/69640/researchers-say-asteroid-has-1-in-1000-chance-of-hitting-earth-in-2182/

48. Probability intuition

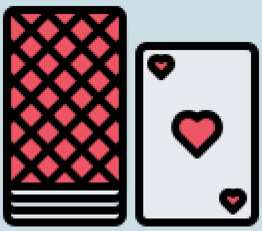


The following two activities will test your probability intuition and ability to justify your probability estimate.

You might find it useful to use a tree diagram or other theoretical calculations to support the result you obtained through simulation.

Think about whether there is a difference between your intuition (what you expect to happen) and your probability estimate? Is there anything else that needs to be taken into account?

Trick cards



In a game of chance there are three cards:

- a trick card with card-backs on both sides,
- another trick card with 2 of hearts on both sides
- a normal card with 2 of hearts on one side and a card-back on the other

The dealer shuffles the cards and shows you a card-back. What is the probability that the other side is the 4 of hearts?

You can use the following Scratch program to get a feel for how the game works.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=599#h5p-23

1. Play the game at least 10 times and note the number of times a card-back was shown at first, and then out of those, how many times the other side was a card back.

2. Can you draw a tree diagram illustrating what is happening as the card is drawn and then flipped over?

Disorderly plane seating

A plane has its tickets organised so that the first person in line sits in seat 1, the second person in line in seat 2, and so on.

However today the first person in line is crazy, and so chooses a random seat. If everyone following him sits in their assigned seat if it's empty, or a random seat if it's not, what are the odds the last passenger can sit in his or her correct seat?

1. Try the program out at least 10 times, and note the number of times the last person got to sit in the last seat.

An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=599#h5p-24

2. Can you draw a tree diagram illustrating what happens? Try it for the case of just 3 passengers.

Creating a computer simulation for the trick card problem (simulating sequential events)

In the card trick problem, the dealer first draws a card and then shows a random side. The outcome of the first event (drawing a card) alters the probability of outcomes for the sequential events. We already know how to model such situations using tree diagrams, and that will be an important strategy when it comes to looking at the theoretical probability. However, it's also useful to model these scenarios using a simulation so that we can investigate and test our theoretical calculations. A simulation can also give us an intuitive feel for what happens.

The following video goes through setting up a simulation of the card-trick problem in Scratch.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=599#oembed-1

Transcript

You can see the code at the project page. Note that you could have a different structure or slightly different probabilities.

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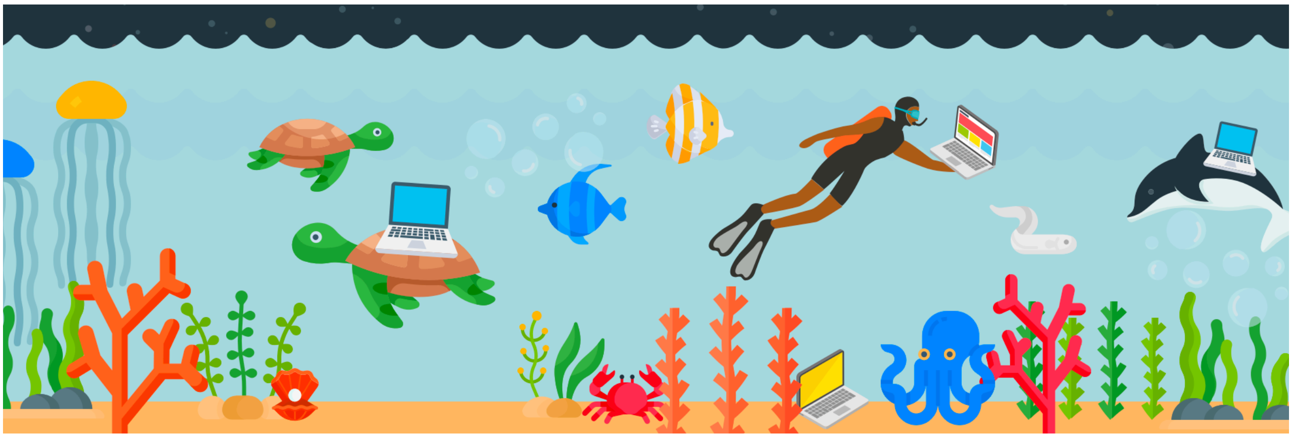
49. The normal distribution



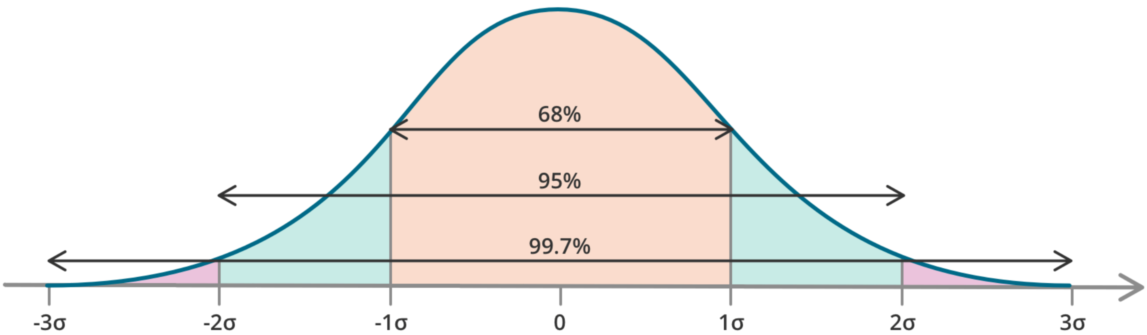
You've probably heard of the bell curve, right?

The normal distribution (also referred to as the "bell curve" or the Gaussian distribution) is a model for the likelihood of observations (i.e. if you go out and collect data). When data follows a normal distribution, there are more observations densely packed towards the centre, and the further away from this centre you get, the rarer the observations. The exact mathematical model that describes this distribution can be attributed to developments by a number of mathematicians including DeMoivre, Laplace, Adrain and Gauss, although it is usually seen as Gauss's model (hence "Gaussian" distribution). In the last 200 years it has become paramount as a fundamental model in statistics.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=538#oembed-1

Transcript

The diagram below summarises the percentages of observations we expect if we have a mean of 0. Sigma (σ) is the mathematical symbol that's used to represent the standard deviation variable.



The bell curve makes use of the concepts of the mean, and the standard deviation from the mean.

Mean

Finding the mean is simply a matter of adding all the observations together, and then dividing it by the number of observations. For example, imagine we wanted to find out the mean length of leaves of a particular tree. We might pick 100 leaves, measure each leaf in mm, add all these measurements together and then divide it by the number of leaves we picked. It'd look something like this:

$$mean\ leaf\ length = \frac{21mm + 17mm + 20mm\ldots}{100}$$

Standard deviation

Standard deviation is complex to calculate, and you don't need to be able to do it to follow along in this book.

For now, it's enough to know that standard deviation is a measure of the average difference of each observation to the mean. In our tree leaf example, it tells us how likely it is that we will find a tree that's different or similar to the mean. This is assuming – and this may be a big assumption – that the length of leaves on the tree follows the normal distribution (if it does, it should look a little bit like a bell curve if we plot it on a histogram).

If a quantity, such as the length of leaves on a tree, does follow a normal distribution, then:

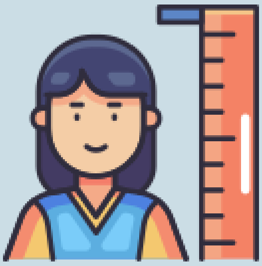
- 68% of observations (leaf length measurements) will be within one standard deviation of the centre (that is, the mean);
- 95% of observations will be within two standard deviations of the centre; and
- 99.7% of observations will be within three standard deviations of the centre.

Let's take another example. Let's consider heights of women. Suppose the average height of an Australian female adult is 162cm with a standard deviation of 7cm. Then we expect 68% of the population to be between 155 and 169cm tall (162 – 7 and 162 + 7), we expect 95% of the population to be between 148 and 176cm tall, and 99.7% of the population to be between 141 and 183cm tall.

We can align this with our probability thinking. If we randomly select a woman from the population, there's a high chance she'll be between 155 and 169cm. It's only in 0.3% of cases (or a 3/1000 chance) that we'd randomly choose someone that's above 181cm or below 141cm.

Let's have a go at using the normal distribution model to make predications about the probability of something occurring.

Exercises



Suppose the average height of an Australian woman is 162cm with a standard deviation of 7cm.

Suppose, then, that two women were randomly selected from the Australian population. What is the probability that one of them will be below 141cm tall and the other one will be above 183cm tall?

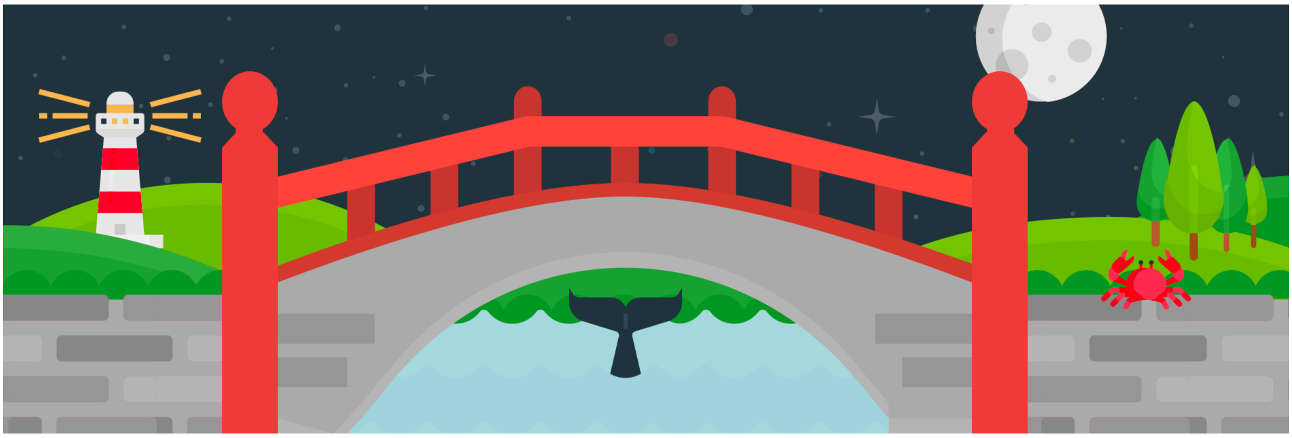
An interactive H5P element has been excluded from this version of the text. You can view it online here:
https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=538#h5p-22

Notation

In order to represent a normal model symbolically, we typically use the notation $N(\mu, \sigma)$ where N means normal model, μ (mu, pronounced 'moo') is the mean and σ (sigma) is the standard deviation. So the female heights model we described before would be $N(162, 7)$.

Note that this is a continuous model, so we have probabilities for **ranges** of values, rather than any single value. For example, we know that there is a 34% chance (half of 68) that a female adult will be between 162-169cm, however strictly speaking the probability of someone being exactly 163cm would be essentially 0 using this model. Although we could use a calculator or look-up table to work out what the probability of someone being between 162.5 and 163.5 would be.

50. Where does the Normal Distribution Model come from?

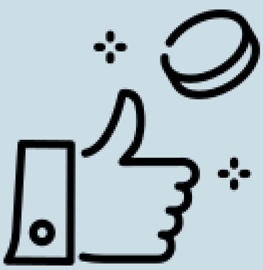


Knowing where the model comes from will help us to use it properly.

We say 'model' because these percentages usually only approximately apply to data we observe in the real world. It was developed by DeMoivre in 1733 and used to approximate what is known as a binomial distribution.

Let's look at it in terms of tossing a coin.

Exercises



A large classroom of children toss a coin three times each. Each child records their outcome – eg, THH (tails, heads, heads). What is the likely distribution of the number of times that the coin landed on heads?

What if the children tossed the coin four times each? What if they tossed it ten times each?

There are a number of ways we could solve this problem, such as using probability trees, running a computer simulation, or even trialling it in a classroom.

Let's start with the first toss.

We can all agree that there's a one in two chance of landing on heads if we flip a coin once. So there's a $\frac{1}{2}$ probability of landing heads once and tails once.

What about the probability of landing no heads (ie, two tails) or two heads?

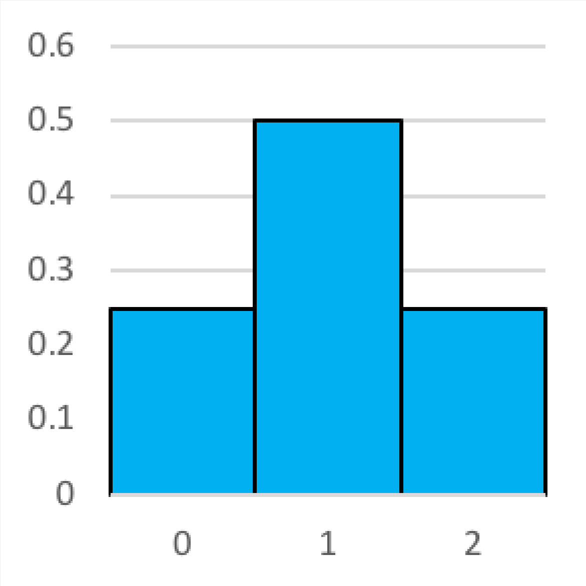
If we have landed one head already, there's then one in two chance of landing another on the next flip.

This gives us $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ of landing two heads in a row. In other words, it's probable that we'll land two heads in a row once in every four attempts, on average. The same logic applies to landing two tails in a row, so there's also a 1 in 4 (or 0.25) chance of landing no heads.

Reading a histogram

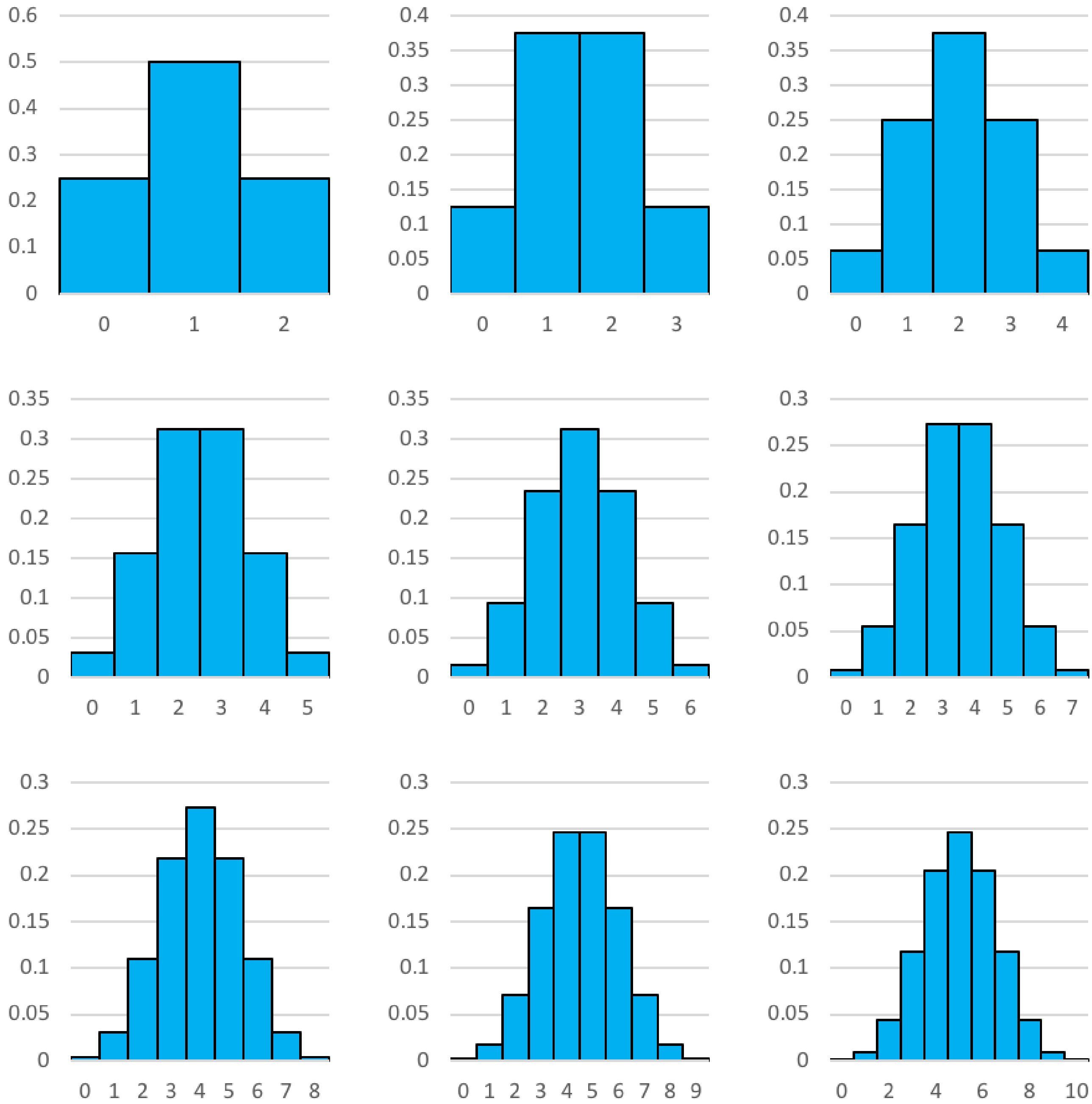
Here we see the histogram of the probability of landing a certain number of heads if you flipped two coins.

As we know, there's a $\frac{1}{4}$ or 0.25 chance of landing 0 heads or 2 heads. There's a $\frac{1}{2}$ or 0.5 probability of landing a single head. It would be easy to convert this to percentages, too. There's a 25% chance of landing 0 or 2 heads, and a 50% chance of landing 1 head.



- The number of heads rolled in a set (a set consisting of two coins being flipped) is either 0, 1 or 2. This is represented on the horizontal x-axis.
- The probability of rolling that number of heads in a set is represented on the vertical y-axis.

Let's have a look at what the histograms look like when we increase the number of coin tosses. Reading left to right, we have 2 coin tosses, 3 tosses, 4 tosses and so on until we reach 10 tosses. You'll no doubt notice that it looks a whole lot like the normal distribution.



Knowing these kinds of distributions is handy when we're playing games of chance (backgammon anyone?). However, it's truly incredible how often these kinds of normal distribution patterns occur when we measure things in everyday life.

A reasonable conjecture as to why we observe these kinds of distributions in real life then, is because the myriad of factors that go into a woman's height or the length of a leaf can be broken down into multiple small probability outcomes that in the end will determine the ultimate characteristic. For example, at a very basic level, we might consider the height of the mother and the height of the father, if both parents are tall, is like tossing two heads on a coin.

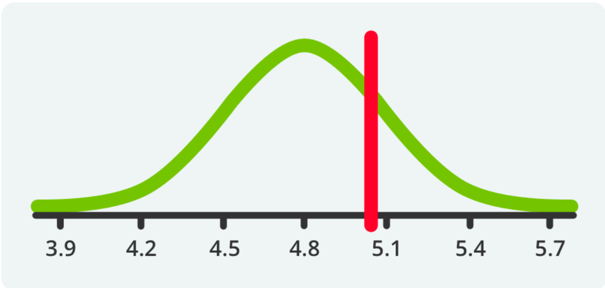
Check out this video of a fields medalist who loves the normal distribution.

One or more interactive elements has been excluded from this version of the text. You can view them online here: https://oercollective.caul.edu.au/mathematical-reasoning-investigation/?p=549#oembed-1

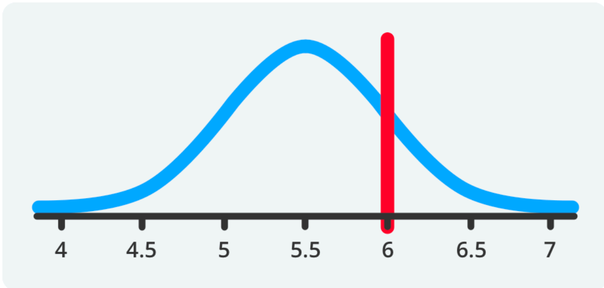
We can use the normal model as a basis for comparison.

One of the most useful things about the normal model is that we can use it as a basis for comparing results with respect to different variables using a common measure. For example, what is more unusual, a 6m long great white shark or a 5m long saltwater crocodile? If by unusual we mean how rarely it seems to occur, then we need to estimate the distribution of lengths of each of the species.

Suppose the crocodile lengths follow a normal distribution given by N(4.8, 0.3) and the shark lengths follow the model N(5.5, 0.5), we can use these values to determine our expectation of seeing the individuals of these lengths.



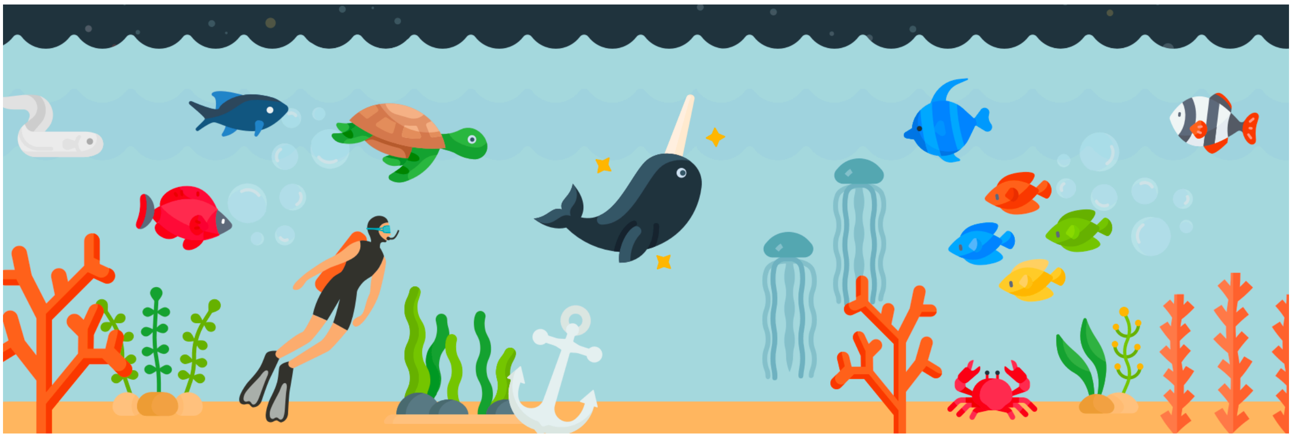
CROCODILES



SHARKS

We can see from the position of the 5m crocodile with respect to it's normal model that perhaps it is less strange than the 6m shark, which sits a whole standard deviation above the mean, whereas the crocodile is less than a whole standard deviation above the mean.

51. Mean of a distribution



The normal distribution is great, but...

It's important to remember it's approximate and that not everything follows a normal distribution. Firstly, recall that we can have discrete and continuous data. For discrete data, where there are only a finite number of variables a value can take, we usually represent probability distributions using tables or histograms.

For example, imagine we figured out the probability of the number of coins that would land on heads if we tossed four coins. Our table should end up looking something like this:

X	0	1	2	3	4
$Pr(X)$	0.0625	0.25	0.375	0.25	0.0625

On the top line we have the possible results (0 heads through to 4 heads) and on the bottom line we have the probability of this result occuring.

Note that a probability distribution will always add to one, whereas a frequency distribution will usually be whole numbers. A probability distribution indicates the likelihood of that particular value occurring, whereas a frequency distribution indicates the number of times it actually did occur, and sometimes we will estimate probability distributions based on frequency distributions of collected data (by turning the frequencies into percentages).

We can determine the mean of a discrete distribution by multiplying each of the probabilities by the values they correspond with. So in the above example, we would have:

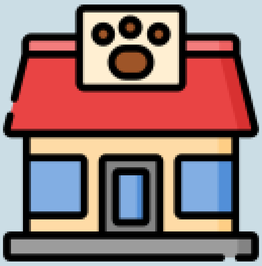
0(0.0625) + 1(0.25) + 2(0.375) + 3(0.25) + 4(0.0625)
= .25 + 0.75 + 0.75 + 0.25
= 2

This is the same idea as a frequency distribution, e.g. if we tossed 4 coins 16 times and obtained the exact number of expected results, we would have

X	0	1	2	3	4
$Freq$	1	4	6	4	1

In other words, we got 0 once, we got 1 head 4 times, we got 2 heads 6 times, 3 heads 4 times and 4 heads once, then writing these all out we have the sixteen trials: 0, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 4. We should find that the average is 2 here as well.

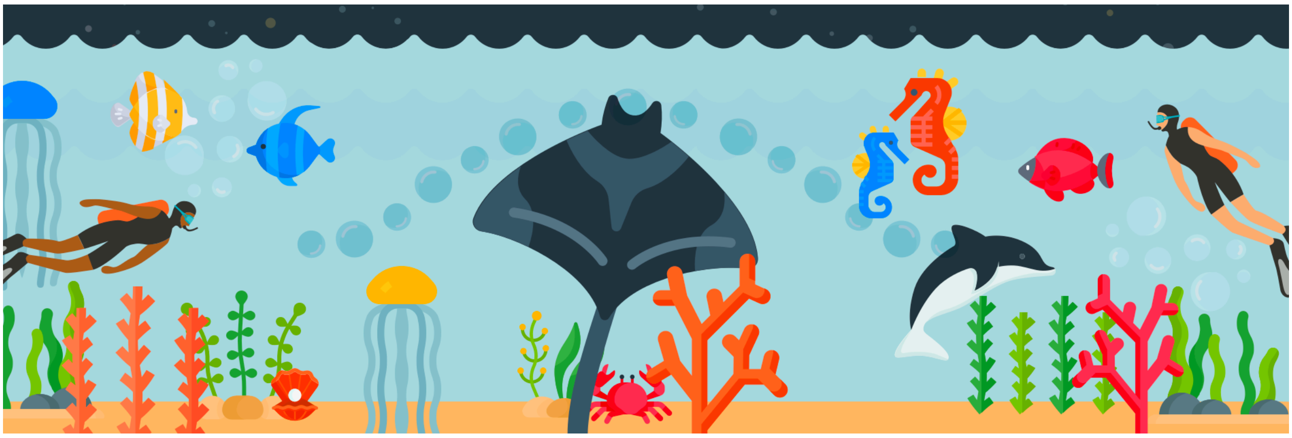
Animal shelter



An animal shelter charity considers paying a marketing company for an online advertisement.

The charity knows that from past experience, 70% of those who visit their website (by following an advertising link) won't donate anything, 20% will make the minimum donation of $10 and 10% will donate $50. What is the average donation expected to be made by people clicking on the advertisement? How many clicks would be required for it to be worth paying $5000 for the ad?

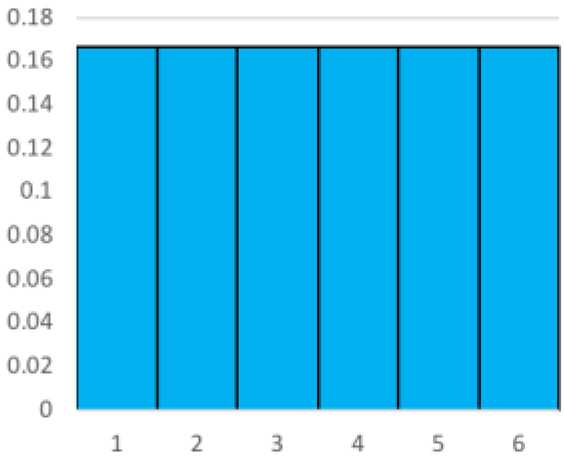
52. Other distributions



A uniform distribution is what we get when each of the possible outcomes is equally likely.

For example, if rolling a die, we expect each of the numbers to occur at a similar frequency, and the probability distribution would be.

Roll	1	2	3	4	5	6
Pr(Roll)	1/6	1/6	1/6	1/6	1/6	1/6



Each of the bars in the histogram are equal and if we did a number of trials we would expect these to be almost equal as well. This is because there is no greater likelihood of obtaining numbers closer to the middle. However, the mean of this distribution will be the value in the centre:

$$(1 \times \frac{1}{6}) + (2 \times \frac{1}{6}) + (3 \times \frac{1}{6}) + (4 \times \frac{1}{6}) + (5 \times \frac{1}{6}) + (6 \times \frac{1}{6}) = \frac{21}{6} = 3.5$$

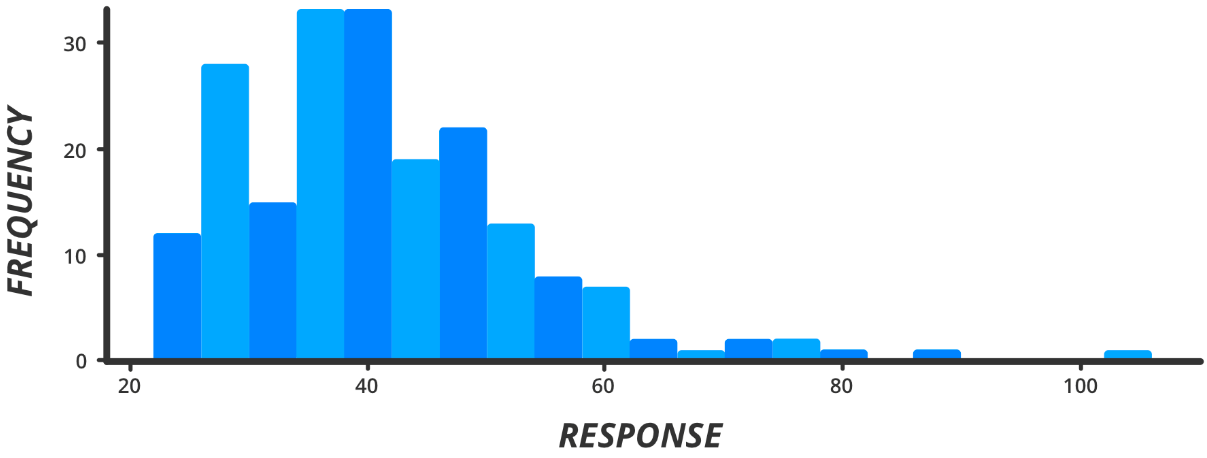
This mean doesn't tell us the roll we expect to get when we toss a single die (how can you roll a 3.5?), however if we rolled a die any number of times, we would expect the average of all the rolls to be 3.5. So, if you roll 10 dice, you should get a combined score that is close to 35. Once we're repeating the experiment (and if we were to record results), we actually would end up with a distribution that starts looking like the normal distribution again!

This is a result of the Central limit theorem, an important result in statistics but one that we will not focus on too much for the moment.

Skewed distributions

In our previous studies of statistics we talked about distributions as describing the 'shape' of the histogram, where we could have symmetric, or positively/negatively skewed. Positive skew means the tail is longer for higher values and there could be a few very high values, whereas negative skew occurs if there are a few low lying values and "long tail" for lower values.

This is an example graph that is positively skewed. Note that if we were to calculate the mean, these few high and outlying values would push the mean up. In the data below, the mean is 41.34 while the median is 39.69.



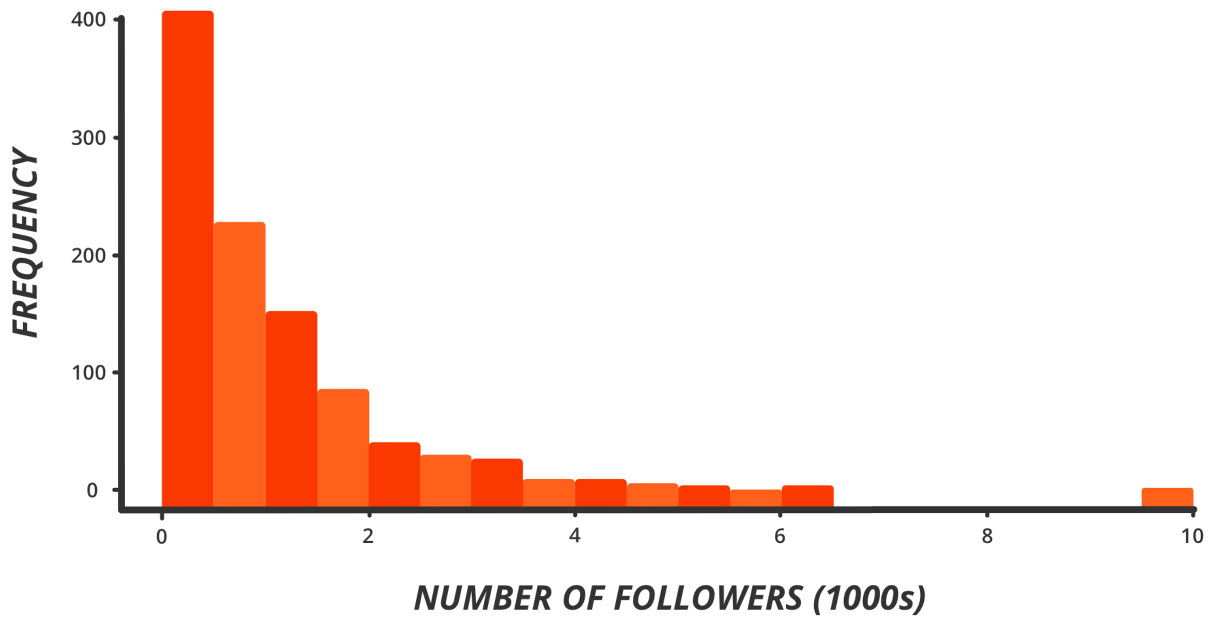
In reality, many *approximately normal* distributions may exhibit a degree of positive or negative skew, particularly in cases where it is easy to have very high values but impossible to have symmetrically low values.

For example, weights for male adults would exhibit a positive skew, because even if the mean is, say 75kg, there are a number of individuals that would be over 120kg (45kg above the mean), however it's much less likely to have adults that are under 30kg (45kg below the mean). Similarly, if we are timing 100m sprints, while we may have a group of people with a mean of say, 20 seconds, it would be easy for people to take 40 seconds or longer if they felt like it, but impossible for anyone to complete the race in 1 second.

Exponential distribution

There are actually a number of other mathematical models, like the normal distribution, that are given in terms of parameters and model real life phenomena. We will look at just one here: the exponential distribution. An exponential distribution is used to describe a number of processes, in particular the time between independent events (e.g. the rate at which people join a queue) however they may also approximate populations that have a majority of very small values and a few outlying high values, for example, the wealth of populations and the number of followers on twitter or Instagram will exhibit behavior similar to exponential distributions (or sometimes the less extreme 'power-law' distribution).

For example, the following graph is what a sample of 1000 twitter accounts might look like in terms of the number of followers.



Of course, there would also be some twitter accounts with followers in the millions. Similarly with wealth, we have a majority of people earning under $90000 per annum, then some earning between $90000 and $150000, and fewer earning between 150k and 300k, then some individuals in the many hundreds of 1000s and some in the millions.

Build your intuition

Would the following measurements most likely have a normal or a skewed distribution? Why or why not?

- The number of shoes manufactured by shoe size?
- The price of shoes on sale at a stocktake sale by shoe size?
- The weight of an Indian elephant living in the wild?
- The amount of weight a girder could hold, depending on the density of the material the girder is made out of?
- A person's life expectancy, depending on the year they were born?
- Sale of movie tickets, by days after the release of the movie
- The internal temperature of a cake, by time elapsed since it has been taken out of the oven.

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Conclusion

You've reached the end of the book. That's an achievement in itself.

Now is a good time to think back over what you've learnt and reflect on what it means for you.

Throughout the book we've asked you to have a go at two fundamentally different types of problems.

The first type we've called 'routine problems.' These are the problems you'd usually find in a textbook: they have an established methodology for solving and a very clear answer at the end of it. For many of us, that's what we think of when we hear the word 'maths.' It's important not to dismiss this kind of learning lightly, because having a grasp of some of these methods and techniques gives us tools for modelling and understanding the world. As you will no doubt have noticed, knowing how to apply these methods can make solving problems a whole lot easier. Algebra is probably the most obvious example of this, because it is such a powerful tool to help us figure out the relationship between things.

The next type we've called 'non-routine problems.' These are the problems for which there's no clear cut method, and it's not always straightforward to know if we've arrived at the precise answer. Sometimes there is no precise answer. For these problems, we need to figure out the method for ourselves. This can be a powerful way to learn mathematics, not only because it links what we're learning to real world applications, but because it forces us to really understand and be able to justify the method we've chosen.

This, in turn, promotes mathematical reasoning. This centrally important, critical thinking skill teaches us how to make reasoned conjectures, test them and then come to reasoned conclusions. The process of doing so will mean that most of us need to challenge our assumptions, keep a mind that is open and strategic at the same time, justify our own thought process and think about what it means to solve problems in the first place.

If there's one thing that both routine and non-routine mathematical problems can teach us, it's the skill of patiently working through problems creatively, systematically and methodically.

Your final problem – create a new one!

Pose a problem

All throughout the book, you've been asked to develop an understanding of mathematics using problems. We hope that, by creating your own methods to solve non-routine problems, you understand the principles underpinning your solutions at a different level to problems you've learnt to solve from a routine textbook. We also hope it's helped you to practice that creative process of problem solving and mathematical reasoning.

Now it's your turn to pose a problem. Think for a minute about what makes something a "problem" and not just a routine skills question is not always straight forward and sometimes it depends on the level of the student. For example, asking a late high-school student to tell you the volume of a sphere is more-or-less a routine problem, but for a student in year 7 or 8 it could be quite perplexing. As a rough guide, a problem should

- not have a straight-forward solution;

- require the solver to define terms or set the scope of the solution (i.e., it can be ill-defined);
- have a solution that could be verified or approached in a number of ways;
- be generalisable to other contexts or situations;
- require the solver to use some form of abstract thinking.

After attempting a number or problems yourself, you might have some other opinions that apply – e.g. that the problem should be “interesting” and motivate the solver to **want to solve it**.

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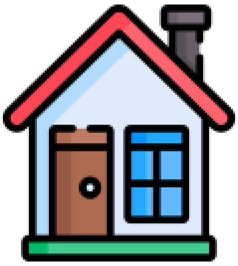

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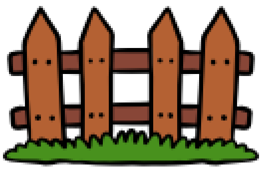

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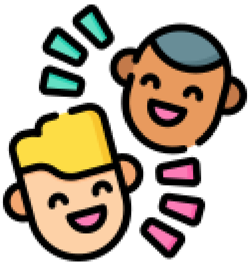

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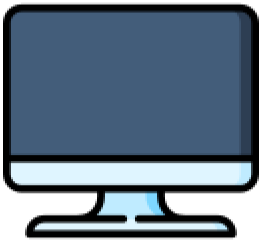

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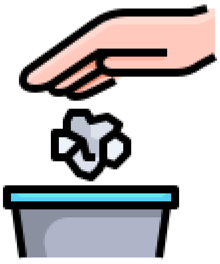

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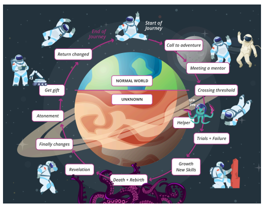

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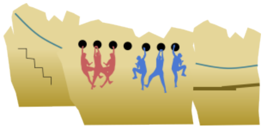

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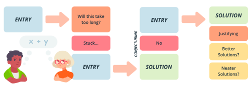

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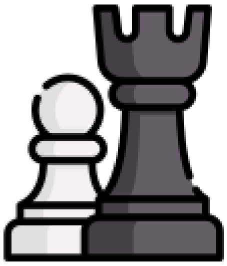

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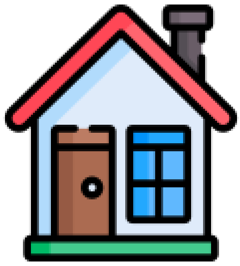

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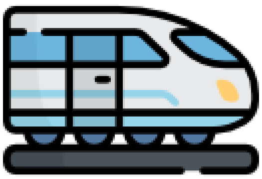

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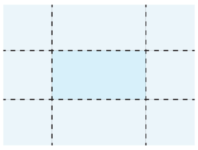

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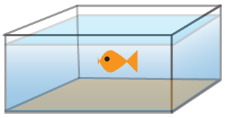

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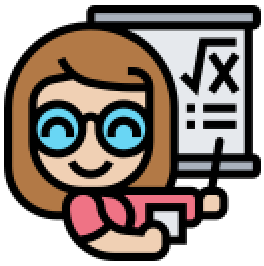

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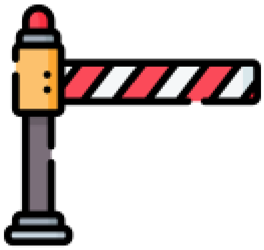

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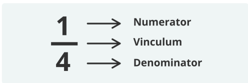

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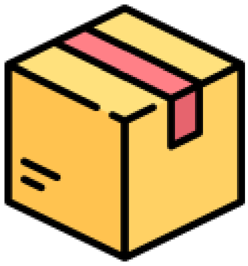

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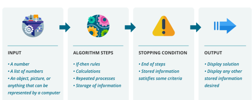
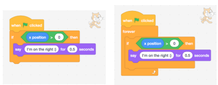
Algorithms



Lea Piskiewicz



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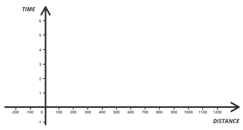
Interpreting visual information



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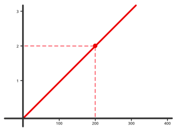

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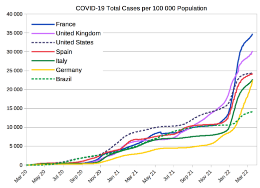

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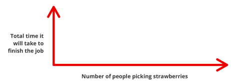

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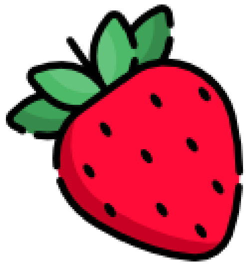

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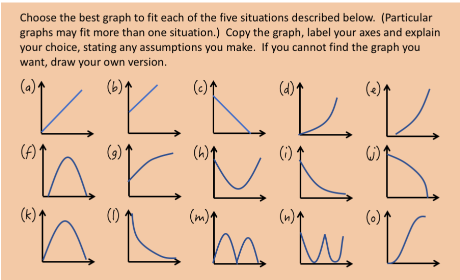

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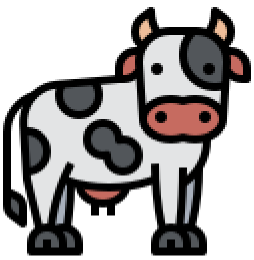
Sequences and progressions



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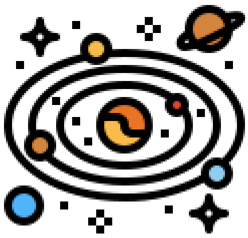

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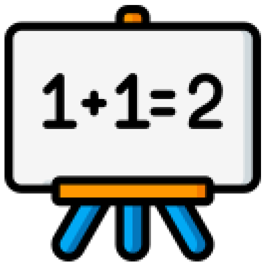

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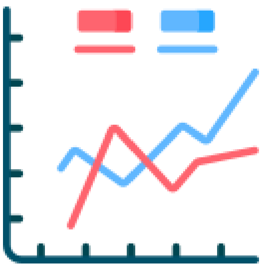

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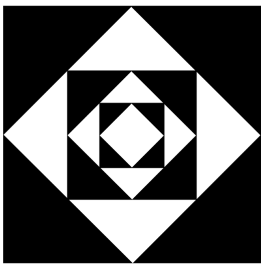
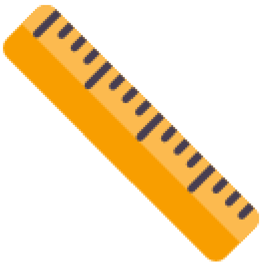


Simon James

Measurement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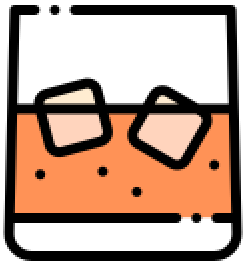
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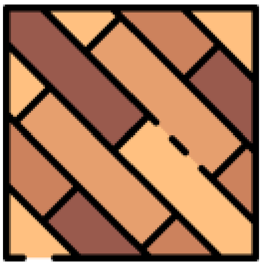
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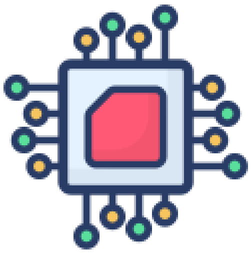
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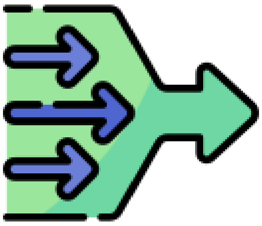
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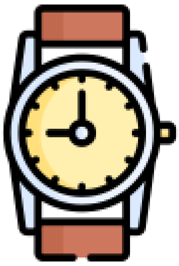
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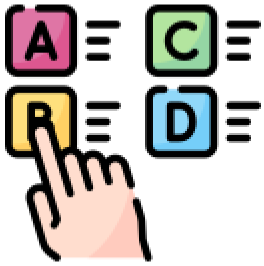

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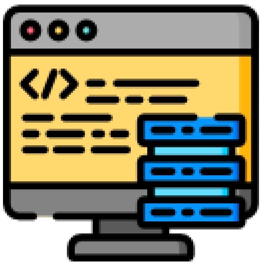
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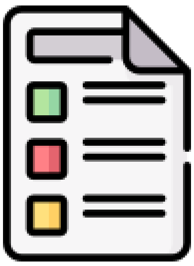
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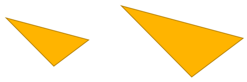
Geometry



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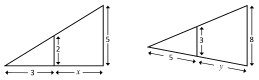
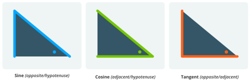


Simon James, Erin Cheffers, Lea Piskiewicz



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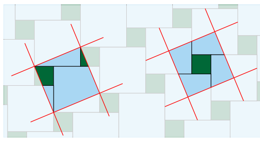

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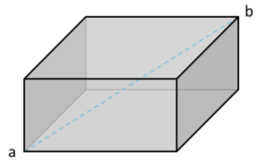

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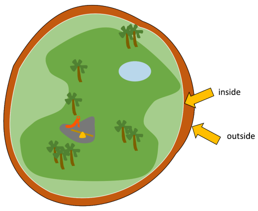

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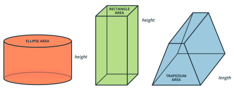
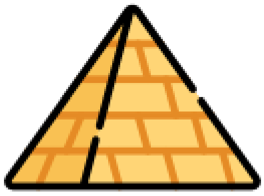


Erin Cheffers



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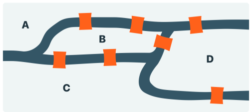
Graphs and networks



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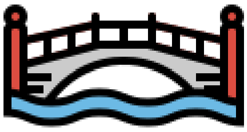

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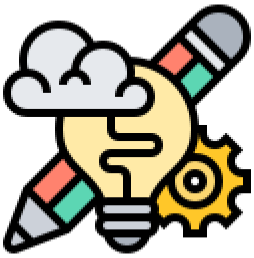

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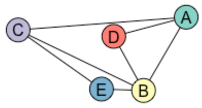

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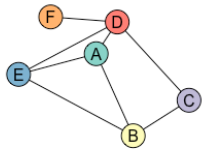

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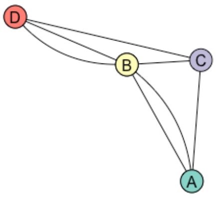

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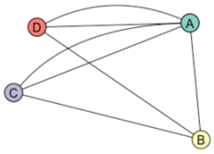

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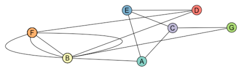

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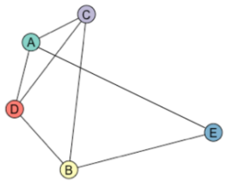

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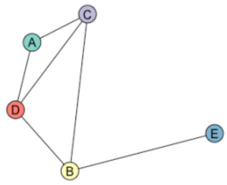

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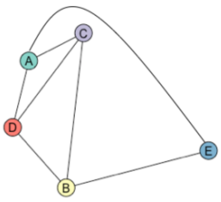

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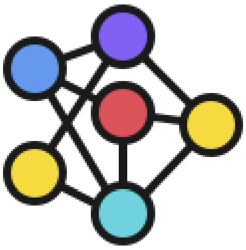

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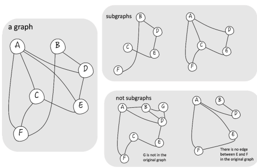

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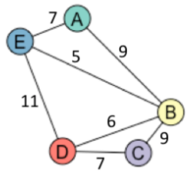

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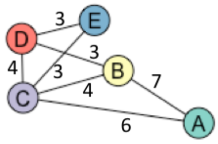

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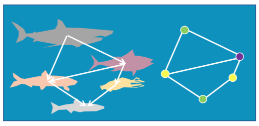

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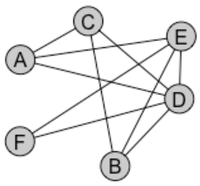

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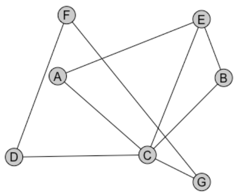

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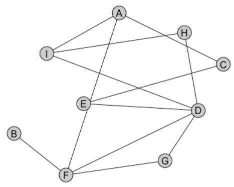

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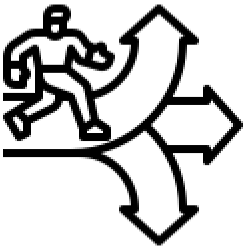
Probability



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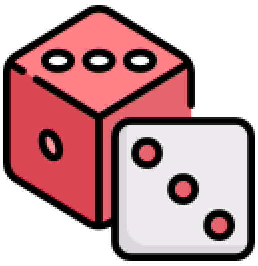

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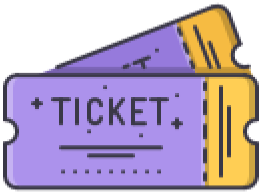

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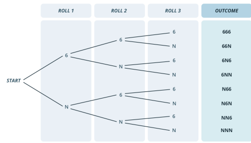

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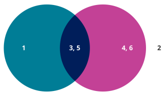
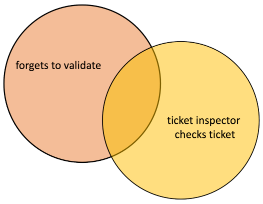


Lea Piskiewicz



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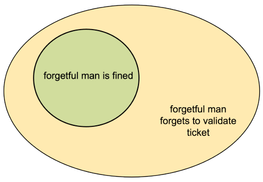
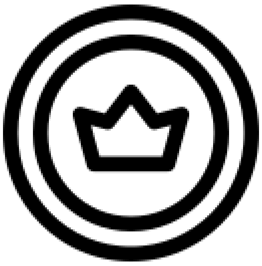


Simon James

Simulation

Leaves icons created by cahiwak – Flaticon



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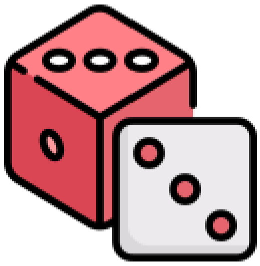

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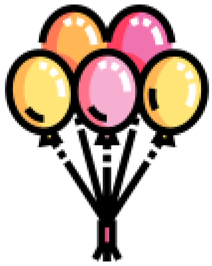

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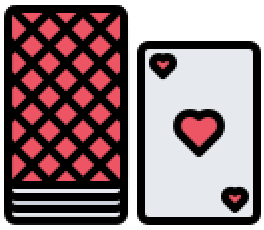

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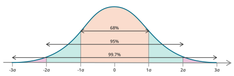

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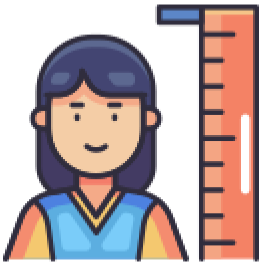

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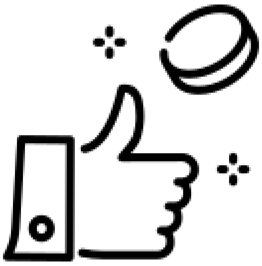

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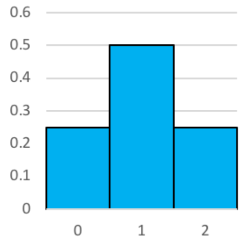
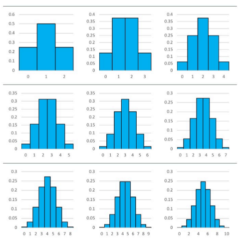


Simon James



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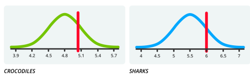

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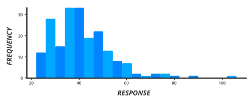

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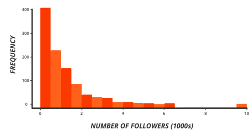

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