S. Palani

Basic System Analysis

Second Edition

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This Book is Submitted to The Lotus Feet of
LORD VENGATESWARA

Preface

The book Basic System Analysis presents a comprehensive treatment of signals and linear systems for the undergraduate level study. It is a rich subject with diverse applications such as signal processing, control systems, and communication systems. It provides an integrated treatment of continuous-time and discrete-time forms of signals and systems. These two forms are treated side by side. Even though continuous-time and discrete-time theory have many mathematical properties common between them, the physical processes that are modeled by continuous-time systems are very much different from the discrete-time systems counterpart.

I have written this book with the material I have collected during my long experience of teaching signals and systems to the undergraduate level students in national level reputed institutions. The book in the present form is written to meet the requirements of undergraduate syllabus of Indian Universities in general and B.Tech. EEE branch of Uttar Pradesh Technological University in particular. The organization of the chapters is as follows.

Chapter 1 deals with the representation of signals and systems. It motivates the reader as to what signals and systems are and how they are related to other areas such as communication systems, control systems, and digital signal processing. In this chapter, various terminologies related to signals and systems are defined. Further, mathematical description, representation, and classifications of signals and systems are explained.

Chapter 2 deals with the Fourier representation of continuous-time signals. Continuous-time periodic signals are represented by trigonometric Fourier series, polar Fourier series, and exponential Fourier series.

It is not possible to find Fourier series representation of non-periodic signals. In Chap. 3, Fourier transform is introduced which can represent periodic as well as non-periodic signals. In this chapter, the Fourier transform for continuous-time signal is explained.

The Laplace transform is a very powerful tool in the analysis of continuous-time signals and systems. In Chap. 4, the Laplace transform method is explained and its properties derived. The use of Laplace transform to solve differential equation is described.

Chapter 5 is devoted to the z-transform and its application to discrete-time signals and systems. The properties of z-transform and techniques for inversion are introduced in this chapter. The use of z-transform for solving difference equation is explained.

Chapter 6 is devoted to state space modeling and analysis of continuous-time and discrete-time systems. Formation of vector matrix differential/difference equation is also explained in this chapter.

In Chap. 7, application of MATLAB and Python programs to solve problems is discussed.

The notable features of this book include the following:

1. The syllabus content of signals and systems for undergraduate level has been covered.
2. The organization of the chapter is sequential in nature.
3. Large number of numerical examples have been worked out.
4. Learning objectives and summary are given in each chapter.
5. For the students to practice, short and long questions with answers are given at the end of each chapter.
6. In this edition, a new chapter titled "Application of MATLAB and PYTHON Programs" has been included. Here many applications to real-life practical systems and exposure to computational tools are discussed by solving numerical problems. Some useful special computational concepts are also presented which will be useful to readers.

I take this opportunity to thank Shri Sunil Saxena, Managing Director, Ane Books Pvt. Ltd, India, for coming forward to publish the book. I would like to express our sincere thanks to Shri A. Rathinam, General Manager (South), Ane Books Pvt. Ltd., who took the initiatives to publish the book in a short span of time. I would like to express my sincere thanks to Mr. V. Ashok who has done a wonderful job to key the voluminous book like this in a very short time and beautifully too. I would also like to thank my wife Dr. S. Manimegalai, M.B.B.S., M.D., who was the source of inspiration while preparing this book.

Tiruchirappalli, India Dr. S. Palani

Contents

About the Author

Dr. S. Palani obtained his B.E. degree in Electrical Engineering in 1966 from the University of Madras, M.Tech. in Control Systems Engineering from the Indian Institute of Technology Kharagpur in 1968, and Ph.D. in Control Systems Engineering from the University of Madras in 1982. He has a wide teaching experience of over four decades. He started his teaching career in 1968 at the erstwhile Regional Engineering College (now National Institute of Technology), Tiruchirappalli, in the department of EEE and occupied various positions. As Professor and Head, he took the initiative to start the Instrumentation and Control Engineering Department. After a meritorious service of over three decades in REC, Tiruchirappalli, he joined Sudharsan Engineering College, Pudukkottai, as Founder Principal. He established various departments with massive infrastructure.

He has published more than a hundred research papers in reputed international journals and has won many cash awards. Under his guidance, 17 research scholars were awarded Ph.D. He has carried out several research projects worth several lakhs of rupees funded by the Government of India and AICTE. As Theme Leader of the Indo–UK, REC Project on energy, he has visited many universities and industries in the UK. He is the author of the books titled Control Systems Engineering, Signals and Systems, Digital Signal Processing, Linear System Analysis, and Automatic Control Systems. His research areas are the design of controllers for dynamic systems, digital signal processing, and image processing.

Chapter 1
Representation of Signals and Systems

Chapter Objectives

- To define various terminologies related to signals and systems.
- To classify signals and systems.
- To give mathematical description and representation of signals and systems.
- To perform basic operations on CT signals.
- To classify CT signals as periodic and non-periodic, odd and even and power and energy signals.
- To classify systems as linear and non-linear, time invariant and time varying, static and dynamic, causal and non-casual, stable and unstable, invertible and non-invertible.
- To find the force–voltage and force–current electric analogous circuit for mechanical system.
- To find the time response of first- and second-order systems.

1.1 Introduction

The concepts of signals and systems play a very important role in many areas of science and technology. These concepts are very extensively applied in the field of circuit analysis and design, long-distance communication, power system generation and distribution, electron devices, electrical machines, biomedical engineering, aeronautics, process control, speech, and image processing to mention a few. **Signals represent some independent variables which contain some information about the behavior of some natural phenomenon.** Voltages and currents in electrical and electronic circuits, electromagnetic radio waves, human speech, and sounds produced by animals are some of the examples of signals. **When these signals are operated**

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on some objects, they give out signals in the same or modified form. These objects are called systems. A system is, therefore, defined as the interconnection of objects with a definite relationship between objects and attributes. Signals appearing at various stages of the system are attributes. $R,\ L,\ C$ components, spring, dash-pots, mass, etc. are the objects. The electrical and electronic circuits comprising of $R,\ L,\ C$ components and amplifiers, the transmitter and receiver in a communication system, the petrol and diesel engines in an automobile, chemical plants, nuclear reactor, human beings, animals, a government establishment, etc. are all examples of systems.

1.2 Terminologies Related to Signals and Systems

Before we give mathematical descriptions and representations of various terminologies related to signals and systems, the following terminologies which are very frequently used are defined as follows:

1.2.1 Signal

A signal is defined as a physical phenomenon which carries some information or data. The signals are usually functions of independent variable time. There are some cases where the signals are not functions of time. The electrical charge distributed in a body is a signal which is a function of space and not time.

1.2.2 System

A system is defined as the set of interconnected objects with a definite relationship between objects and attributes. The interconnected components provide desired function. Objects are parts or components of a system. For example, switches, springs, masses, dash-pots, etc. in a mechanical system and inductors, capacitors, and resistors in an electrical system are the objects. The displacement of mass, spring, and dash-pot and the current flow and the voltage across the inductor, capacitor, and resistor are the attributes. There is a definite relationship between the objects and attributes. The voltages across $R,\ L,\ C$ series components can be expressed as $v_R = iR$, $v_L = L\frac{di}{dt}$, and $v_C = \frac{1}{C}\int i dt$. If this series circuit is excited by the voltage source $e_i(t)$, the $e_i(t)$ is the input attribute or the input signal. If the voltage across any of the objects $R,\ L$, and C is taken then such an attribute is called the output signal. The block diagram representation of input and output (voltage across the resistor) signals and the system is shown in Fig. 1.1.

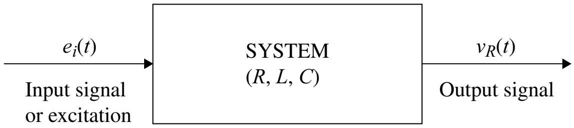


Fig. 1.1 Block diagram representation of signals and systems

1.3 Continuous- and Discrete-Time Signals

Signals are broadly classified as follows:

1. Continuous-time signal (CT signal).
2. Discrete-time signal (DT signal).

The signal that is specified for every value of time t is called continuous-time signal and is denoted by $x(t)$. On the other hand, the signal that is specified at discrete value of time is called discrete-time signal. The discrete-time signal is represented as a sequence of numbers and is denoted by $x[n]$ where n is an integer. Here time t is divided into n discrete time intervals. The continuous-time signal (CT) and discrete-time signal (DT) are represented in Figs. 1.2 and 1.3 respectively.

It is to be noted that in continuous-time signal representation the independent variable t which has unit as sec is put in the parenthesis $(\cdot)$ and in discrete-time signal the independent variable n which is an integer is put inside the square parenthesis $[\cdot]$. Accordingly, the dependent variables of the continuous-time signal/system are

Fig. 1.2 CT signal

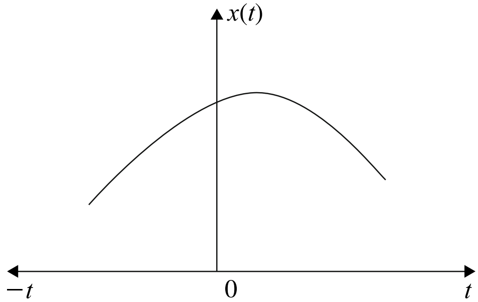
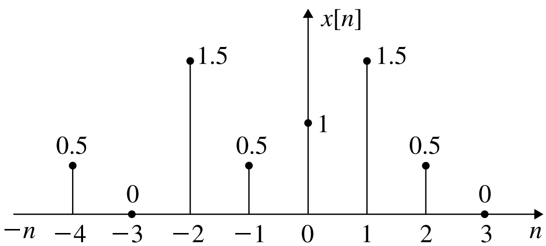


Fig. 1.3 DT signal



denoted as $x(t)$, $g(t)$, $u(t)$, etc. Similarly the dependent variables of discrete-time signals/systems are denoted as $x[n]$, $g[n]$, $u[n]$, etc.

A discrete-time signal $x[n]$ is represented by the following two methods:

1.
$$x[n] = \begin{cases} \left(\frac{1}{a}\right)^n & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{1.1}$$

Substituting various values of n where $n \geq 0$ in Eq. (1.1) the sequence for $x[n]$ which is denoted by $x\{n\}$ is written as follows:

$$x\{n\} = \left\{1, \frac{1}{a}, \frac{1}{a^2}, \ldots, \frac{1}{a^n}\right\}$$

2. The sequence is also represented as given below:

$$x\{n\} = \{3,\ 2,\ \underset{\uparrow}{5},\ 4,\ 6,\ 8,\ 2\}$$

The arrow indicates the value of $x[n]$ at $n = 0$ which is 5 in this case. The numbers to the left of the arrow indicate to the negative sequence $n = -1, -2$, etc. The numbers to the right of the arrow correspond to $n = 1, 2, 3, 4$, etc. Thus, for the above sequence $x[-1] = 2$, $x[-2] = 3$, $x[0] = 5$, $x[1] = 4$, $x[2] = 6$, $x[3] = 8$, and $x[4] = 2$. If no arrow is marked for a sequence, the sequence starts from the first term in the extreme left. Consider the sequence

$$x\{n\} = \{5,\ 3,\ 4,\ 2\}$$

Here $x[0] = 5$, $x[1] = 3$, $x[2] = 4$, and $x[3] = 2$. There is no negative sequence here.

1.4 Basic Continuous-Time Signals

Basic continuous-time signals play a very important role in signals and systems analysis. The following are the basic continuous-time signals which serve as a basis to represent other signals. The basic continuous-time signals are

1. Unit impulse function.
2. Unit step function.
3. Unit ramp function.
4. Unit parabolic function.
5. Unit rectangular pulse (or gate) function.
6. Unit area triangular function.

7. Unit signum function.
8. Unit Sinc function.
9. Sinusoidal signal.
10. Real exponential signal.
11. Complex exponential signal.

The mathematical description and graphical representation of the above signals are discussed below. Similar to continuous-time signals, basic discrete-time signals are also available. The descriptions of these signals will immediately follow this.

1.4.1 Unit Impulse Function

The unit impulse function is also known as **Dirac delta** function which is represented in Fig. 1.4. The unit impulse function is denoted as $\delta(t)$ and its mathematical description is given below:

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ 1 & t = 0 \end{cases} \tag{1.2}$$

1.4.1.1 Importance of Impulse Function

1. By applying impulse signal to a system one can get the impulse response of the system. From impulse response, it is possible to get the transfer function of the system.
2. For a linear time invariant system, if the area under the impulse response curve is finite, then the system is said to be stable.
3. Form the impulse response of the system, one can easily get the step response and ramp response by integrating it once and twice respectively.
4. Impulse signal is easy to generate and apply to any system.

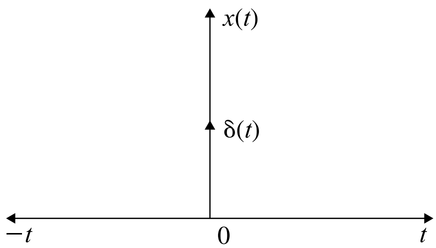


Fig. 1.4 Unit impulse function

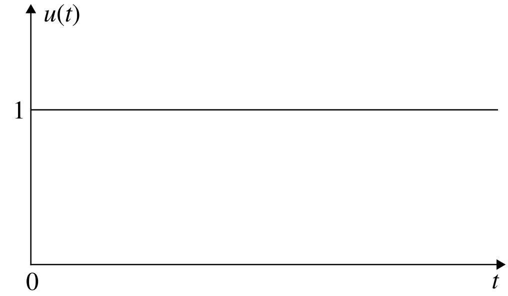


Fig. 1.5 Unit step function

1.4.2 Unit Step Function

The unit step function is shown in Fig. 1.5. The function is defined as follows:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{1.3}$$

The step function is denoted by $u(t)$. Any causal signal which begins at $t = 0$ (which has a value of zero for $t < 0$) is multiplied by the signal by $u(t)$. For example, **a causal exponentially decaying signal e^{-at} ($t \geq 0$) is represented as $x(t) = e^{-at}u(t)$. Similarly e^{-at} ($t < 0$) is represented as $x(t) = e^{-at}u(-t)$.**

1.4.2.1 Importance of Step Function

1. Step function is easy to generate and apply to the system.
2. By differentiating the step response impulse response can be obtained. By integrating the step response, ramp response can be obtained.
3. Step signal is considered as a white noise which is drastic. If the system response is satisfactory for a step signal, it is likely to give satisfactory response to other types of signals.
4. Application of step signal is equivalent to the application of numerous sinusoidal signals with a wide range of frequencies.

1.4.3 Unit Ramp Function

The unit ramp function is represented in Fig. 1.6. It is defined by the following mathematical equation:

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{1.4}$$

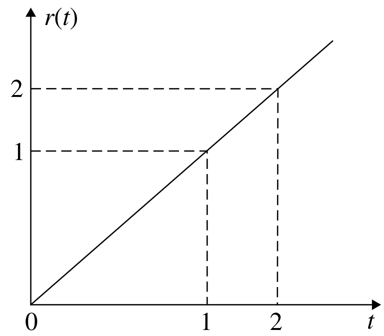


Fig. 1.6 Unit ramp function

For a causal signal ($t \geq 0$), the ramp function can also be expressed as

$$r(t) = t\,u(t) \tag{1.5}$$

1.4.3.1 Relationships Between Impulse, Step, and Ramp Signals

1. Integrating the unit step signal $u(t)$ we get

$$\int u(t)dt = \int dt = t \tag{1.6}$$

   By integrating the unit step function, unit ramp function is obtained. In the reverse process, by differentiating a ramp function, a step function is obtained.
2. The continuous-time unit step function is the running integral of the unit impulse function which is expressed as

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

$$\frac{du(t)}{dt} = \delta(t) \tag{1.7}$$

3. By differentiating the ramp function twice, the impulse function is obtained

$$r(t) = t$$
$$\frac{dr(t)}{dt} = 1 = u(t) \tag{1.8}$$

$$\frac{d^2r(t)}{dt^2} = \frac{du(t)}{dt} = \delta(t) \tag{1.9}$$

Thus, the impulse function is obtained by differentiating the ramp function twice. By the reverse process, by integrating the impulse function twice, the ramp function is obtained which is mathematically expressed as follows:

$$r(t) = \int\int \delta(t)\, dt \tag{1.10}$$

The relationships between the impulse, step, and ramp signals are represented below:

$$\delta(t) \xrightarrow{\text{integrate}} u(t) \xrightarrow{\text{integrate}} r(t)$$

$$r(t) \xrightarrow{\text{differentiate}} u(t) \xrightarrow{\text{differentiate}} \delta(t)$$

1.4.4 Unit Parabolic Function

The unit parabolic function $x(t)$ is represented in Fig. 1.7. The mathematical expression is given below:

$$x(t) = \frac{1}{2}t^2 \qquad t \geq 0 \tag{1.11}$$

If the parabolic function is differentiated, unit ramp function is obtained. Thus

$$\frac{dx(t)}{dt} = t \qquad t \geq 0$$

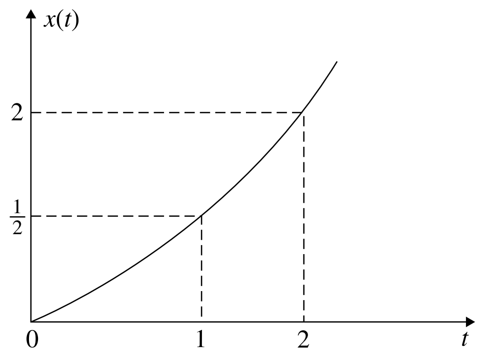


Fig. 1.7 Unit parabolic function

1.4.5 Unit Rectangular Pulse (or Gate) Function

The unit area rectangular pulse which is also called gate function is represented in Fig. 1.8. Mathematically it is described as follows:

$$x(t) = \begin{cases} \left(\frac{1}{a}\right) & \text{for } |t| \le \frac{a}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1.12}$$

The above equation is also written in the following form:

$$x(t) = \frac{1}{a} \qquad -\frac{a}{2} \le t \le \frac{a}{2}$$

The function is written as $x(t) = \text{rect}(t)$.

1.4.6 Unit Area Triangular Function

The unit area triangular function is represented in Fig. 1.9. It is symbolically written as $x(t) = \text{tri}(t)$. It is defined as

$$\text{tri}(t) = \begin{cases} [1 - |t|] & |t| \le 1 \\ 0 & |t| > 1 \end{cases} \tag{1.13}$$

The above equation can be written in the following form also:

$$\begin{aligned} \text{tri}(t) &= [1 + t] \qquad -1 \le t \le 0 \\ &= [1 - t] \qquad 0 \le t \le 1 \end{aligned}$$

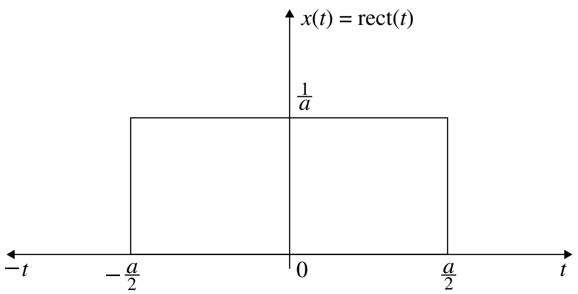


Fig. 1.8 Unit area rectangular pulse (or gate) function

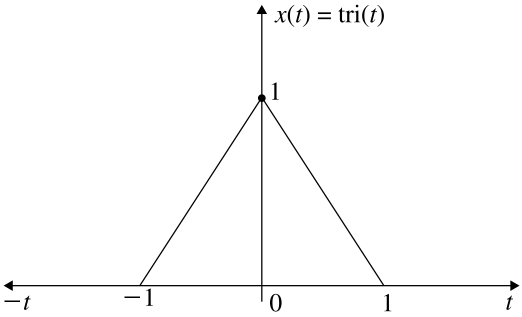


Fig. 1.9 Unit area triangular function

1.4.7 Unit Signum Function

The signum function is written in the abbreviated form as sgn(t). It represents the characteristics of an ideal relay. This is shown in Fig. 1.10. It is defined by the following equations:

$$\mathrm{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases} \tag{1.14}$$

1.4.8 Unit Sinc Function

The unit sinc function is represented in Fig. 1.11. It is defined as

$$\mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t} \qquad -\infty < t < \infty. \tag{1.15}$$

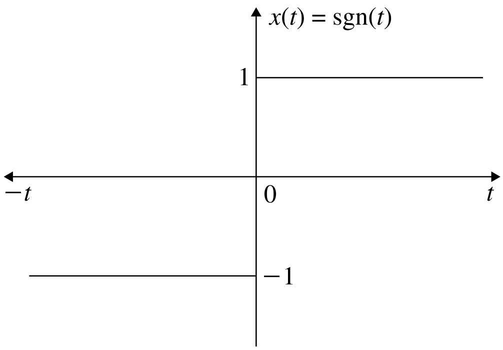


Fig. 1.10 Representation of unit signum function

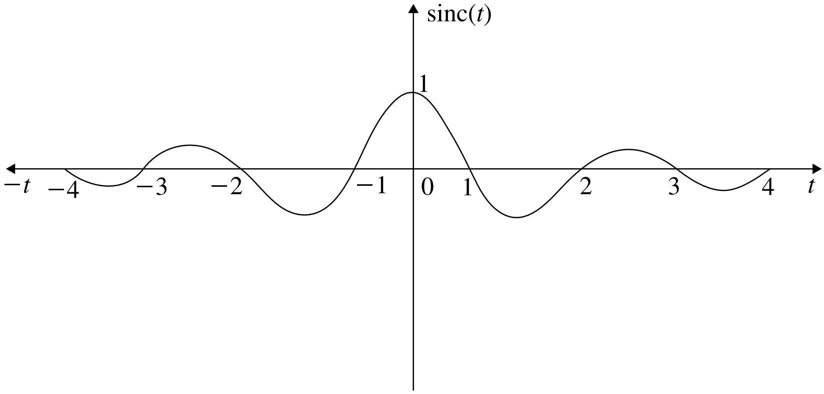


Fig. 1.11 Representation of unit sinc function

1.4.9 Sinusoidal Signal

The sinusoidal signal is represented in Fig. 1.12. It is defined as

$$x(t) = A\sin(\omega t - \phi) \tag{1.16}$$

where $A =$ peak amplitude, $\omega =$radian frequency, $\phi =$phase shift.

1.4.10 Real Exponential Signal

Let

$$x(t) = e^{st} \tag{1.17}$$

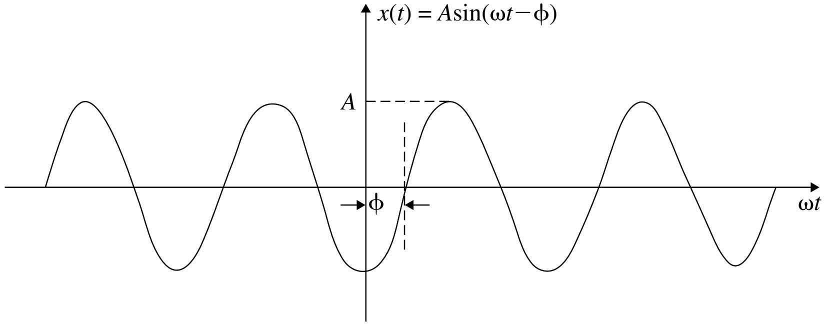


Fig. 1.12 Representation of sinusoidal signal

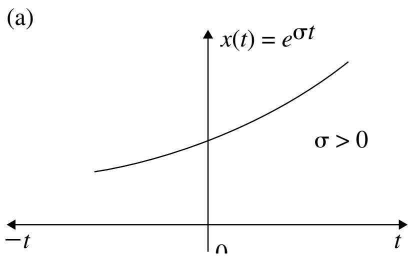


Fig. 1.13 Representation of real exponential signals. **a** Growing exponential; **b** Decaying exponential

where $s = \sigma + j\omega$ is a complex number. The signal $x(t)$ in Eq. (1.17) is called general complex exponential. Equation (1.17) is written in the following form:

$$\begin{aligned} x(t) &= e^{(\sigma+j\omega)t} \\ &= e^{\sigma t} e^{j\omega t} \\ &= e^{\sigma t}(\cos \omega t + j \sin \omega t) \end{aligned} \tag{1.18}$$

If $\omega = 0$,

$$x(t) = e^{\sigma t} \tag{1.19}$$

Equation (1.19) is real exponential. The plot of $x(t)$ with respect to t for $\sigma > 0$ and $\sigma < 0$ is shown in Fig. 1.13a and b respectively. For $\sigma > 0$, the signal is exponentially growing and for $\sigma < 0$, it is exponentially decaying.

1.4.11 Complex Exponential Signal

The signal $x(t)$ in Eq. (1.18) is the general complex exponential which has real part as $e^{\sigma t} \cos \omega t$ and the imaginary part $e^{\sigma t} \sin \omega t$. For $\sigma = 0$, the signal $x(t)$ is a sinusoid. For $\sigma > 0$, $x(t)$ is a sinusoid which is exponentially building and is shown in Fig. 1.14a. For $\sigma < 0$, the signal $x(t) = e^{-\sigma t}(\cos \omega t + j \sin \omega t)$ is exponentially decaying and is shown in Fig. 1.14b.

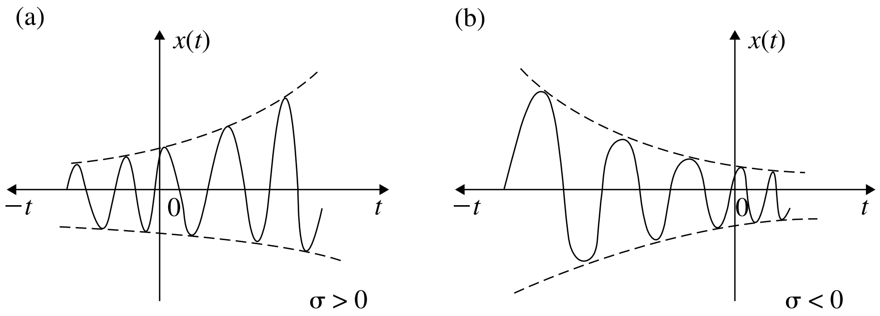


Fig. 1.14 Complex exponential signals. **a** Exponentially growing ($\sigma > 0$); **b** Exponentially decaying ($\sigma < 0$)

1.5 Basic Operations on Continuous-Time Signals

The basic operations performed on continuous-time signals are given below:

1. Addition of CT signals.
2. Multiplications of CT signals.
3. Amplitude scaling of CT signals.
4. Time scaling of CT signals.
5. Time shifting of CT signals.
6. Reflection or folding of CT signals.
7. Inverted CT signal.

1.5.1 Addition of CT Signals

Consider the signals $x_1(t)$ and $x_2(t)$ which are shown in Fig. 1.15a and b. The amplitude of these two signals at each instant of time is added to get their sum. The following table is prepared.

From Table 1.1, $x(t) = x_1(t) + x_2(t)$ is plotted and is shown in Fig. 1.15c.

1.5.2 Multiplication of CT Signals

Consider the two signals $x_1(t)$ and $x_2(t)$ shown in Fig. 1.15a and b respectively. These signals $x_1(t)$ and $x_2(t)$ are multiplied to get $x(t)$

$$x(t) = x_1(t) \times x_2(t)$$

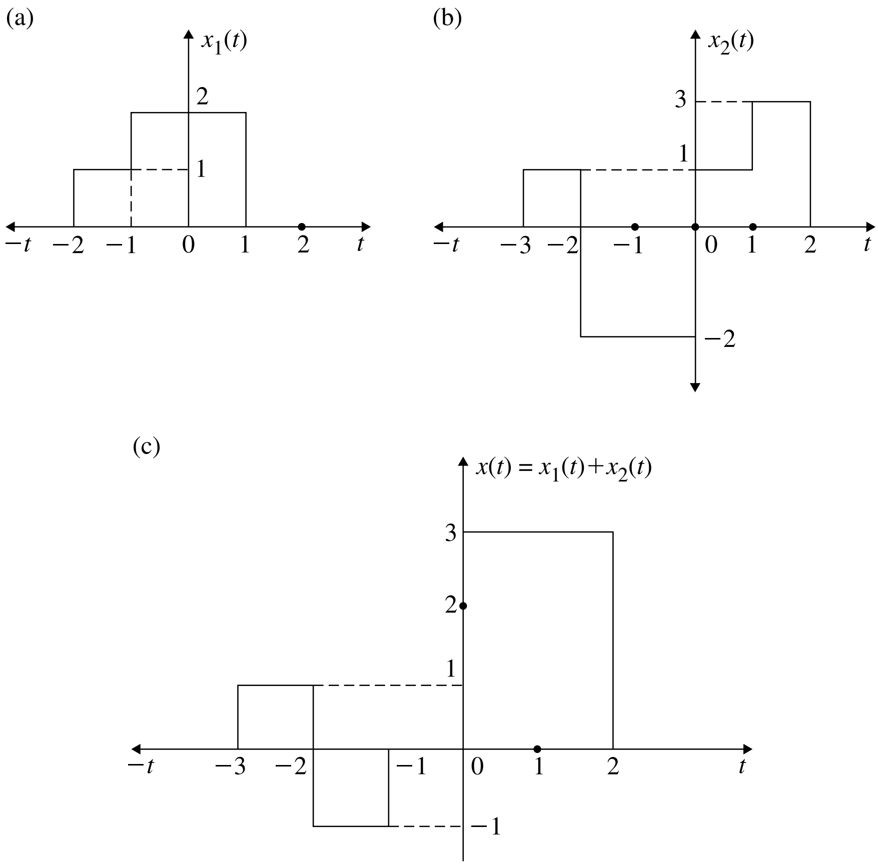


Fig. 1.15 Addition of two CT signals (*Contd.*) Addition of two CT signals

Table 1.1 Sum of two signals $x_1(t)$ and $x_2(t)$

t	−3	−2	−1	0	1	2
$x_1(t)$	0	1	2	2	0	0
$x_2(t)$	1	−2	−2	1	3	0
$x(t) = x_1(t) + x_2(t)$	1	−1	0	3	3	0

The functions $x_1(t)$ and $x_2(t)$ at different time intervals are determined from figure and multiplied. Table 1.2 is prepared to get $x(t)$ at different time intervals. Table 1.2 is transformed to plot $x(t) = x_1(t) \times x_2(t)$ which is shown in Fig. 1.16.

Table 1.2 Product of two signals $x_1(t)$ and $x_2(t)$

t	−3	−2	−1	0	1	2
$x_1(t)$	0	1	2	2	0	0
$x_2(t)$	1	−2	−2	1	3	0
$x(t) = x_1(t) \times x_2(t)$	0	−2	−4	2	0	0

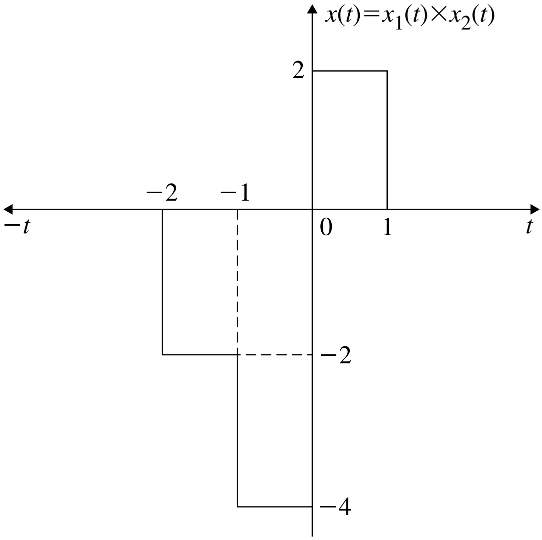


Fig. 1.16 Multiplications of two CT signals

1.5.3 Amplitude Scaling of Signals

Consider the signals $x(t)$ sketched and shown in Fig. 1.17a. This signal when multiplied by a factor A is expressed as $Ax(t)$. At any time t, the amplitude of $x(t)$ is multiplied by A. This type of signal transformation is called amplitude scaling. The signal $3x(t)$ is shown in Fig. 1.17b. At any instant t, $x(t)$ is multiplied by a factor 3.

Consider the signal $\frac{x(t)}{2}$. At any time t, the amplitude of $x(t)$ shown in Fig. 1.17a is divided by the factor 2. The above transformation is plotted in Fig. 1.17c.

1.5.4 Time Scaling of CT Signals

The compression or expansion of a signal in time is known as time scaling. Consider the signal $x(t)$ shown in Fig. 1.18a. The signal is time compressed and shown in Fig. 1.18b as $x(4t)$. For any given magnitude of $x(t)$, the time is divided by the factor 4. The time expanded signal $x(\frac{t}{4})$ is shown in Fig. 1.18c. Here, for any given

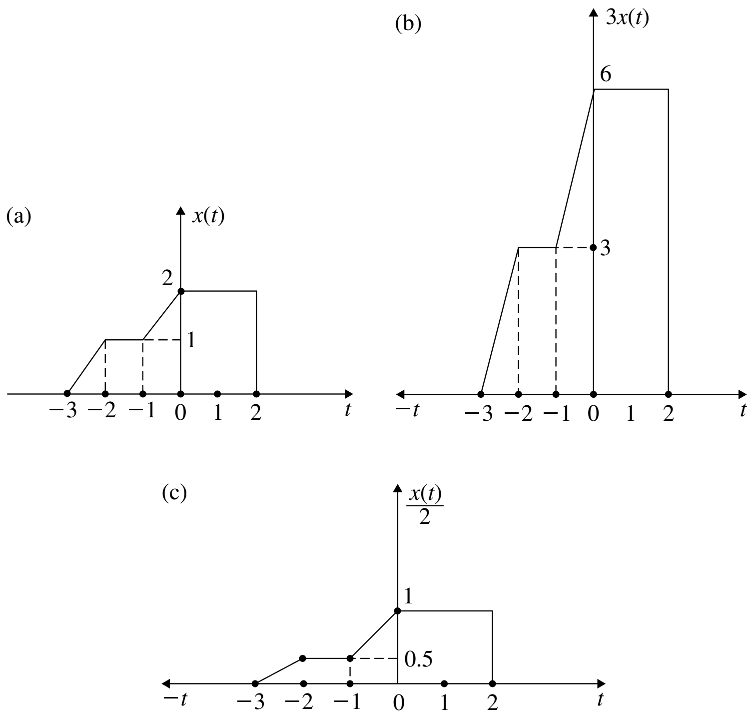


Fig. 1.17 Amplitude scaling. **a** $x(t)$; **b** $3x(t)$ and **c** $\frac{x(t)}{2}$

magnitude of $x(t)$, the time is multiplied by the factor 4. In general, for any given amplitude of $x(t)$, $x(at)$ is time compressed by a factor a and $x(\frac{t}{a})$ is time expanded by a factor a.

1.5.5 *Time Shifting of CT Signals*

Consider the signal $x(t) = u(t)$, the unit step function. The step function is shown in Fig. 1.19a as $u(t)$. The transformation $t = t - t_0$ where t_0 is any arbitrary constant amounts to shifting $u(t)$ to the right by t_0 unit if t_0 is positive and is denoted as $u(t - t_0)$. If t_0 is negative, the function is shifted to the left by t_0 unit and is denoted as $u(t + t_0)$. The right shifted $u(t - t_0)$ is shown in Fig. 1.19b and left shifted $u(t + t_0)$ is shown in Fig. 1.19c. The signal $u(-t)$ is shown in Fig. 1.19d and is obtained by folding $u(t)$ shown in Fig. 1.19a. $u(-t) = 1$ for $t < 0$. If we fold across the vertical axis, the signal to the right of the vertical axis is transformed to its left and *vice versa*. That is why it is called **folded signal**. The signal $u(-t - t_0)$ is obtained by shifting

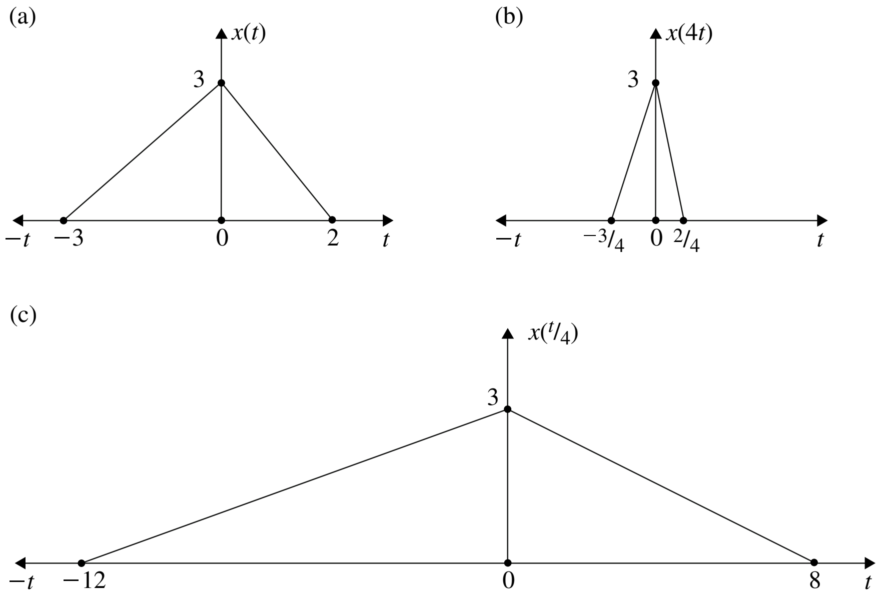


Fig. 1.18 Time scaling of CT signals

the signal $u(-t)$ to the left by t_0 unit as shown in Fig. 1.19e. The signal $u(-t+t_0)$ is obtained by shifting the signal $u(-t)$ to the right by t_0 unit and is shown in Fig. 1.19f.

Summary of Shifting of CT Signal

1. **It $x(t)$ is given, then $x(t+t_0)$ is plotted by shifting $x(t)$ to the left by t_0.**
2. **It $x(t)$ is given, then $x(t-t_0)$ is plotted by shifting $x(t)$ to the right by t_0.**
3. **It $x(-t)$ is given, then $x(-t-t_0)$ is plotted by shifting $x(-t)$ to the left by t_0.**
4. **It $x(-t)$ is given, then $x(-t+t_0)$ is plotted by shifting $x(-t)$ to the right by t_0.**
5. **In general for $x(t+t_0)$ and $x(-t-t_0)$ the time shift is made to the left of $x(t)$ and $x(-t)$ respectively by t_0. For $x(t-t_0)$ and $x(-t+t_0)$ the time shift is made to the right of $x(t)$ and $x(-t)$ respectively by t_0.**

1.5.6 Signal Reflection or Folding

Consider the signal $x(t)$ shown in Fig. 1.20a. The signal $x(-t)$ is obtained by putting a mirror along the vertical axis. The signal to the right of the vertical axis gets reflected to the left and *vice versa*. Alternatively, if we make a folding across the vertical axis,

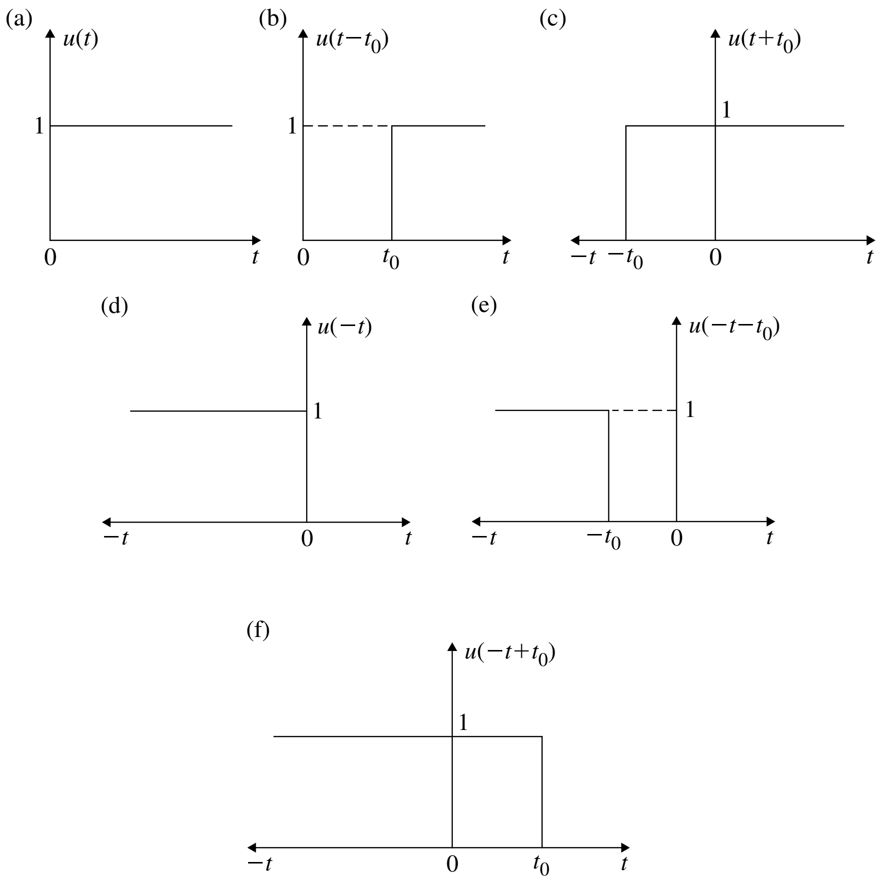


Fig. 1.19 Representation of time shifting CT signals

the signal in the right of the vertical axis is printed in the left and *vice versa*. The signal so obtained is $x(-t)$.

1.5.7 Inverted CT Signal

Consider the CT signal $x(t)$ shown in Fig. 1.21a. The inverted signal $-x(t)$ is obtained by inverting its amplitude. By this the signal above the horizontal axis (time axis) comes below the axis and *vice versa*. Alternatively, if a mirror is put along the horizontal axis, the signal above the axis gets reflected below the axis and *vice versa*.

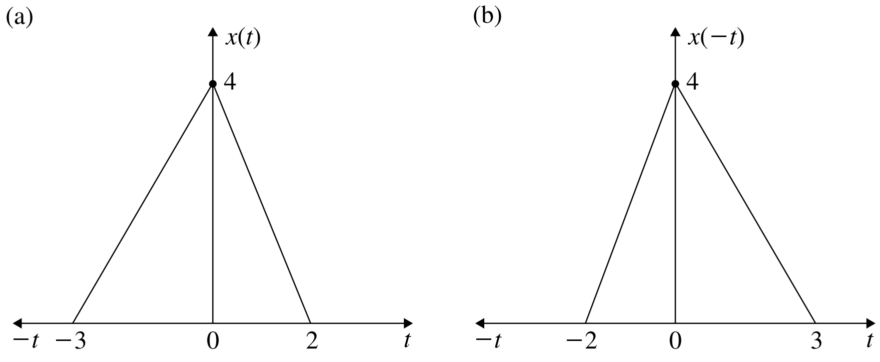


Fig. 1.20 CT signal reflection or folding

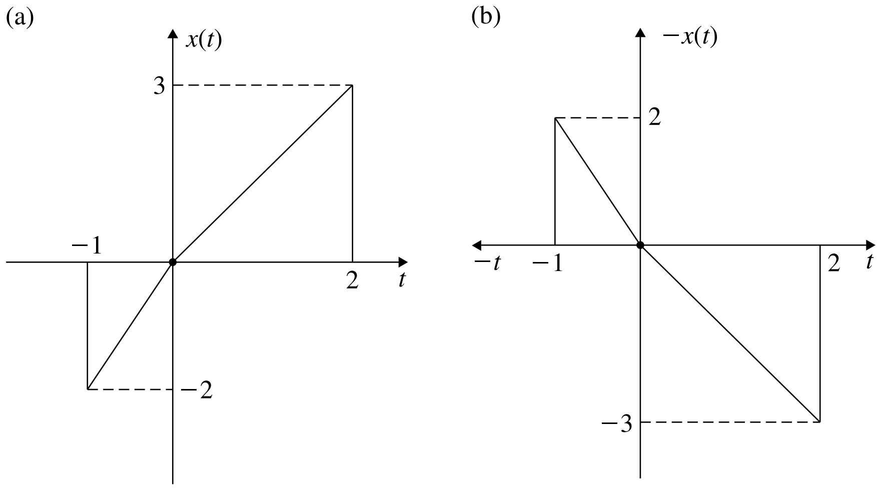


Fig. 1.21 Inverted CT signal

1.5.8 Multiple Transformation

The transformation, namely, amplitude scaling, time reversal, time shifting, time scaling, etc. when applied simultaneously, the sequence of operation is important. If not followed correctly, it would give erroneous results.

Consider the following signal:

$$y(t) = Ax\left(\frac{-t - t_0}{a}\right)$$

The sequence of transformation is as follows:

1. $y(t)$ is written in the following form:

$$y(t) = Ax\left(-\frac{t}{a} - \frac{t_0}{a}\right)$$

2. Plot $x(t)$.
3. Plot $Ax(t)$ using amplitude scaling.
4. Plot $Ax(-t)$ using time reversal.
5. Plot $Ax(-t - \frac{t_0}{a})$ by shifting $Ax(-t)$ to the left by $\frac{t_0}{a}$ (time shifting).
6. Plot $Ax(-\frac{t}{a} - \frac{t_0}{a})$ by time expansion.

The following examples illustrate the above sequence of operation.

Example 1.1 Consider the signal $y(t) = 5x(-3t + 1)$ where $x(t)$ is shown in Fig. 1.2a. Plot $y(t)$ and $-y(t)$.

Solution

1. The given signal $x(t)$ is represented in Fig. 1.22a.
2. The signal $x(t)$ is amplitude scaled and plotted in Fig. 1.22b.
3. $5x(-t)$ is obtained by folding $5x(t)$ in Fig. 1.22b and is plotted in Fig. 1.22c.
4. $5x(-t)$ is time shifted by one unit to the right and $5x(-t + 1)$ is obtained and shown in Fig. 1.22d.
5. $5x(-t + 1)$ is time compressed by a factor 3 and $5x(-3t + 1)$ is obtained. This is shown in Fig. 1.22e.
6. $5x(-3t + 1)$ amplitude inverted to get $-5x(-3t + 1)$. This is shown in Fig. 1.22f.

Example 1.2 For a signal $x(t)$ shown in Fig. 1.23a, sketch

(a) $x(3t + 2)$

(b) $x\left(\frac{-t}{2} - 1\right)$

(*Anna University, June 2007*)

Solution To plot $x(3t + 2)$

1. $x(t)$ is represented in Fig. 1.23a. $x(t)$ is moved to the left by $t = 2$ and is shown in Fig. 1.23b.
2. By time compression by a factor 3, from Fig. 1.23b, $x(3t + 2)$ is obtained and is shown in Fig. 1.23c.

Solution To plot $x(-(\frac{t}{2}) - 1)$

1. By folding $x(t)$ represented in Fig. 1.23a, $x(-t)$ is obtained and is shown in Fig. 1.23d.

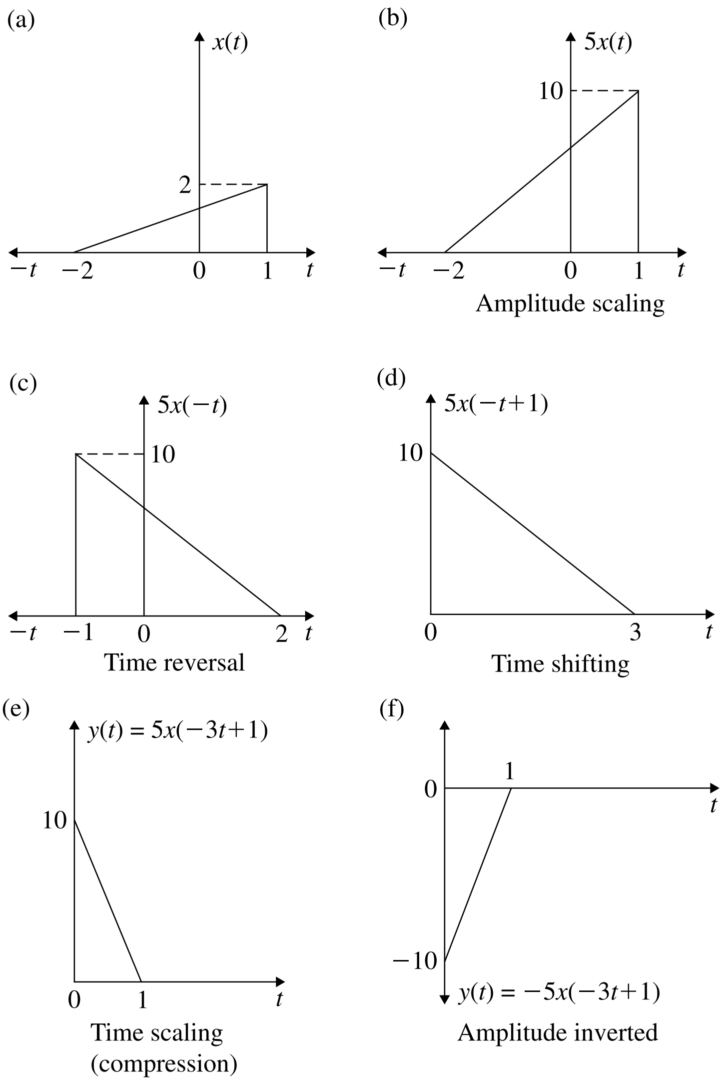


Fig. 1.22 Basic operations on CT signal

2. $x(-t-1)$ is obtained by shifting $x(-t)$ by $t = 1$ to the left. $x(-t-1)$ is sketched as shown in Fig. 1.23e.
3. By time expansion, the time of the signal $x(-t-1)$ is multiplied by the factor 2, and $x(-\frac{t}{2}-1)$ is obtained. This is shown in Fig. 1.23f.

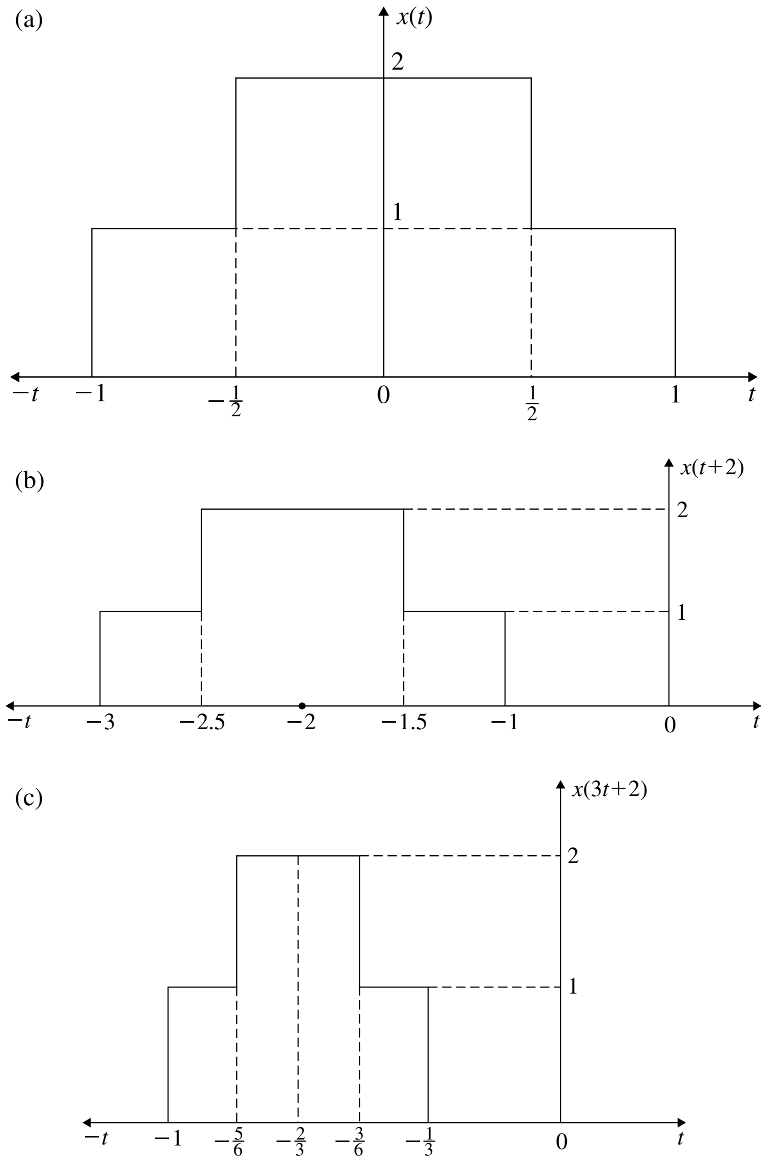


Fig. 1.23 **a** Plot of $x(t)$. **b** Time shifted $x(t)$. **c** Time compressed $x(t)$. **d** Folded $x(t)$. **e** Time shifted $x(-t)$. **f** Time expansion of $x(-t-1)$ to get $x(-\frac{t}{2}-1)$

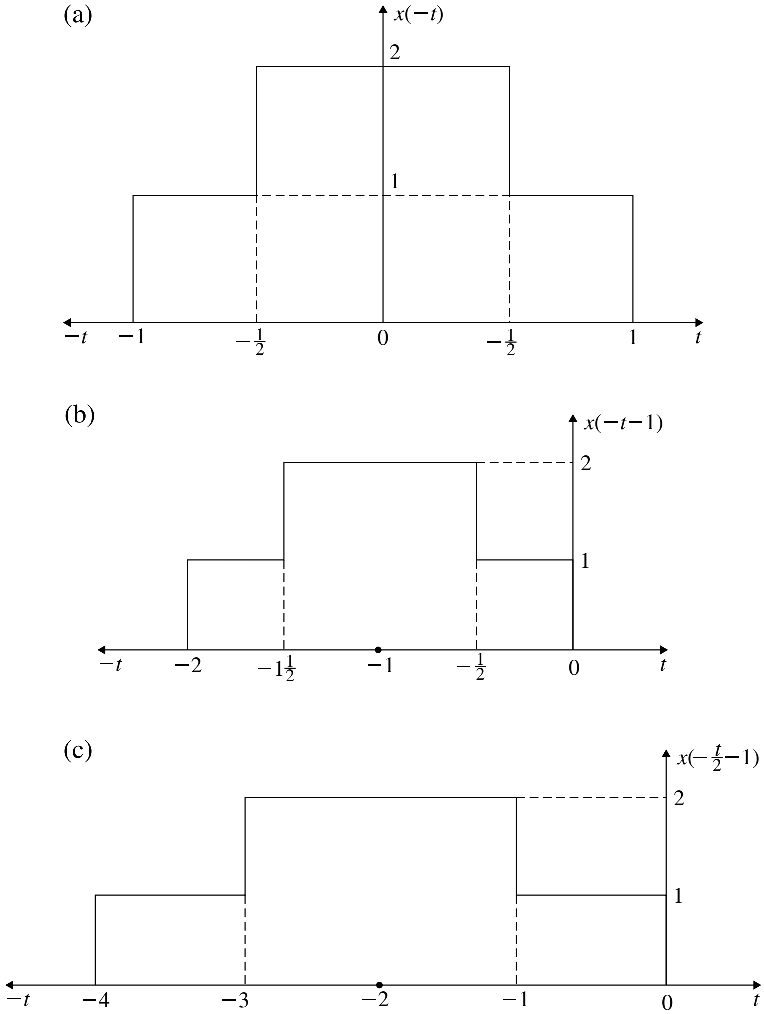


Fig. 1.23 (continued)

Example 1.3 The rectangular signal $x(t)$ is shown in Fig. 1.24a. Sketch the following signals:

(a) $x(t-3)$

(b) $2x(t)$

(c) $-3x(t)$

(d) $x(t-2)+3x(t)$

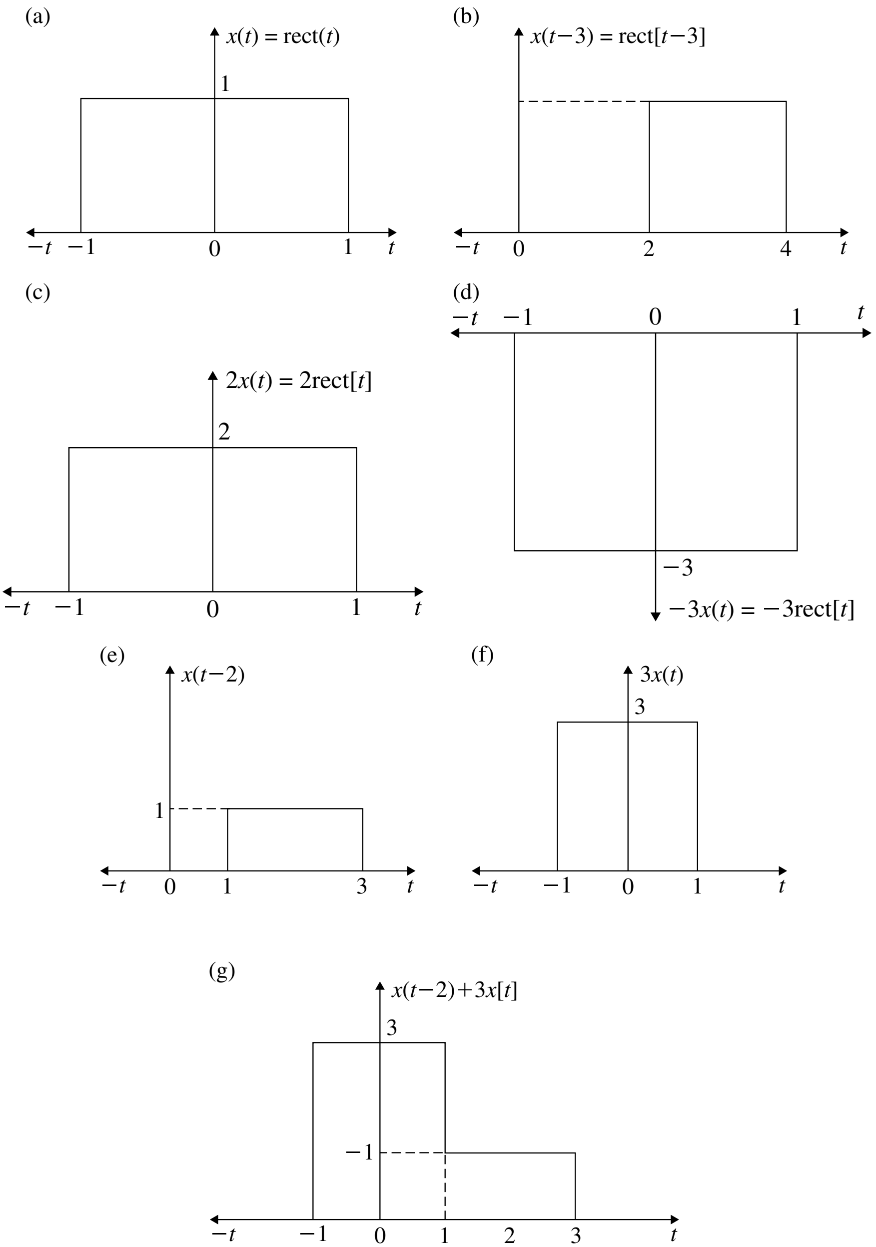


Fig. 1.24 **a** $x(t) = \text{rect}(t)$ signal and **b** Representation of $x(t-3) = \text{rect}[t-3]$. **c** Representation of $2x(t) = 2\,\text{rect}[t]$ and **d** Representation of $-3x(t) = -3\text{rect}[t]$. **e** Representation of $x(t-2)$ and **f** Representation of $3x(t)$. **g** Representation of $x(t-2) + 3x(t)$

Solution

(a) **To represent the signal $x(t-3)$**
$x(t-3)$ is obtained by time shifting $x(t)$ by 3 unit of time towards right. This is shown in Fig. 1.24b.

(b) **To represent the signal $2x(t) = 2\,\text{rect}[t]$**
This is amplitude scaled signal. The amplitude of $x(t) = \text{rect}[t]$ is multiplied by the factor 2 and is shown in Fig. 1.24c.

(c) **To represent the signal $-3x(t) = -3\text{rect}[t]$**
The signal $x(t)$ is amplitude inverted and multiplied by a factor 3. This is shown in Fig. 1.24d.

(d) **To represent the signal $x(t-2) + 3x(t)$**
The time delayed $x(t-2)$ is obtained by shifting $x(t)$ to the right by a factor 2. This is represented in Fig. 1.24e. The signal $x(t)$ is amplitude multiplied by a factor 3 and $3x(t)$ is obtained. This is shown in Fig. 1.24f. By adding the signals shown in Fig. 1.24e and f, $x(t-2) + 3x(t)$ is obtained and is represented in Fig. 1.24g.

Example 1.4 Consider the triangular wave form $x(t)$ shown in Fig. 1.25a. Sketch the following wave forms:

(a) $x(2t+3)$

(b) $x\left(\dfrac{t+3}{2}\right)$

(c) $x\left(\dfrac{t}{2}-3\right)$

(d) $x(-2t+3)$

(e) $x(-2t-3)$

Solution

(a) **To sketch $x(2t+3)$**
Figure 1.25a shows $x(t) = \text{tri}(t)$. By time shifting by $t = 3$ towards left, $x(t+3)$ is obtained and this is sketched in Fig. 1.25b. $x(t+3)$ is time compressed by a factor of 2 to get $x(2t+3)$. This is sketched in Fig. 1.25c.

(b) **To sketch $x\left(\frac{t+3}{2}\right)$**
The signal $x\left(\frac{t+3}{2}\right)$ is written as $x\left(\frac{t}{2}+1.5\right)$. The signal $x(t)$ is time shifted to the left by 1.5 unit to get $x(t+1.5)$. This is sketched in Fig. 1.25d. $x(t+1.5)$ is time expanded by a factor 2 to get $x\left(\frac{t}{2}+1.5\right)$ which is nothing but $x\left(\frac{t+3}{2}\right)$. This is sketched in Fig. 1.25e.

(c) **To sketch $x\left(\frac{t}{2}-3\right)$**
$x(t-3)$ is obtained from $x(t)$ by time shifting the signal $x(t)$ to the right by 3 unit and is shown in Fig. 1.25f. By time expansion of $x(t-3)$ by a factor 2, $x\left(\frac{t}{2}-3\right)$ is obtained and sketched as shown in Fig. 1.25g.

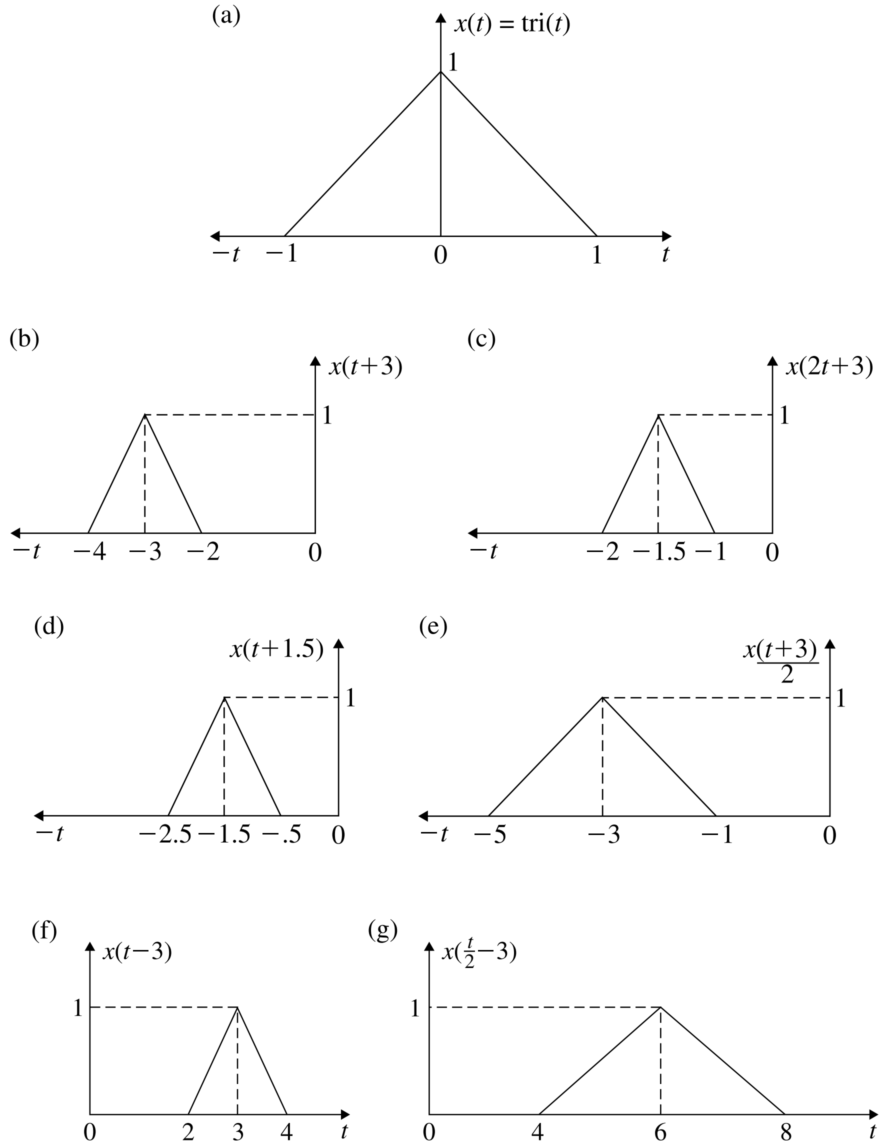


Fig. 1.25 **a** $x(t) = \text{tri}(t)$. **b** $x(t+3)$; **c** $x(2t+3)$. **d** $x(t+1.5)$; **e** $x\left(\frac{t+3}{2}\right)$. **f** $x(t-3)$; **g** $x\left(\frac{t}{2}-3\right)$. **h** $x(-t)$; (i) $x(-t+3)$; **j** $x(-2t+3)$. **k** $x(-t-3)$; **l** $x(-2t-3)$

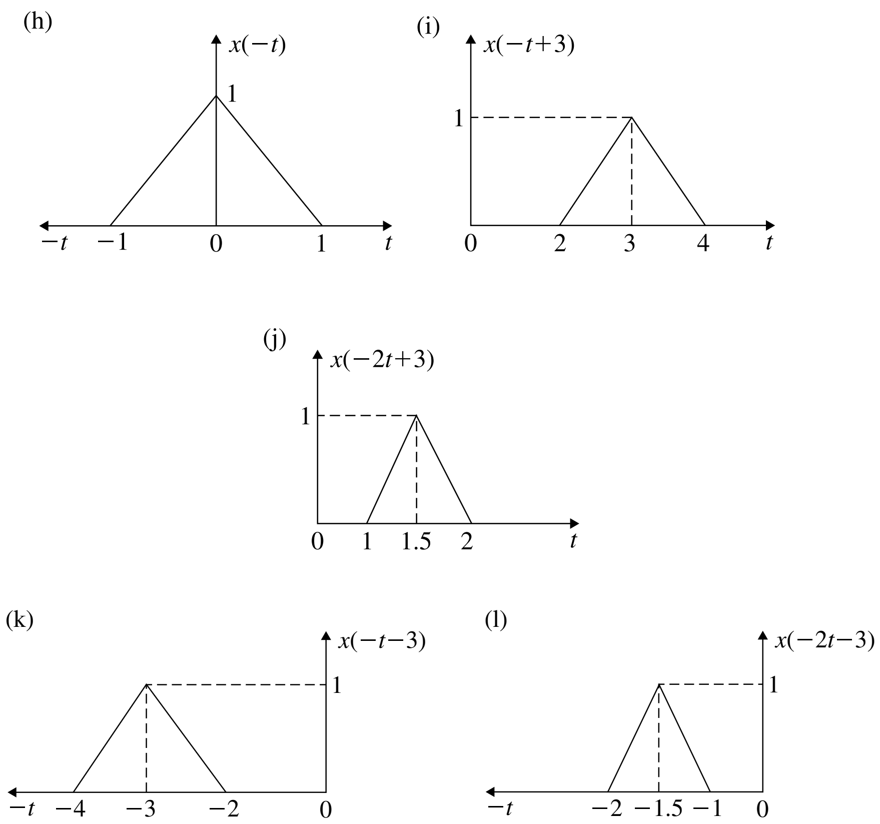


Fig. 1.25 (continued)

(d) **To sketch the signal $x(-2t+3)$**
Signal $x(-t)$ is obtained by folding $x(t)$ and it is shown in Fig. 1.25h. $x(-t)$ is time shifted to the right by 3 unit to get $x(-t+3)$. This is shown in Fig. 1.25i. The signal $x(-t+3)$ is time compressed by a factor 2 to get $x(-2t+3)$. This is sketched in Fig. 1.25j.

(e) **To sketch the signal $x(-2t-3)$**
$x(-t)$ is shown in Fig. 1.25h. From Fig. 1.25h, $x(-t)$ is time shifted towards left by 3 units to get $x(-t-3)$. This is shown in Fig. 1.25k. $x(t-3)$ is time compressed by a factor 2 to get $x(-2t-3)$. This is sketched in Fig. 1.25l.

Example 1.5 A continuous-time signal $x(t)$ is shown in Fig. 1.26a. Sketch and label carefully each of the following signals:

(a) $x(t-1)$

(b) $x(2-t)$

(c) $x(t)\left[\delta\left(t+\frac{3}{2}\right)-\delta\left(t-\frac{3}{2}\right)\right]$

(d) $x(2t+1)$

(*Anna University, April 2008*)

Solution

(a) **To sketch $x(t-1)$**
$x(t-1)$ is the time delayed signal of $x(t)$ by one unit. $x(t)$ is shifted to the right by $t=1$ and it is sketched as shown in Fig. 1.26b.

(b) **To sketch $x(2-t)$**
The folded signal of $x(t)$ is $x(-t)$ and is shown in Fig. 1.26c. $x(-t)$ is right shifted by 2 unit to get $x(2-t)$ and is shown in Fig. 1.26d.

(c) **To sketch $x(t)[\delta(t+\frac{3}{2})-\delta(t-\frac{3}{2})]$**
$\delta(t+\frac{3}{2})$ and $\delta(t-\frac{3}{2})$ are shown in Fig. 1.26e, which occur as unit impulses at $t=-\frac{3}{2}$ and $t=\frac{3}{2}$ respectively. At $t=-\frac{3}{2}$, $x(t)=-\frac{1}{2}$ and $\delta(t+\frac{3}{2})=1$. Using the property of impulse $x(t)\delta(t-t_0)=x(t_0)\delta(t-t_0)$, we get $x(t)\delta(t+\frac{3}{2})=-\frac{1}{2}$. Similarly at $t=\frac{3}{2}$, $x(t)=\frac{1}{2}$ and $-\delta(t-\frac{3}{2})=-1$. Hence, $x(t)\delta(t-\frac{3}{2})=-\frac{1}{2}$. This is sketched as shown in Fig. 1.26f.

(d) **To sketch $x(2t+1)$**
From Fig. 1.26a, $x(t+1)$ is derived by shifting $x(t)$ to the left by $t=1$. This is shown in Fig. 1.26g. By time compression of $x(t+1)$ by a factor 2, $x(2t+1)$ is obtained and sketched as shown in Fig. 1.26h.

Example 1.6 Represent the signal $x(t)=5u(4-t)$.

Solution

1. The unit step signal when its amplitude is multiplied by a factor 5, it becomes $5u(t)$. When this is time reversed, it becomes $5u(-t)$ and is shown in Fig. 1.27a.
2. $5u(-t)$ is time shifted to the right by $t=4$ and is sketched as $5u(4-t)$ in Fig. 1.27b.

Example 1.7 Sketch the signal $x(t)=[u(t)-u(t-a)]$ where $a>0$.

Solution

1. The unit step signal $u(t)$ is shown in Fig. 1.28a.
2. The unit step signal with a time delay a and amplitude inverted is shown in Fig. 1.28b.
3. If the above two step signals are added, a pulse signal is obtained and is sketched as shown in Fig. 1.28c which gives $u(t)-u(t-a)$. The above signal is defined as

$$x(t)=1 \qquad 0\le t\le a$$

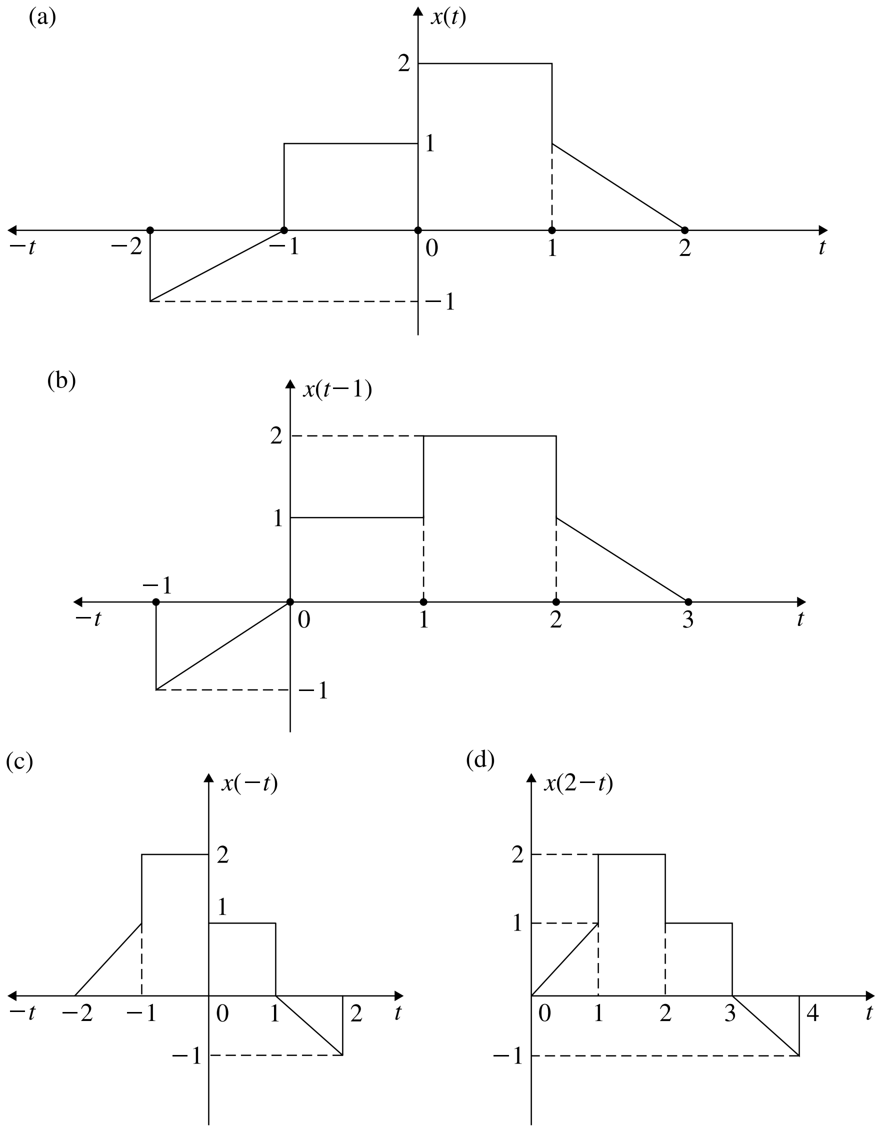


Fig. 1.26 **a** $x(t)$ plot. **b** $x(t-1)$ plot. **c** $x(-t)$ and **d** $x(2-t)$. **e** $\delta(t+\frac{3}{2})$, $-\delta(t-\frac{3}{2})$ and **f** $x(t)[\delta(t+\frac{3}{2}) - \delta(t-\frac{3}{2})]$. **g** $x(t+1)$ and **h** $x(2t+1)$

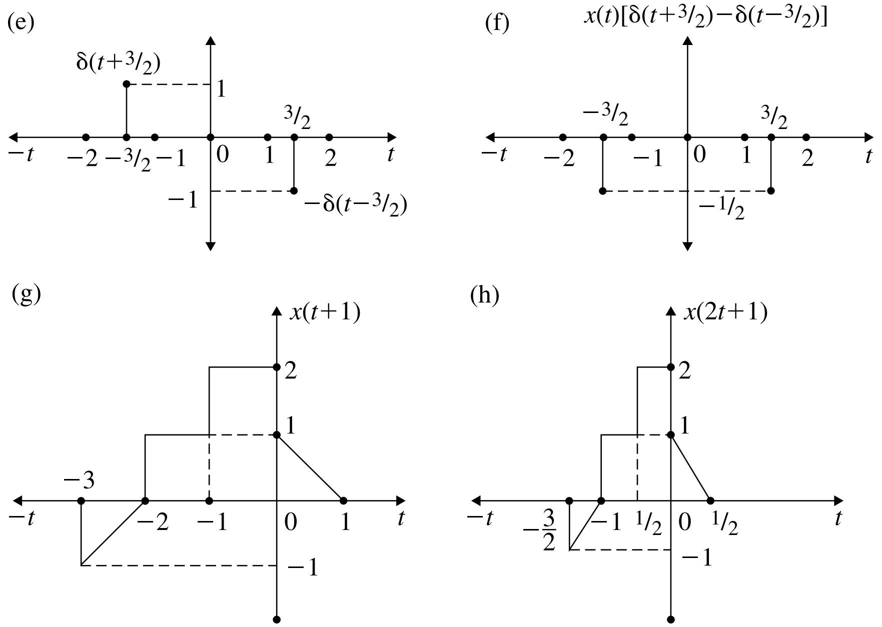


Fig. 1.26 (continued)

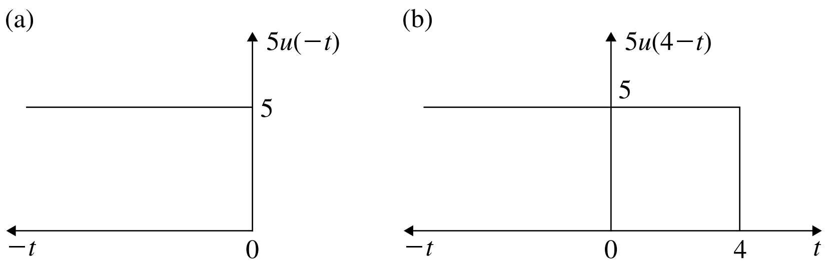


Fig. 1.27 Time shifted step signal

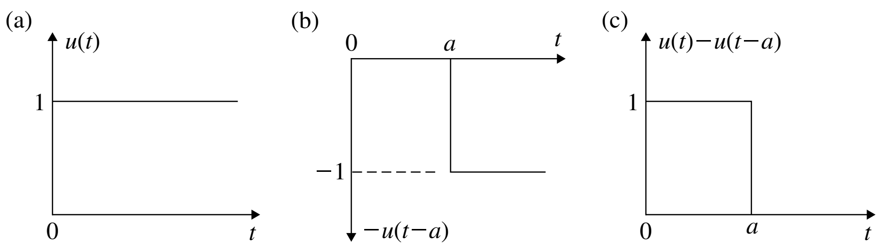


Fig. 1.28 Pulse signal from two step signals

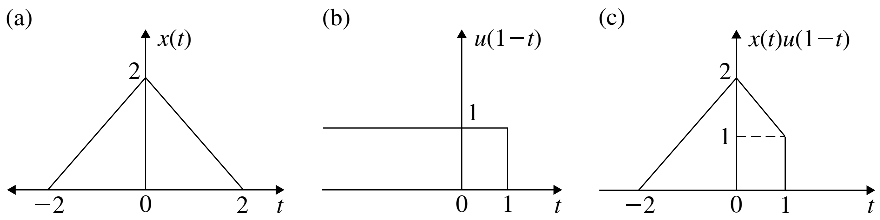


Fig. 1.29 Product of triangular and time delayed step signals

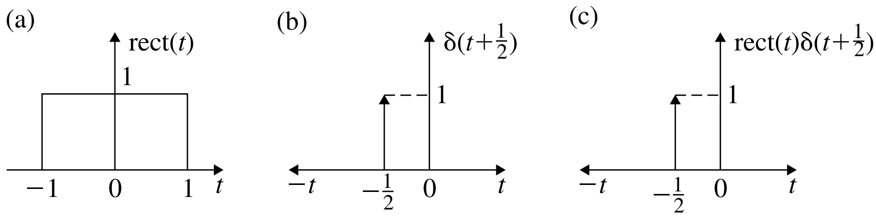


Fig. 1.30 Product of rectangular and time advanced impulse

Example 1.8 Consider the signal $x(t)$ shown in Fig. 1.29a. Sketch the signal $x(t)u(1-t)$.

Solution

1. The signal $x(t)$ is shown in Fig. 1.29a. The signal $u(1-t)$ is shown in Fig. 1.29b.
2. The signal $x(t)$ is multiplied by the factor 1 for the intervals $-2 \leq t \leq 0$ and $0 \leq t \leq 1$. During these time intervals, the slopes of the straight lines of the triangles are $+1$ and -1 respectively. Hence, $x(t)$ is retained as it is. At $t = 1$, $x(t) = 1$ and $u(1-t) = 1$. Hence, $x(t)u(1-t) = 1$.
3. For $t > 1$, $u(1-t) = 0$ and hence $x(t)u(1-t) = 0$. This is sketched in Fig. 1.29c.

Example 1.9 Consider the signal rect(t). Sketch the signal rect(t) $\delta(t + \frac{1}{2})$.

Solution

1. The rectangular pulse rect(t) is shown in Fig. 1.30a.
2. The time advanced impulse $\delta(t + \frac{1}{2})$ is defined as follows:

$$\delta\left(t + \frac{1}{2}\right) = 1 \quad \text{if } t = -\frac{1}{2}$$
$$= 0 \quad \text{otherwise}$$

This is sketched in Fig. 1.30b.

3. At $t = -\frac{1}{2}$, the magnitude of rect$(t) = 1$. Hence, using the property $x(t)\ \delta(t + t_0) = x(t_0)$, we sketch $x(t)\delta(t + t_0)$ as an impulse at $t = -\frac{1}{2}$ which is shown in Fig. 1.30c.

Example 1.10

$$x(t) = 10e^{-3t+4}$$

Determine $x(t+2)$, $x(-t+2)$, and $x(\frac{t}{4} - 5)$.

Solution

$$x(t) = 10e^{-3t+4}$$

1. For $t = t + 2$,

$$x(t+2) = 10e^{-3(t+2)+4}$$

$$x(t+2) = 10e^{-3t-2}$$

2. For $t = -t + 2$,

$$x(-t+2) = 10e^{-3(-t+2)+4}$$

$$x(-t+2) = 10e^{3t-2}$$

3. For $t = (\frac{t}{4} - 5)$,

$$x\left(\frac{t}{4} - 5\right) = 10e^{-3(\frac{t}{4}-5)+4}$$

$$x\left(\frac{t}{4} - 5\right) = 10e^{-\frac{3}{4}t+19}$$

Example 1.11 Decompose the signal $x(t)$ shown in Fig. 1.31a in terms of basic signals such as delta, step, and ramp.

(*Anna University, December 2007*)

Solution

1. The given signal $x(t)$ is shown in Fig. 1.31a.
2. The signals $u(t) + u(t-1) - 3u(t-2)$ are shown in Fig. 1.31b and their sum is shown in Fig. 1.31c.
3. The signals $r(t-3)$ and $r(t-4)$ are shown in Fig. 1.31d.

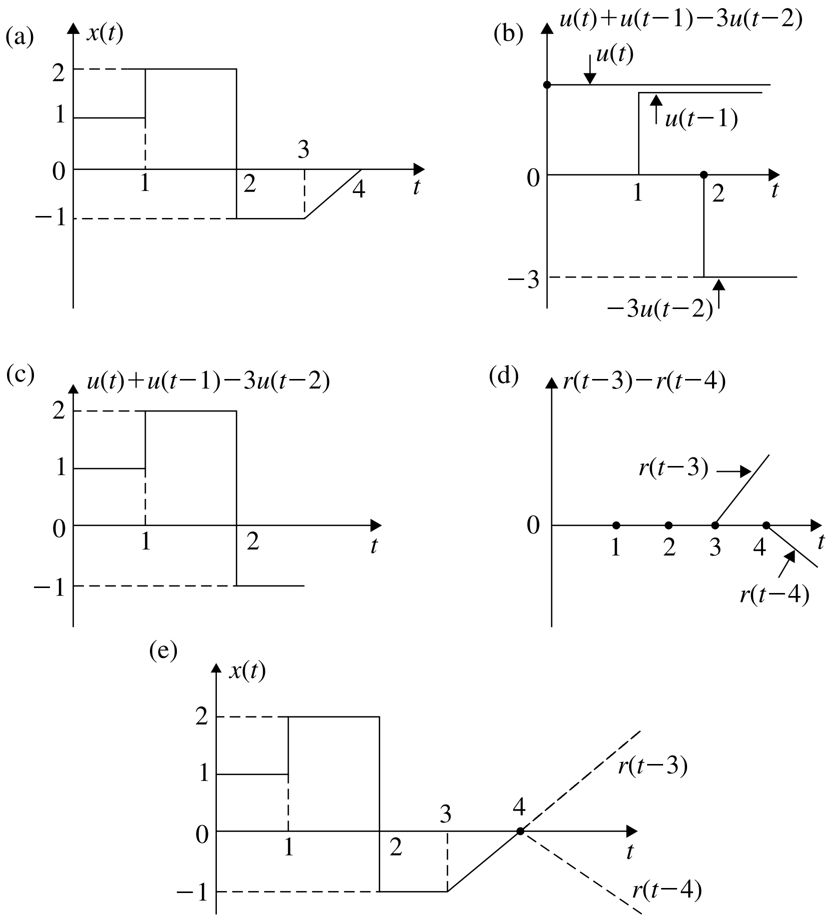


Fig. 1.31 Composite signal expressed in terms of basic signals

4. Signals in Fig. 1.31c and d are summed up and they are shown in Fig. 1.31e which is nothing but $x(t)$. Hence

$$x(t) = u(t) - u(t-1) - 3u(t-2) + r(t-3) - r(t-4) + u(t-3)$$

or

$$x(t) = u(t) + u(t-1) - 3u(t-2) + (t-4)[u(t-3) - u(t-4)] + u(t-3)$$

Example 1.12 Sketch the signals

$$\text{(a)} \quad x(t) = -4\text{sgn}\, 3t \qquad \text{(b)} \quad x(t) = 5\text{sinc}\, 10t$$

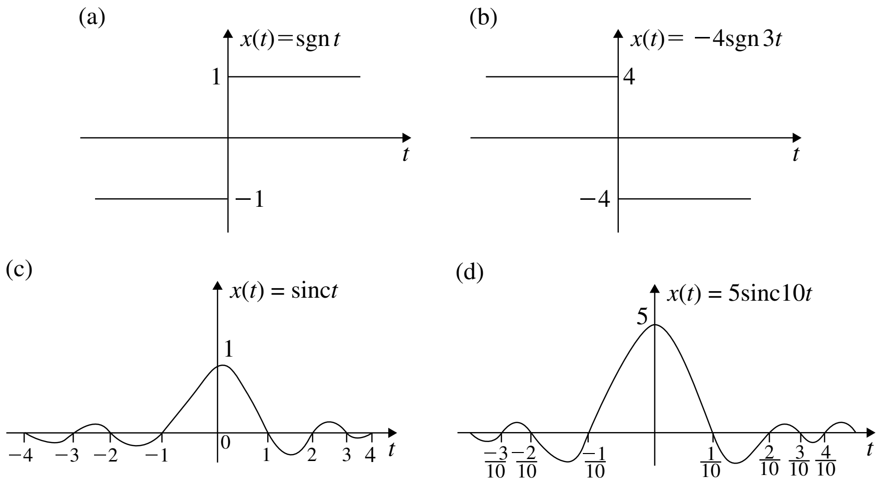


Fig. 1.32 Representation of signum and sinc functions

Solution

(a) $\boldsymbol{x(t) = -4\text{sgn}3t}$
The signal sgn t is shown in Fig. 1.32a. The signum function is inverted and multiplied by a factor 4. The time compression by a factor 3 does not apply in this case as the signal remains constant for $-\infty < t < \infty$. The signal is shown in Fig. 1.32b.

(b) $\boldsymbol{x(t) = 5\text{sinc}10t}$
The signal sinc t is sketched in Fig. 1.32c. The sinc function amplitude is multiplied by the factor 5 and the time is compressed by the factor 10. $x(t) = 5\sin 10t$ is represented in Fig. 1.32d.

1.6 Classification of Signals

Signals which are classified in the broad category of continuous- and discrete-time signals are further classified as follows:

1. Deterministic and non-deterministic (random) signals.
2. Periodic and non-periodic (aperiodic) signals.
3. Odd and even signals.
4. Power and energy signals.

1.6.1 Deterministic and Non-deterministic Continuous Signals

Deterministic signals are signals which are characterized mathematically. The amplitude of such signals at any time interval t can be determined at all time t. Consider the signals described by the following equations:

$$x(t) = A$$
$$x(t) = A \sin \omega t$$

The above signals represent a step signal and a sinusoidal signal respectively and they are shown in Fig. 1.33a and b. At any instant of time t the amplitude of the step signal which is deterministic can be easily determined. On the other hand consider the sinusoidal signal polluted with noise shown in Fig. 1.33b. The magnitude of such a signal cannot be easily determined since the noise variation is random.

1.6.2 Periodic and Non-periodic Continuous Signals

Consider the continuous-time signal described by the following equation:

$$x(t + nT_0) = x(t) \text{ for all } t \tag{1.20}$$

where n is any integer value. **A continuous-time signal $x(t)$ is said to be periodic with period T_0 if it repeats itself in a minimum positive interval. The minimum positive interval over which a function repeats is called fundamental period T_0**. The fundamental frequency f is expressed as

$$f_0 = \frac{1}{T_0} \tag{1.21}$$

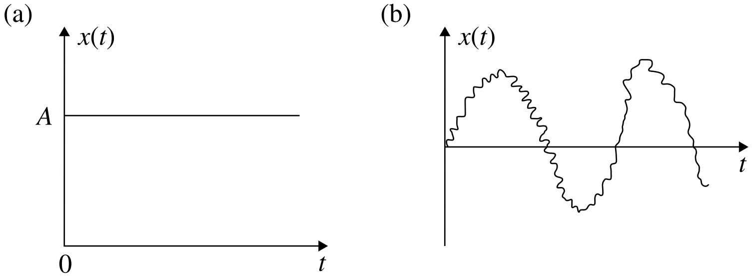


Fig. 1.33 Continuous. **a** Deterministic signal; **b** Random signal

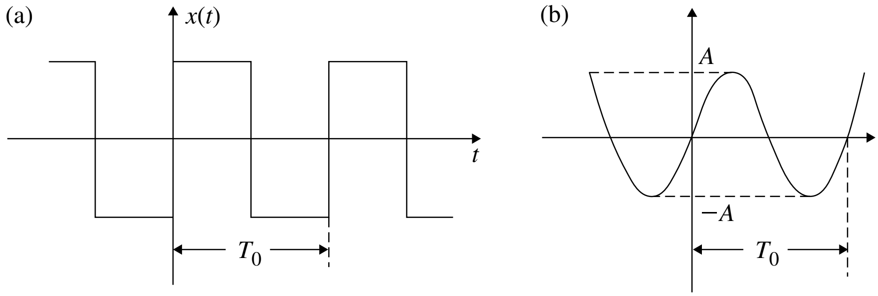


Fig. 1.34 Examples of periodic signals. **a** Rectangular wave; **b** Sine wave

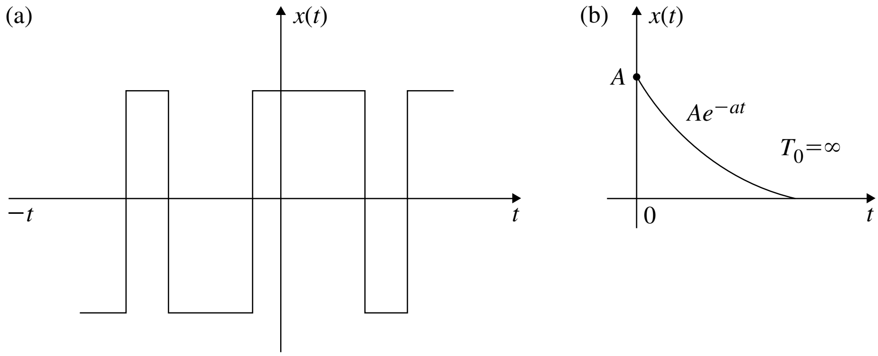


Fig. 1.35 Non-periodic signals. **a** Rectangular; **b** Exponential decay

where f_0 is expressed in cycles per sec. The fundamental radian frequency is expressed as

$$\begin{aligned}\omega_0 &= 2\pi f_0 \\ &= \frac{2\pi}{T_0}\end{aligned} \tag{1.22}$$

Here ω is expressed in rad./s. The periodic rectangular wave and sine wave are shown in Fig. 1.34a and b respectively.

Any continuous-time signal which is not periodic is said to be non-periodic or aperiodic signal. Figure 1.35a represents a non-periodic rectangular wave and Fig. 1.35b represents an exponential decay. The non-periodic signal does not repeat itself with respect to time.

1.6.3 Fundamental Period of Two Periodic Signals

Consider the periodic signal of two periodic functions with two different fundamental periods as given below:

$$x(t) = A_1 \sin\left(2\pi \frac{t}{T_1}\right) + A_2 \sin\left(2\pi \frac{t}{T_2}\right) \tag{1.23}$$

where T_1 and T_2 are the fundamental periods of two sine waves. The fundamental period of the composite signal $x(t)$ is given by the shortest time by which these signals have an integer number. If each of these two signals repeats exactly an integer number of times in some minimum time interval, then they will repeat exactly an integer number of times again in the next time interval. This is calculated as the least common multiple [LCM] of the two fundamental periods. Thus, the fundamental period of a periodic signal which is composed of more than one periodic signal is obtained by taking least common multiple of the fundamental periods of all the signals. The fundamental frequency of the sum of the signals is the greatest common divisor of the two frequencies. **It is to be remembered that if any of the composite signal is non-periodic, then the overall function is also non-periodic.**

Instead of sum of two functions, if a signal is a product of two functions, the method of finding the fundamental period remains the same. Consider the following composite signal:

$$x(t) = A \sin\left(2\pi \frac{t}{T_1}\right) \sin\left(2\pi \frac{t}{T_2}\right) \tag{1.24}$$

The fundamental periods of the two sine functions are T_1 and T_2. The fundamental period of $x(t)$ is calculated as the least common multiple of T_1 and T_2. The sum of product of two or more periodic signals is periodic **iff** (if and only if) the ratio of their fundamental periods is rational. The following steps are followed to determine this:

1. Determine the fundamental period of the individual signal in the sum or product.
2. Find the ratio of the fundamental period of the first signal with the fundamental period of every other signal.
3. If these ratios are rational, then the sum or the product of the composite signal is periodic.
4. The fundamental period of the composite signal is determined by taking the least common multiple of the fundamental period. Alternatively, the greatest common divisor of the fundamental frequency of each signal gives the fundamental frequency of the composite signal.

For example if T_1, T_2, and T_3 are the fundamental periods of three signals which are the sums of the composite signal then the ratio $\frac{T_1}{T_2}$ and $\frac{T_1}{T_3}$ should be an integer multiple or rational. $\frac{T_1}{T_2} = \frac{5}{3}$ is an integer or rational number. On the other hand $\frac{T_1}{T_2} = \frac{5}{3.17}$ is not an integer number and it is not rational.

Sinusoidal and complex exponentials are examples of continuous-time periodic signals. Consider the following sinusoidal signal:

$$x(t) = A\sin(\omega_0 t + \theta) \tag{1.25}$$

$$\begin{aligned} x(t+T_0) &= A\sin(\omega_0(t+T_0)+\theta) \\ &= A\sin(\omega_0 t + \omega_0 T_0 + \theta) \end{aligned} \tag{1.26}$$

A sine function repeats itself when its total argument is increased or decreased by any integer multiple of 2π radians. Thus, in Eq. (1.32) if we put $\omega_0 T_0 = 2\pi$,

$$x(t+T_0) = A\sin(\omega_0 t + \theta) = x(t)$$

In other words the fundamental period of a sine function is

$$T_0 = \frac{2\pi}{\omega_0} \tag{1.27}$$

Now consider the complex exponential

$$x(t) = e^{j\omega_0 t}$$

$$x(t+T_0) = e^{j\omega_0(t+T_0)} \tag{1.28}$$

$$= e^{j\omega_0 t}e^{j\omega_0 T_0} \tag{1.29}$$

If we put $e^{j\omega_0 T_0} = 1$, Eq. (1.35) becomes

$$x(t+T_0) = e^{j\omega_0 t} = x(t)$$

Thus, the condition for the complex exponential to be periodic is that

$$e^{j\omega_0 T_0} = 1$$

$$\text{or} \quad \omega_0 T_0 = 2\pi \qquad [e^{j2\pi} = \cos 2\pi + j\sin 2\pi = 1]$$

$$T_0 = \frac{2\pi}{\omega_0} \tag{1.30}$$

Example 1.13 Test the periodicity of the following signals:

(a) $x(t) = 3\cos\left(5t + \frac{\pi}{6}\right)$

(b) $x(t) = e^{j10t}$

(c) $x(t) = \tan(5t + \theta)$

(d) $x(t) = 1$

(*Anna University, May 2006*)

Solution

(a) $\boldsymbol{x(t) = 3\cos\left(5t + \frac{\pi}{6}\right)}$

$$\omega_0 = 5\,\text{rad./s.}$$

Using Eq. (1.33), we get

$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{5}\ \text{s.}$$

The given signal is periodic with the fundamental period $T_0 = \frac{2\pi}{5}$ s.

(b) $\boldsymbol{x(t) = e^{j10t}}$

$$\omega_0 = 10\,\text{rad./s.}$$

Using Eq. (1.36), we get

$$\begin{aligned} T_0 &= \frac{2\pi}{\omega_0} \\ &= \frac{2\pi}{10} = 0.2\pi\ \text{s.} \end{aligned}$$

The given signal is periodic with the fundamental period

$$T_0 = 0.2\pi\ \text{s.}$$

(c) $\boldsymbol{x(t) = \tan(5t + \theta)}$

$$\begin{aligned} x(t + T_0) &= \tan(5(t + T_0) + \theta) \\ &= \tan(5t + 5T_0 + \theta) \end{aligned}$$

The tangent function repeats itself for every π rad. of its total argument. Thus, if $5T_0 = \pi$,

$$\begin{aligned} x(t + T_0) &= \tan(5t + \theta) \\ &= x(t) \end{aligned}$$

Hence

$$T_0 = \frac{\pi}{5}\ \text{s.}$$

(d) $x(t)$ is a d.c. signal and it does not repeat itself. Hence, it is not periodic.

Example 1.14 If $x_1(t)$ and $x_2(t)$ are periodic signals of period T_1 and T_2, show that the sum $x(t) = x_1(t) + x_2(t)$ is a periodic signal if $T_1/T_2 = n/m$ which is a rational number.

Solution For the signals $x_1(t)$ and $x_2(t)$ to be periodic, the following equations hold good:

$$\begin{aligned} x_1(t) &= x_1(t + mT_1) \\ x_2(t) &= x_2(t + nT_2) \end{aligned}$$

Now,

$$\begin{aligned} x(t) &= x_1(t) + x_2(t) \\ x(t + T) &= x_1(t + T) + x_2(t + T) \\ &= x_1(t + mT_1) + x_2(t + nT_2) \end{aligned}$$

From the above equations, we get

$$\begin{aligned} T &= mT_1 = nT_2 \\ \frac{T_1}{T_2} &= \frac{n}{m} = \text{a rational number.} \end{aligned}$$

Example 1.15 If $x_1(t)$ and $x_2(t)$ are the periodic signals with fundamental periods T_1 and T_2 respectively, show that the product $x(t) = x_1(t)x_2(t)$ will be periodic if $\frac{T_1}{T_2}$ is a rational number.

Solution For the periodic signals $x_1(t)$ and $x_2(t)$, the following equations are written:

$$\begin{aligned} x_1(t) &= x_1(t + T_1) = x_1(t + mT_1) \\ x_2(t) &= x_2(t + T_2) = x_2(t + nT_2) \\ x(t) &= x_1(t + mT_1)x_2(t + nT_2) \end{aligned}$$

Also

$$x(t + T) = x_1(t + T)x_2(t + T)$$

From the above two equations, we get

$$\begin{aligned} T &= mT_1 = nT_2 \\ \frac{T_1}{T_2} &= \frac{n}{m} = \text{a rational number} \end{aligned}$$

Example 1.16 Test whether the following signals are periodic. If periodic determine the fundamental period and frequency.

(a) $x(t) = e^{j(\pi t-2)}$

(b) $x(t) = \cos^2 t$

(c) $x(t) = E_v \cos 4\pi t$

(d) $x(t) = e^{(j\pi-2)t}$

Solution

(a) $\boldsymbol{x(t) = e^{j(\pi t-2)}}$

$$\begin{aligned} x(t) &= e^{j(\pi t-2)} \\ &= e^{-j2} e^{j\pi t} \end{aligned}$$

The signal is a complex exponential with e^{-j2} being a constant. Comparing this with standard complex exponential, we get

$$\begin{aligned} e^{j\pi t} &= e^{j\omega_0 t} \\ \omega_0 &= \pi \\ T_0 &= \frac{2\pi}{\omega_0} = \frac{2\pi}{\pi} \\ T_0 &= 2\,\text{s.} \\ f_0 &= \frac{1}{T_0} = \frac{1}{2} \\ f_0 &= 0.5\,\text{Hz.} \end{aligned}$$

The signal is a periodic one with fundamental period $T_0 = 2$ s. and fundamental frequency $f_0 = 0.5\,\text{Hz}$.

(b) $\boldsymbol{x(t) = \cos^2 t}$

$$\begin{aligned} \cos^2 t &= \frac{1}{2}[1 + \cos 2t] \\ &= \frac{1}{2} + \frac{1}{2}\cos 2t \\ &= x_1(t) + x_2(t) \end{aligned}$$

where

$$x_1(t) = \frac{1}{2} \text{ which is a d.c. signal}$$

and

$$x_2(t) = \frac{1}{2}\cos 2t$$

For $x_1(t)$, the fundamental radian frequency

$$\omega_0 = 2$$
$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{2} = \pi \text{ s.}$$

The fundamental frequency $f_0 = \frac{1}{T_0} = \frac{1}{\pi}$ Hz.

(c) $\boldsymbol{x(t) = E_v \cos 4\pi t}$

The even function of $x(t)$ is

$$\begin{aligned}
E_v x(t) &= \frac{1}{2}[x(t) + x(-t)] \\
&= \frac{1}{2}[\cos 4\pi t + \cos(-4\pi t)] \\
&= \cos 4\pi t \\
\omega_0 &= 4\pi \\
T_0 &= \frac{2\pi}{\omega_0} = \frac{2\pi}{4\pi} = 0.5 \text{ s.} \\
f_0 &= \frac{1}{T_0} = \frac{1}{0.5} = 2 \text{ Hz}
\end{aligned}$$

(d) $\boldsymbol{x(t) = e^{(j\pi - 2)t}}$

$$\begin{aligned}
x(t) &= e^{(j\pi - 2)t} \\
&= e^{-2t} e^{j\pi t}
\end{aligned}$$

The function $e^{j\pi t}$ is periodic with fundamental period 2 s. as seen in problem (a). However the function e^{-2t} is non-periodic and becomes zero at $t \to \infty$. Hence, the composite signal $x(t)$ is aperiodic.

Example 1.17 Consider the following continuous-time signal:

$$x(t) = 2\cos 3\pi t + 7\cos 9t$$

Find the periodicity of the signal.

(*Anna University, May 2005*)

Solution

$$x(t) = x_1(t) + x_2(t)$$

where

$$\begin{aligned}
x_1(t) &= 2\cos 3\pi t \\
x_2(t) &= 7\cos 9t
\end{aligned}$$

If T_1 is the fundamental period of $x_1(t)$,

$$\begin{aligned}
\omega_1 &= 3\pi \\
T_1 &= \frac{2\pi}{\omega_1} = \frac{2\pi}{3\pi} = \frac{2}{3} \text{ (rational)} \\
x_2(t) &= 7\cos 9t \\
\omega_2 &= 9 \\
T_2 &= \frac{2\pi}{\omega_2} = \frac{2\pi}{9} \text{ (not rational)} \\
\frac{T_1}{T_2} &= \frac{2}{3}\frac{9}{2\pi} = \frac{3}{\pi} \text{ (not rational)}
\end{aligned}$$

The signal $x(t)$ is not periodic.

Example 1.18 Find the fundamental period and frequency of the following signals:

$$\begin{aligned}
&\text{(a)} \qquad x(t) = 5\sin 24\pi t + 7\sin 36\pi t \\
&\text{(b)} \qquad x(t) = 5\cos \pi t \sin 3\pi t
\end{aligned}$$

Solution

(a) **Method 1:**

$$\begin{aligned}
x(t) &= 5\sin 24\pi t + 7\sin 36\pi t \\
&= x_1(t) + x_2(t)
\end{aligned}$$

where

$$\begin{aligned}
x_1(t) &= 5\sin 24\pi t \\
x_2(t) &= 7\sin 36\pi t
\end{aligned}$$

Let T_1 and T_2 be the fundamental periods of $x_1(t)$ and $x_2(t)$ respectively.

$$\begin{aligned}
\omega_1 &= 24\pi \\
T_1 &= \frac{2\pi}{\omega_1} = \frac{2\pi}{24\pi} = \frac{1}{12} \text{ (rational)} \\
\omega_2 &= 36\pi \\
T_2 &= \frac{2\pi}{\omega_2} = \frac{2\pi}{36\pi} = \frac{1}{18} \text{ (rational)} \\
\frac{T_1}{T_2} &= \frac{1}{12} \times 18 = \frac{3}{2} \text{ (rational)}
\end{aligned}$$

The composite signal is a periodic signal. Since T_1 and T_2 are rational, $x(t)$ is periodic. The fundamental period is obtained as follows. From the ratio of $\frac{T_1}{T_2}$,

$$2T_1 = 3T_2 = T_0$$
$$T_0 = \frac{2}{12} = \frac{1}{6}\ \text{s.}$$

or

$$T_0 = \frac{3}{18} = \frac{1}{6}\ \text{s.}$$
$$f_0 = \frac{1}{T_0} = 6\ \text{Hz.}$$

$$T_0 = \frac{1}{6}\ \text{s.}$$
$$f_0 = 6\ \text{Hz.}$$

Method 2:
In this method, the least common multiple (LCM) for T_1 and T_2 is obtained which gives T_0. In case, T_1 and T_2 are fractions, they are made integers by multiplying by a least number. For T_1 and T_2 thus obtained, LCM is found. T_0 is obtained by dividing by the same number which was chosen to make T_1 and T_2 as integers. In the above example,

(1)
$$T_1 = \frac{1}{12} \quad \text{and} \quad T_2 = \frac{1}{18}$$

By multiplying T_1 and T_2 by 36, $T_1 = 3$ and $T_2 = 2$.
(2) The LCM for the new T_1 and T_2 is easily obtained as 6.
(3) T_0 is obtained by dividing LCM by 36.

$$T_0 = \frac{\text{LCM}}{36} = \frac{6}{36} = \frac{1}{6}\ \text{s.}$$

$$T_0 = \frac{1}{6}\ \text{s.}$$
$$f_0 = 6\,\text{Hz.}$$

(b)

$$x(t) = 5\cos\pi t \sin 3\pi t = x_1(t)x_2(t)$$

where

$$x_1(t) = 5\cos\pi t$$
$$x_2(t) = \sin\pi t$$

Let T_1 and T_2 be the fundamental periods of $x_1(t)$ and $x_2(t)$ respectively. The following equations are obtained for T_1 and T_2.

$$\begin{aligned}\omega_1 &= \pi \\ T_1 &= \frac{2\pi}{\omega_1} = \frac{2\pi}{\pi} = 2\text{ s. (rational)} \\ \omega_2 &= 3\pi \\ T_2 &= \frac{2\pi}{\omega_2} = \frac{2\pi}{3\pi} = \frac{2}{3}\text{ s. (rational).} \\ \frac{T_1}{T_2} &= 2 \times \frac{3}{2} = 3\text{ (rational).}\end{aligned}$$

The composite signal $x(t)$ is periodic and the fundamental period T_0 is given by

$$T_0 = T_1 = 3T_2 = 2\,\text{s.}$$

$$\begin{aligned}T_0 &= 2\,\text{s.} \\ f_0 &= 0.5\,\text{Hz}\end{aligned}$$

The same results are obtained by finding LCM for T_1 and T_2. By multiplying T_1 and T_2 by 3, they are made integers. The new $T_1 = 6$ and $T_2 = 2$. The LCM for this is 6. Hence, $T_0 = \frac{LCM}{3} = \frac{6}{3} = 2\,\text{s.}$ and $f_0 = \frac{1}{T_0} = 0.5\,\text{Hz}$.

Example 1.19 Find whether the following signal is periodic. If periodic, determine the fundamental period and frequency. Also determine the fundamental period of each function in the composite signal in the time of the fundamental period.

$$x(t) = \sin(2\pi t - \pi) - 5\cos\left(3\pi t + \frac{\pi}{4}\right) - 8\cos\left(5\pi t - \frac{\pi}{8}\right)$$

Solution

$$x(t) = x_1(t) + x_2(t) + x_3(t)$$

where

$$\begin{aligned}x_1(t) &= \sin(2\pi t - \pi) \\ x_2(t) &= -5\cos\left(3\pi t + \frac{\pi}{4}\right) \\ x_3(t) &= -8\cos\left(5\pi t + \frac{\pi}{8}\right)\end{aligned}$$

Let T_1, T_2, and T_3 be the fundamental periods of $x_1(t)$, $x_2(t)$, and $x_3(t)$ respectively.

$$\omega_1 = 2\pi$$
$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{2\pi} = 1 \text{ s. (rational)}$$
$$\omega_2 = 3\pi$$
$$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{3\pi} = \frac{2}{3} \text{ s. (rational)}$$
$$\omega_3 = 5\pi$$
$$T_3 = \frac{2\pi}{\omega_3} = \frac{2\pi}{5\pi} = \frac{2}{5} \text{ s. (rational)}$$
$$\frac{T_1}{T_2} = \frac{1 \times 3}{2} = \frac{3}{2} \text{ s. (rational)}$$
$$\frac{T_1}{T_3} = \frac{1 \times 5}{2} = \frac{5}{2} \text{ s. (rational)}$$

Hence, the composite signal $x(t)$ is periodic. The fundamental period is obtained by taking LCM of T_1, T_2, and T_3 as explained below:

(1)

$$T_1 = 1; \quad T_2 = \frac{2}{3}; \quad T_3 = \frac{2}{5}$$

Multiply by 15 to make them integers. The new periods are obtained as $T_1 = 15,\ T_2 = 10$, and $T_3 = 6$.

(2) The LCM is obtained as

$$\begin{array}{r|l} 5 & 15,\ 10,\ 6 \\ \hline 3 & 3,\ \ 2,\ \ 6 \\ \hline 2 & 1,\ \ 2,\ \ 2 \\ \hline & 1,\ \ 1,\ \ 1 \end{array}$$

The LCM $= 5 \times 3 \times 2 = 30$.

(3)

$$T_0 = \frac{\text{LCM}}{15} = \frac{30}{15} = 2\,\text{s.}$$

$$T_0 = 2\,\text{s.}$$
$$f_0 = \frac{1}{T_0} = 0.5\,\text{Hz.}$$

The fundamental period of $x_1(t)$ during $T_0 = 2\,\text{s.}$ is

$$T_{01} = \frac{T_0}{T_1} = \frac{2}{1} = 2$$

The fundamental period of $x_2(t)$ during $T_0 = 2$ s. is

$$T_{02} = \frac{T_0}{T_2} = \frac{2}{2} \times 3 = 3$$

The fundamental period of $x_3(t)$ during $T_0 = 2$ s. is

$$T_{03} = \frac{T_0}{T_3} = \frac{2}{2} \times 5 = 5$$

1.6.4 Odd and Even Functions of Continuous-Time Signals

One of the properties of signals is their symmetry when the time is reversed. They are classified as even and odd signals. A continuous-time signal $x(t)$ is said to be an even signal if it satisfies the following condition:

$$x(-t) = x(t) \text{ for all } t \tag{1.31}$$

It is identical under folding about the origin. A signal $x(t)$ is said to be an odd signal if it satisfies the condition,

$$x(-t) = -x(t) \text{ for all } t \tag{1.32}$$

An odd signal must necessarily be zero at $t = 0$. While even signals are symmetric about the vertical axis odd signals are anti-symmetric (asymmetric) about the time origin. Consider the following signal:

$$\begin{aligned} x(t) &= A \cos \omega t \\ x(-t) &= A \cos(-\omega t) \\ &= A \cos \omega t = x(t) \end{aligned}$$

The above even signal is shown in Fig. 1.36.

Consider the following signal:

$$\begin{aligned} x(t) &= A \sin \omega t \\ x(-t) &= A \sin(-\omega t) \\ &= -A \sin \omega t \\ &= -x(t) \end{aligned}$$

The above odd signal is shown in Fig. 1.37. The odd function is zero at $t = 0$ as shown in Fig. 1.37.

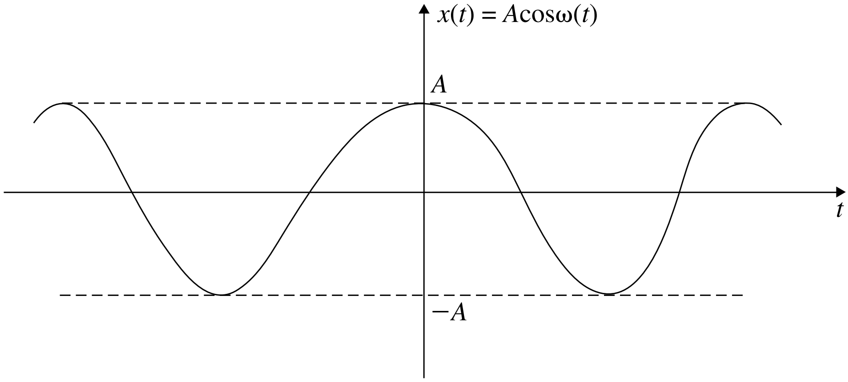


Fig. 1.36 Representation of an even (symmetric) function

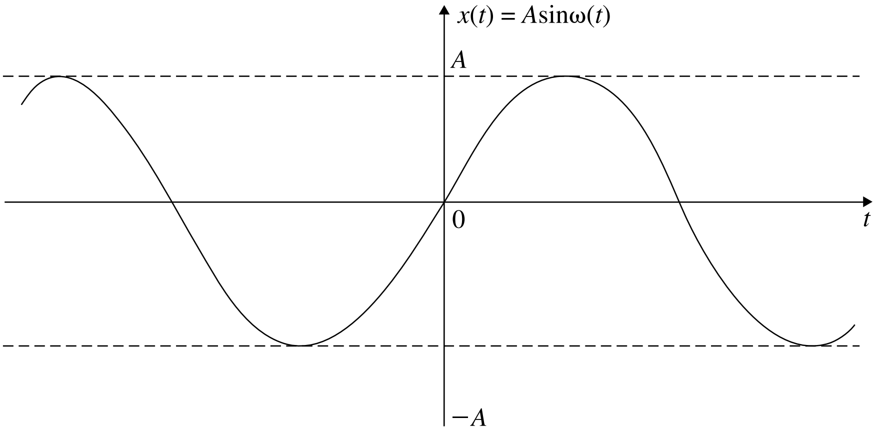


Fig. 1.37 Representing of an odd (anti-symmetric) function

1.6.4.1 Even and Odd Components of a Signal

A continuous-time signal $x(t)$ can be expressed in terms of odd and even components. Let $x_e(t)$ and $x_0(t)$ represent the even and odd components of $x(t)$. We may write $x(t)$ as

$$x(t) = x_e(t) + x_0(t) \tag{1.33}$$

Putting $t = -t$ in Eq. (1.39) we get

$$x(-t) = x_e(-t) + x_0(-t) \tag{1.34}$$

For an even function $x_e(-t) = x_e(t)$ and for an odd function $x_0(-t) = -x_0(t)$. Equation (1.34) is written as

$$x(-t) = x_e(t) - x_0(t) \tag{1.35}$$

Adding Eqs. (1.33) and (1.35) the following equation is obtained:

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \tag{1.36}$$

Subtracting Eq. (1.35) from Eq. (1.33), we get

$$x_0(t) = \frac{1}{2}[x(t) - x(-t)] \tag{1.37}$$

Example 1.20 Show that the even function has its odd part zero.

Solution From Eq. (1.42) the even function of $x(t)$ can be written as

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

For an even function $x(-t) = x(t)$. Hence, the above equation can be written as

$$x_e(t) = \frac{1}{2}[x(t) + x(t)] = x(t)$$

From Eq. (1.43) the odd function of $x(t)$ can be written as

$$\begin{aligned} x_0(t) &= \frac{1}{2}[x(t) - x(-t)] \\ &= \frac{1}{2}[x(t) - x(t)] = 0 \end{aligned}$$

Thus, it is proved that for an even function the odd part is zero.

Example 1.21 Show that the odd function has its even part zero.

Solution Let $x(t)$ be an odd function. For an odd function, $x(-t) = -x(t)$. The even function of $x(t)$ can be written as

$$\begin{aligned} x_e(t) &= \frac{1}{2}[x(t) + x(-t)] \\ &= \frac{1}{2}[x(t) - x(t)] \\ &= 0 \\ x_0(t) &= \frac{1}{2}[x(t) - x(-t)] \\ &= \frac{1}{2}[x(t) + x(t)] = x(t) \end{aligned}$$

Thus, for an odd function $x(t)$, the even part of $x(t) = 0$.

Example 1.22 Show that the product of two even signals is an even signal.

Solution Let $x_1(t)$ and $x_2(t)$ be the two even signals. Let $x(t)$ be the product of these two signals.

$$x(t) = x_1(t)x_2(t)$$

For an even function, $x(-t) = x(t)$; and $x_1(-t) = x_1(t)$ and $x_2(-t) = x_2(t)$. The above equation is written as follows. Substituting $t = -t$ we get

$$\begin{aligned} x(-t) &= x_1(-t)x_2(-t) \\ &= x_1(t)x_2(t) = x(t) \end{aligned}$$

Thus, $x(t) = x(-t)$ which is even.

Example 1.23 Show that the product of two odd signals is an even signal.

Solution Let $x_1(t)$ and $x_2(t)$ be two odd signals. For the odd signals, $x_1(-t) = -x_1(t)$ and $x_2(t) = -x_2(t)$. Let $x(t)$ be the product of $x_1(t)$ and $x_2(t)$.

$$x(t) = x_1(t)x_2(t)$$

Putting $t = -t$ in the above equation, we get

$$\begin{aligned} x(-t) &= x_1(-t)x_2(-t) \\ &= x_1(t)x_2(t) = x(t) \end{aligned}$$

Thus, it is proved that $x(t) = x(-t)$. The product of two odd signals is an even signal.

Example 1.24 Prove that the product of an odd and an even signal is an odd signal.

Solution Let $x_1(t)$ be an odd signal and $x_2(t)$ be an even signal. Then $x_1(-t) = -x_1(t)$ and $x_2(-t) = x_2(t)$. Let $x(t)$ be the product of $x_1(t)$ and $x_2(t)$.

$$x(t) = x_1(t)x_2(t)$$

Putting $t = -t$ in the above equation, we get

$$\begin{aligned} x(-t) &= x_1(-t)x_2(-t) \\ &= -x_1(t)x_2(t) = -x(t) \end{aligned}$$

Thus, $x(t) = -x(-t)$ which is odd. The product of an odd and an even signal is an odd signal.

Example 1.25 Show that the sum of the two even functions is an even function and the sum of the two odd functions is an odd function.

Solution Let $x(t)$ be expressed as the sum of two functions $x_1(t)$ and $x_2(t)$.

$$x(t) = x_1(t) + x_2(t)$$

Substituting $t = -t$ in the above equation, we get

$$x(-t) = x_1(-t) + x_2(-t) \tag{a}$$

If $x_1(t)$ and $x_2(t)$ are even functions, the above equation is written as

$$\begin{aligned} x(-t) &= x_1(t) + x_2(t) \\ &= x(t) \end{aligned}$$

This shows that $x(t)$ which is the sum of two even functions is an even function. If $x_1(t)$ and $x_2(t)$ are odd functions, equation (a) can be written as

$$\begin{aligned} x(-t) &= x_1(-t) + x_2(-t) \\ &= -(x_1(t) + x_2(t)) \\ &= -x(t) \end{aligned}$$

Thus, $x(t)$ which is the sum of two odd functions is an odd function.

Example 1.26 Find whether the following signals are odd or even. Find the odd and even components.

(a) $x(t) = t^2 - 5t + 10$

(b) $x(t) = t^4 + 4t^2 + 6$

(c) $x(t) = t^3 + 3t$

(d) $x(t) = 10 \sin\left(10\pi t + \frac{\pi}{4}\right)$

(e) $x(t) = e^{j10t}$

Solution

(a) $\boldsymbol{x(t) = t^2 - 5t + 10}$
Put $t = -t$

$$\begin{aligned} x(-t) &= t^2 + 5t + 10 \\ &\neq x(t) \\ &\neq -x(t) \end{aligned}$$

The function is neither even nor odd.

$$\begin{aligned} x_e(t) &= \frac{1}{2}[x(t) + x(-t)] \\ &= \frac{1}{2}[t^2 - 5t + 10 + t^2 + 5t + 10] \end{aligned}$$

$$x_e(t) = (t^2 + 10)$$

$$\begin{aligned} x_0(t) &= \frac{1}{2}[x(t) - x(-t)] \\ &= \frac{1}{2}[t^2 - 5t + 10 - t^2 - 5t - 10] \end{aligned}$$

$$x_0(t) = -5t$$

(b) $\boldsymbol{x(t) = t^4 + 4t^2 + 6}$
Put $t = -t$

$$\begin{aligned} x(-t) &= t^4 + 4t^2 + 6 = x(t) \\ x(t) &= x(-t) \end{aligned}$$

The function is even. The odd part should be zero which can be verified as

$$\begin{aligned} x_0(t) &= \frac{1}{2}[x(t) - x(-t)] \\ &= \frac{1}{2}[t^4 + 4t^2 + 6 - t^4 - 4t^2 - 6] \\ &= 0 \end{aligned}$$

$$x_e(t) = x(t) = t^4 + 4t^2 + 6$$

(c) $\boldsymbol{x(t) = t^3 + 3t}$
Put $t = -t$

$$x(-t) = -(t^3 + 3t) = -x(t)$$

The function is odd. The even component is zero.

$$\begin{aligned} x_0(t) &= t^3 + 3t \\ x_e(t) &= 0 \end{aligned}$$

(d) $\boldsymbol{x(t) = 10\sin\left(10\pi t + \frac{\pi}{4}\right)}$
Put $t = -t$

$$x(-t) = 10\sin\left(-10\pi t + \frac{\pi}{4}\right)$$
$$= -10\sin\left(10\pi t - \frac{\pi}{4}\right)$$
$$= -10\left[\sin 10\pi t\ \cos\frac{\pi}{4} - \cos 10\pi t\ \sin\frac{\pi}{4}\right]$$
$$= \frac{-10}{\sqrt{2}}[\sin 10\pi t - \cos 10\pi t]$$
$$\neq x(t)$$
$$\neq -x(t)$$

The above signal is neither even nor odd.

$$x(t) = 10\left[\sin 10\pi t \cos\frac{\pi}{4} + \cos 10\pi t \sin\frac{\pi}{4}\right]$$
$$= \frac{10}{\sqrt{2}}[\sin 10\pi t + \cos 10\pi t]$$
$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$
$$= \frac{10}{2\sqrt{2}}[\sin 10\pi t + \cos 10\pi t - \sin 10\pi t + \cos 10\pi t]$$

$$x_e(t) = \frac{10}{\sqrt{2}}\cos 10\pi t$$

$$x_0(t) = \frac{1}{2}[x(t) - x(-t)]$$
$$= \frac{10}{2\sqrt{2}}[\sin 10\pi t + \cos 10\pi t + \sin 10\pi t - \cos 10\pi t]$$

$$x_0(t) = \frac{10}{\sqrt{2}}\sin 10\pi t$$

(e) $\boldsymbol{x(t) = e^{j10t}}$

$$x(-t) = e^{-j10t}$$
$$x(t) \neq x(-t)$$
$$x(t) \neq -x(-t)$$

The signal is neither odd nor even.

$$x_e(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[e^{j10t} + e^{-j10t}]$$

$$x_e(t) = \cos 10t$$

$$x_0(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[e^{j10t} - e^{-j10t}]$$

$$x_0(t) = j \sin 10t$$

Note: In all the above cases $\boldsymbol{x_0(t)}$ passes through the origin at $\boldsymbol{t = 0}$.

Example 1.27 Sketch the even and odd components of a step signal shown in Fig. 1.38a.

Solution The step function is shown in Fig. 1.38a. $x(-t)$ is shown in Fig. 1.38b. In Fig. 1.38c, the sum of $x(t)$ and $x(-t)$ is represented. The even function $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$ is shown in Fig. 1.38d. In Fig. 1.38e, $-x(-t)$ is represented. The odd function $x_0(t) = \frac{1}{2}[x(t) - x(-t)]$ is represented in Fig. 1.38f.

Example 1.28 Sketch the even and odd components of the pulse signal shown in Fig. 1.39a.

Solution $x(t)$ is shown in Fig. 1.39a. In Fig. 1.39b, $x(-t)$ is represented. The sum of $x(t) + x(-t)$ is shown in Fig. 1.39c. The even component of $x(t)$ which is $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$ is shown in Fig. 1.39d. In Fig. 1.39e, $-x(-t)$ is shown. The odd component of $x(t)$ which is $x_0(t) = \frac{1}{2}[x(t) - x(-t)]$ is represented in Fig. 1.39f.

Example 1.29 Sketch the even and odd components of the triangular wave shown in Fig. 1.40a.

Solution Figure 1.40a represents the $x(t)$ which is a triangular wave. $x(-t)$ is represented in Fig. 1.40b. $x(t) + x(-t)$ is represented in Fig. 1.40c. From this figure, the even component is obtained by dividing the amplitude by 2 and $x_e(t)$ is shown in Fig. 1.40d. In Fig. 1.40e, $-x(-t)$ is represented which is obtained by inverting Fig. 1.40b. Adding Fig. 1.40a and e, $[x(t) - x(-t)]$ is obtained and represented in Fig. 1.40f. By dividing the amplitude of Fig. 1.40f by 2, $x_0(t)$ which is $\frac{1}{2}[x(t) - x(-t)]$ is obtained and sketched as shown in Fig. 1.40g.

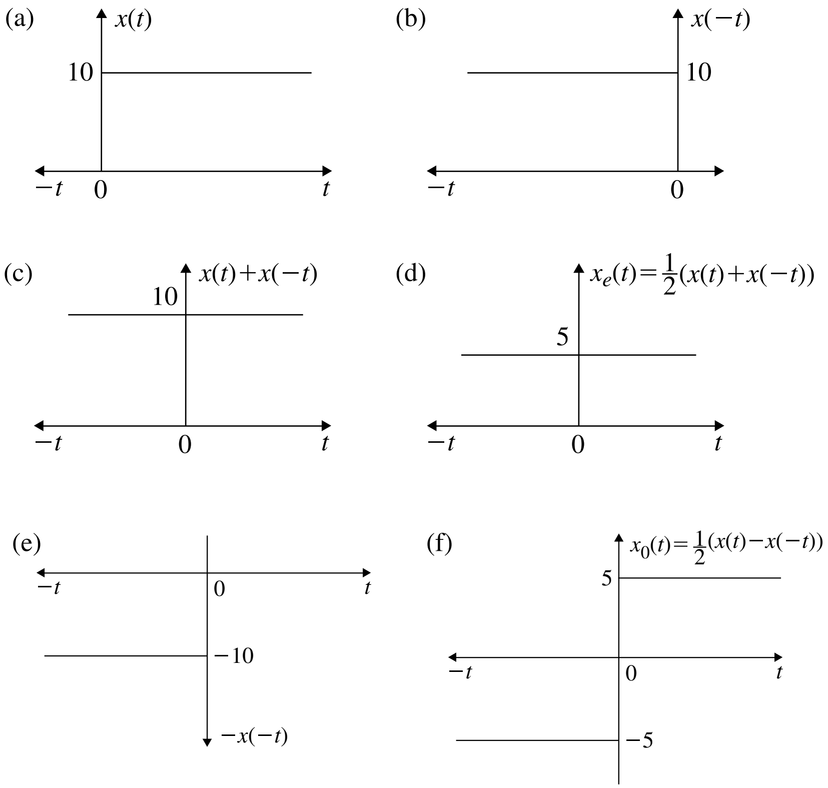


Fig. 1.38 Even and odd components of a step function

Example 1.30 Sketch the even and odd components of exponential signal $x(t) = 10e^{-2t}$.

Solution $x(t) = 10e^{-2t}$ is sketched and shown in Fig. 1.41a. Figure 1.41a is time reversed to get $x(-t)$ and is sketched in Fig. 1.41b. The sum of $x(t)$ and $x(-t)$ is sketched as shown in Fig. 1.41c. The amplitude of Fig. 1.41c is reduced by a factor 2. This gives $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$ and is shown in Fig. 1.41d. Figure 1.41a is inverted and time reversed to get $-x(-t)$ which is sketched in Fig. 1.41e. The sum of Fig. 1.41a and e gives $[x(t) - x(-t)]$ and this is sketched and shown in Fig. 1.41f. The amplitude of Fig. 1.41f is reduced by a factor 2 which gives odd signal $x_0(t) = \frac{1}{2}[x(t) - x(-t)]$. This is shown in Fig. 1.41g.

Example 1.31 Sketch the even and odd parts of the signal shown in Fig. 1.42a.

Solution $x(t)$ is graphically represented in Fig. 1.42a. By time folding of Fig. 1.42a, $x(-t)$ is obtained and is shown in Fig. 1.42b. These figures are graphically added to get $x(t) + x(-t)$ and represented in Fig. 1.42c. To get the even signal of $x(t)$,

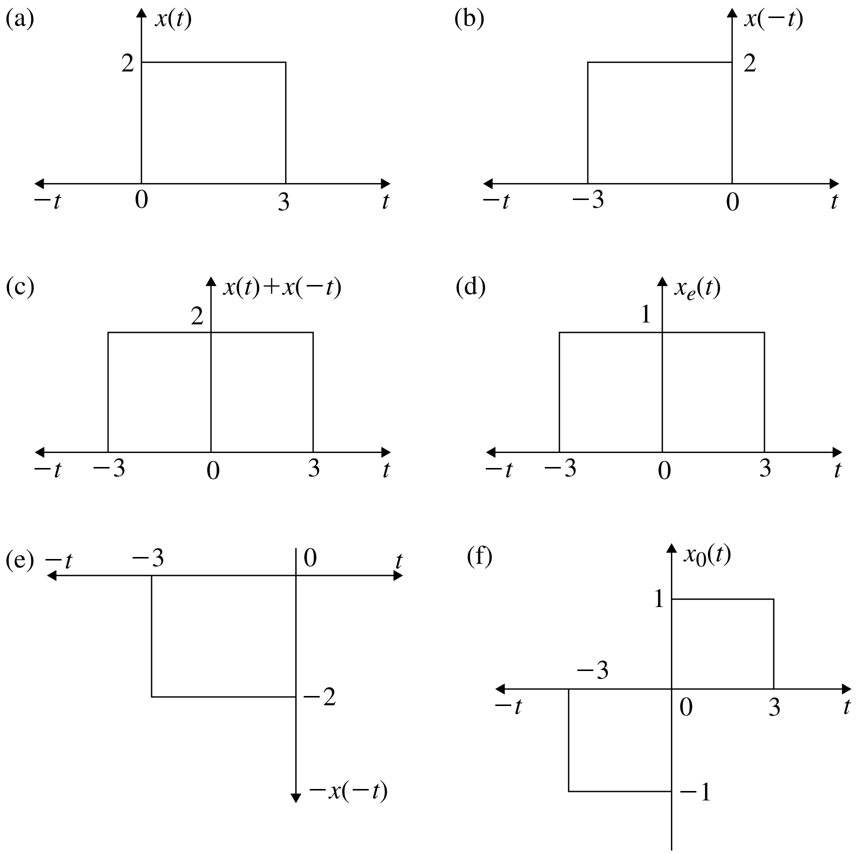


Fig. 1.39 Even and odd components of a pulse signal

the amplitude of the signal is divided by a factor 2 and is represented in Fig. 1.42d. The signal $x(t)$ is time folded and inverted to get $-x(-t)$. This is represented in Fig. 1.42e. Figure 1.42a and e is graphically added to get $x(t) - x(-t)$ which is represented in Fig. 1.42f. The amplitude of the signal in Fig. 1.42f is divided by a factor 2 which gives $x_0(t)$ of $x(t)$. This is represented in Fig. 1.42g.

Note the even component $x_e(t)$ in Fig. 1.42d. It is symmetrical with respect to the vertical axis and when time folded identical mirror images are obtained. Similarly, the odd component $x_0(t)$ represented in Fig. 1.42g passes through the origin at $t = 0$ and it is also anti-symmetry.

Example 1.32 Find the even and odd component of the following signal:

$$x(t) = \cos t + \sin t + \cos t \sin t$$

(*Anna University, May 2007*)

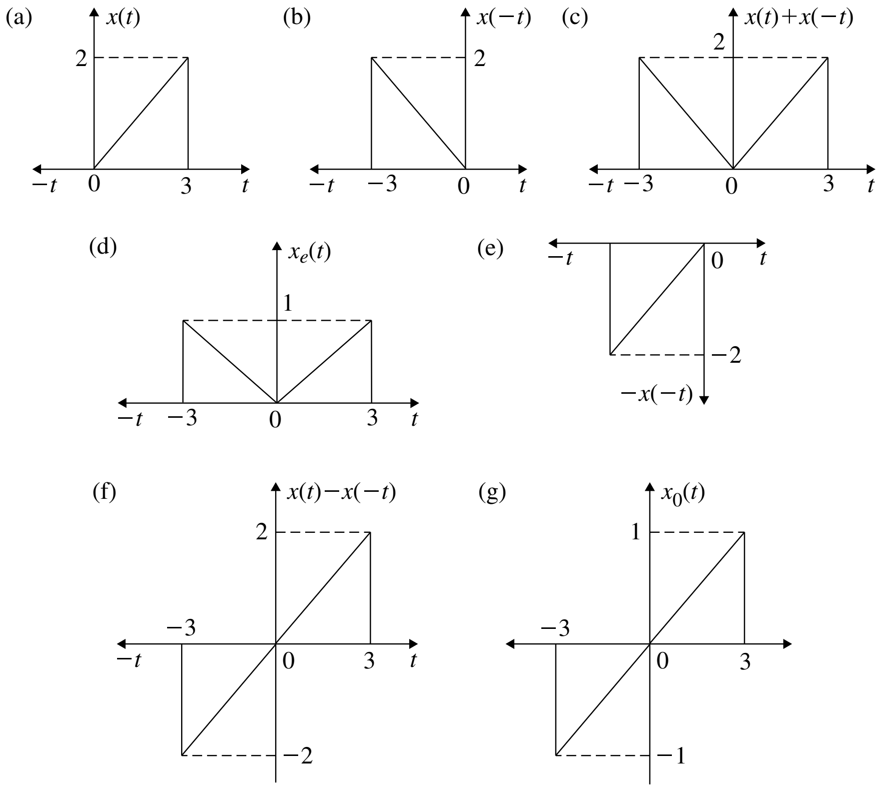


Fig. 1.40 Even and odd components of a triangular wave

Solution

$$x(t) = \cos t + \sin t + \cos t \sin t$$

Put $t = -t$

$$x(-t) = \cos(-t) + \sin(-t) + \cos(-t)\sin(-t) = \cos t - \sin t - \cos t \sin t$$

$$\begin{aligned} x_e(t) &= \frac{1}{2}[x(t) + x(-t)] \\ &= \frac{1}{2}[\cos t + \sin t + \cos t \sin t + \cos t - \sin t - \cos t \sin t] \end{aligned}$$

$$x_e(t) = \cos t$$

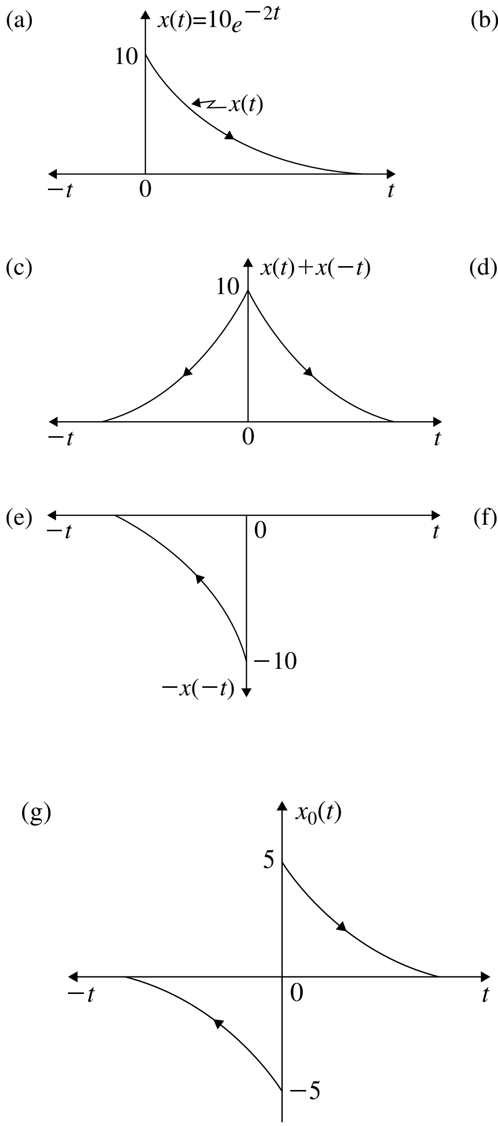


Fig. 1.41 Representation of even and odd function of exponential decay

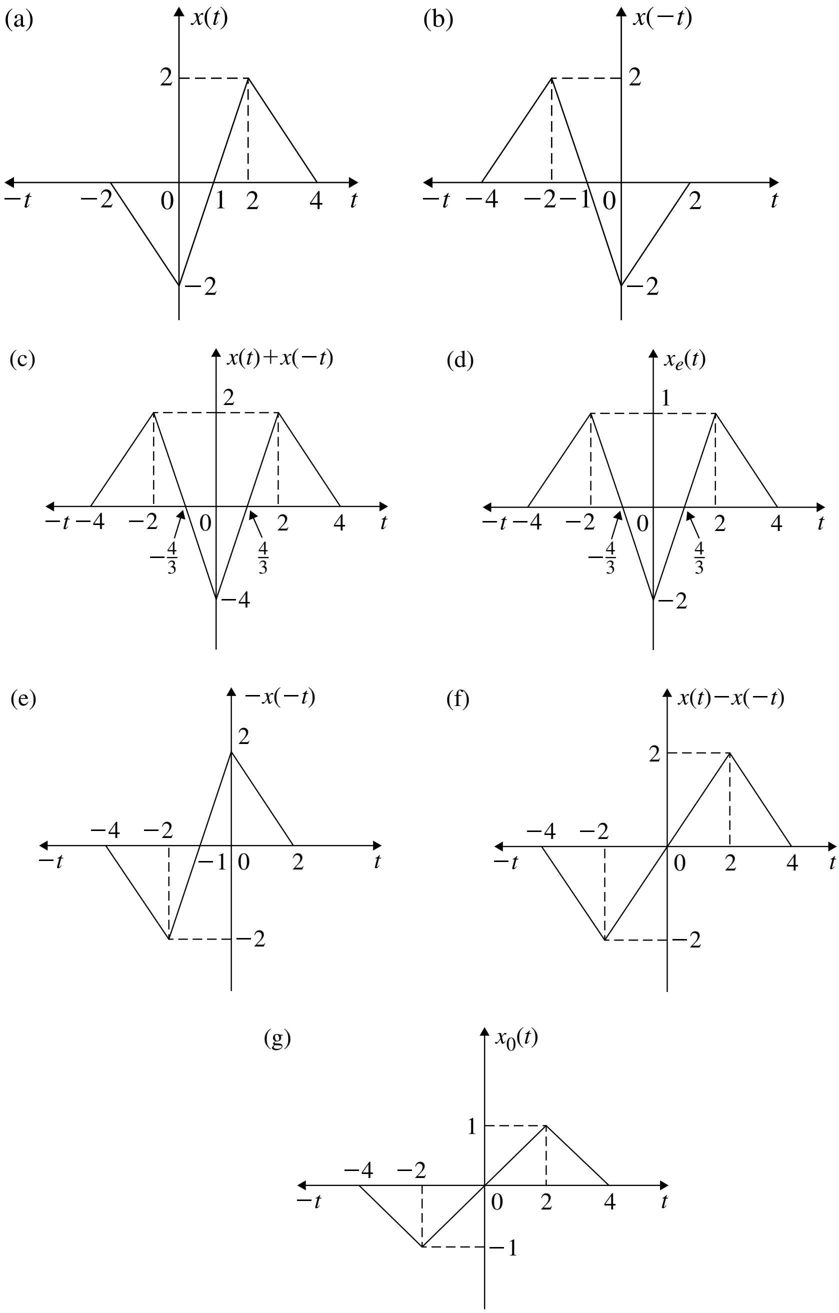


Fig. 1.42 Representation of even and odd signals of Example 1.40

The odd component of $x(t)$ is obtained as follows:

$$x_0(t) = \frac{1}{2}[x(t) - x(-t)]$$
$$= \frac{1}{2}[\cos t + \sin t + \cos t \sin t - \cos t + \sin t + \cos t \sin t]$$

$$x_0(t) = \sin t[1 + \cos t]$$

1.6.5 Energy and Power of Continuous-Time Signals

Consider the electric circuit shown in Fig. 1.43 in which a resistor R is connected across the voltage source $v(t)$. The current flowing through the resistor is $i(t)$. The instantaneous power consumed by the resistor is given by

$$P = i^2(t)R = \frac{v^2(t)}{R} \tag{1.38}$$

If we assume $R = 1$ ohm, the power is expressed as normalized power which is given by

$$P = v^2(t) \tag{1.39}$$

The average power consumption by the circuit over the time $t_1 \leq t \leq t_2$ is given by the following equation:

$$P = \frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} v^2(t)\, dt \tag{1.40}$$

The average energy consumption which is the product of power and time given as

$$E = \int_{t_1}^{t_2} P\, dt = \int_{t_1}^{t_2} v^2(t)\, dt \tag{1.41}$$

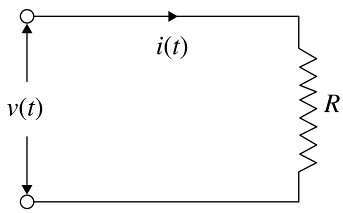


Fig. 1.43 Electric circuit with a resistor

Similar to voltages and currents, many other physical variables such as force, temperature, pressure, and charge are available for other types of systems. As a convention, similar terminologies for power and energy of continuous signal $x(t)$ and discrete signal $x[n]$ are defined and used. However, the "power" and "energy" defined here are not related to physical power and energy. Thus, if $x(t)$ represents a continuous-time signal, then the average power over an infinite time interval T is defined as

$$P = \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \tag{1.42}$$

The expression for the total energy is expressed as

$$E = \underset{T \to \infty}{Lt} \int_{-T}^{T} |x(t)|^2 \, dt \tag{1.43}$$

If the energy signal does not converge, such signals have infinite energy. On the other hand if E converges then the signal has finite energy. From Eqs. (1.42) and (1.43), the following inferences are drawn and given in Table 1.3.

Example 1.33 Find the power, RMS value, and energy of the following signals:

$$\begin{aligned} &\text{(a)} \qquad x(t) = A\,u(t) \\ &\text{(b)} \qquad x(t) = e^{-3t}u(t) \end{aligned}$$

Solution

(a) $\boldsymbol{x(t) = A\,u(t)}$

$$P = \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} A^2 dt$$

For $x(t) = A\,u(t)$, the signal starts at $t = 0$.

Table 1.3 Comparison of power and energy signals

Energy signal	**Power signal**
1. The total energy is obtained using $E = \underset{T \to \infty}{Lt} \int_{-T}^{T} \|x(t)\|^2 \, dt$	1. The average power is obtained using $P = \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} \|x(t)\|^2 \, dt$
2. For the energy signal $0 < E < \infty$, the average power $P = 0$	2. For the power signal $0 < P < \infty$, the energy E should be ∞
3. Non-periodic signals are energy signals	3. Periodic signals are power signals. However all power signals need not be periodic
4. Energy signals are not time limited	4. Power signals exist over infinite time

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} A^2 \int_0^T dt = \frac{A^2}{2T}\left[t\right]_0^T$$
$$= \underset{T\to\infty}{Lt} A^2 \frac{T}{2T} = \frac{A^2}{2}$$

$$P = \frac{A^2}{2} \text{ W}$$

RMS value of power is

$$P_{\text{RMS}} = \sqrt{P} = \frac{A}{\sqrt{2}}$$

$$P_{\text{RMS}} = \frac{A}{\sqrt{2}}$$

Since power is finite, energy E is infinite.

(b) $\boldsymbol{x(t) = e^{-3t}u(t)}$

For this signal t varies from 0 to ∞.

$$E = \underset{T\to\infty}{Lt} \int_0^T (e^{-3t})^2 dt$$
$$= \underset{T\to\infty}{Lt} \int_0^T e^{-6t} dt$$
$$= \underset{T\to\infty}{Lt} \frac{(-1)}{6}\left[e^{-6t}\right]_0^T$$
$$= \frac{1}{6} \underset{T\to\infty}{Lt} \left[1 - e^{-6T}\right]$$

$$E = \frac{1}{6} \text{ J}$$

Since E is finite, power $P = 0$.

Example 1.34 Find the power and energy of the following signals:

(a) $x(t) = A\cos(\omega_0 t + \phi)$

(b) $x(t) = A\sin(\omega_0 t + \phi)$

Solution

(a) $\boldsymbol{x(t) = A\cos(\omega_0 t + \phi)}$
Since the signal is periodic, it is necessarily a power signal and its energy $E = \infty$. The power of the signal is determined as follows:

$$P = \underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T} A^2\cos^2(\omega_0 t + \phi)dt$$

But,

$$\cos^2(\omega_0 t + \phi) = \frac{1 + \cos 2(\omega_0 t + \phi)}{2}$$

$$P = \underset{T\to\infty}{Lt}\frac{A^2}{4T}\int_{-T}^{T}[1 + \cos 2(\omega_0 t + \phi)]dt$$

Now consider the integral

$$\begin{aligned}
&\int_{-T}^{T}\cos 2(\omega_0 t + \phi)dt \\
&= \frac{1}{2\omega_0}[\sin 2(\omega_0 t + \phi)]_{-T}^{T} \\
&= \frac{1}{2\omega_0}[\sin 2(\omega_0 T + \phi) - \sin 2(-\omega_0 T + \phi)] \\
&= \frac{1}{2\omega_0}[\sin 2\phi - \sin 2\phi] \qquad [\because \quad \omega_0 T = 2\pi] \\
&= 0
\end{aligned}$$

$$\begin{aligned}
P &= \frac{A^2}{4}\underset{T\to\infty}{Lt}\frac{1}{T}\big[t\big]_{-T}^{T} \\
&= \frac{A^2}{4}\underset{T\to\infty}{Lt}\frac{1}{T}2T
\end{aligned}$$

$$P = \frac{A^2}{2}$$

(b) $\boldsymbol{x(t) = A\ \sin(\omega_0 t + \phi)}$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} A^2 \sin^2(\omega_0 t + \phi) dt$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{2T} \int_{-T}^{T} \frac{[1 - \cos 2(\omega_0 t + \phi)]}{2} dt$$

$$= \underset{T\to\infty}{Lt} \frac{A^2}{4T} \left[\int_{-T}^{T} dt - \int_{-T}^{T} \cos 2(\omega_0 t + \phi) \right] dt$$

Since $\int_{-T}^{T} \cos 2(\omega_0 t + \phi) dt = 0$

$$P = \underset{T\to\infty}{Lt} \frac{A^2}{4T} [t]_{-T}^{T}$$

$$P = \frac{A^2}{2}$$

Since P is finite, $E = \infty$.

Example 1.35 Find the power and energy of the following signals:

$$x(t) = 5\cos(10t + \phi) + 10\sin(5t + \phi)$$

Solution

$$x(t) = 5\cos(10t + \phi) + 10\sin(5t + \phi) = x_1(t) + x_2(t)$$

where

$$x_{1(t)} = 5\cos(10t + \phi)$$
$$x_2(t) = 10\sin(5t + \phi)$$

Let P_1 and P_2 be the powers of $x_1(t)$ and $x_2(t)$ respectively.

$$P_1 = \frac{A^2}{2} = \frac{25}{2} = 12.5$$
$$P_2 = \frac{A^2}{2} = \frac{100}{2} = 50$$

The average power

$$\begin{aligned} P &= P_1 + P_2 \\ &= 12.5 + 50 \end{aligned}$$

$$P = 62.5 \text{ W}$$

Since the power is finite energy $E = \infty$.

Example 1.36 Find the power and energy of the following signal:

$$x(t) = 5t \qquad -10 < t < 10$$

Solution Energy of the signal E is

$$\begin{aligned} E &= \int_{-10}^{10} (5t)^2 dt = 25 \left[\frac{t^3}{3} \right]_{-10}^{10} \\ &= \frac{25}{3} \times 2000 \end{aligned}$$

$$E = \frac{50000}{3} \text{ J}$$

Power of the signal P is

$$\begin{aligned} P &= \frac{1}{20} \int_{-10}^{10} (5t)^2 dt \\ &= \frac{50000}{3 \times 20} \end{aligned}$$

$$P = \frac{2500}{3} \text{W}$$

Example 1.37 Find the energy and power of the signal:

$$x(t) = u(t) - u(10 - t)$$

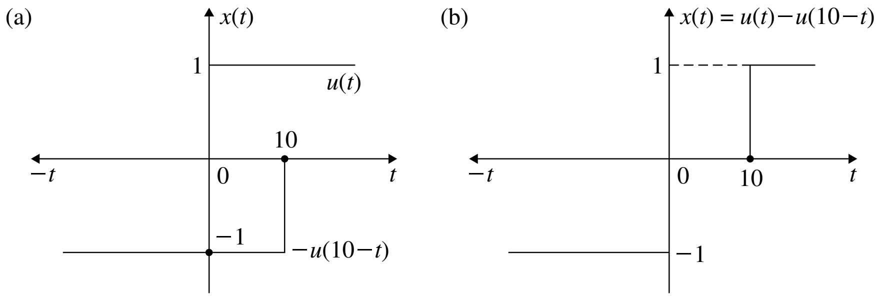


Fig. 1.44 Representation of $x(t) = u(t) - u(10 - t)$

Solution The signals $u(t)$ and $-u(10 - t)$ are represented in Fig. 1.44a. In Fig. 1.44b, $x(t) = u(t) - u(10 - t)$ is sketched. From Fig. 1.44b, the following equation for power is written:

$$
\begin{aligned}
P &= \underset{T\to\infty}{Lt} \frac{1}{2T}\left[\int_{-T}^{0} (-1)^2 dt + \int_{10}^{T} (1)^2 dt\right] \\
&= \underset{T\to\infty}{Lt} \frac{1}{2T}\left\{[t]_{-T}^{0} + [t]_{10}^{T}\right\} \\
&= \frac{1}{2}\underset{T\to\infty}{Lt} \frac{1}{T}[T + T - 10] \\
&= \frac{1}{2}\underset{T\to\infty}{Lt}\left[2 - \frac{10}{T}\right] = 1
\end{aligned}
$$

$$P = 1 \text{ W}$$

If the power is finite, the energy $E = \infty$.

Example 1.38 Determine the power and RMS value of the signal.

$$x(t) = e^{jat} \cos \omega_0 t$$

(*Anna University, 2007*)

Solution

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |e^{jat} \cos \omega_0 t|^2 dt$$

$$e^{jat} = \cos at + j \sin at$$

$$|e^{jat}| = \sqrt{\cos^2 at + \sin^2 at} = 1$$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} \cos^2 \omega_0 t dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{4T} \int_{-T}^{T} (1 + \cos 2\omega_0 t) dt$$

Since $\int_{-T}^{T} \cos 2\omega_0 t dt = 0$, as provided in Example 1.24,

$$P = \underset{T\to\infty}{Lt} \frac{1}{4T} \int_{-T}^{T} dt = \frac{1}{4T} 2T$$

$$P = 0.5 \text{ W}$$

RMS value of power is

$$P_{\text{RMS}} = \frac{1}{\sqrt{2}} = 0.707$$

Example 1.39 Find the power and energy of the following signals:

(a) $x(t) = 10e^{j2\pi t} u(t)$

(b) $x(t) = e^{j(2t+\pi/4)}$

(*Anna University, April 2007*)

Solution

(a) $\boldsymbol{x(t) = 10e^{j2\pi t} u(t)}$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{0}^{T} |10e^{j2\pi t}|^2 dt \qquad [x(t) = 0 \text{ for } t < 0]$$

$$= \frac{100}{2} \underset{T\to\infty}{Lt} \frac{1}{T} \int_0^T dt \qquad |e^{j2\pi t}| = 1$$

$$= 50 \, \frac{1}{T} [T] = 50$$

$$P = 50 \text{ W}$$

Since power is finite, $E = \infty$.

(b) $\boldsymbol{x(t) = e^{j(2t+\pi/4)}}$

$$|x(t)| = \left| e^{j(2t+\pi/4)} \right| = 1$$

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} dt = \frac{1}{2T} 2T = 1$$

$$P = 1$$

since power is finite, $E = \infty$.

Example 1.40 Find the energy of the following signal:

$$x(t) = 5 \text{ tri} \left(\frac{t}{2} \right)$$

Solution The triangular signal $x(t) = \text{tri}(t)$ is shown in Fig. 1.45a. By amplitude multiplication and time expansion, $x(t) = 5 \, \text{tri} \left(\frac{t}{2} \right)$ is obtained and shown in Fig. 1.45b. For Fig. 1.45b the following equation is written:

$$x(t) = \frac{5}{2} t + c \qquad -2 \le t \le 0$$

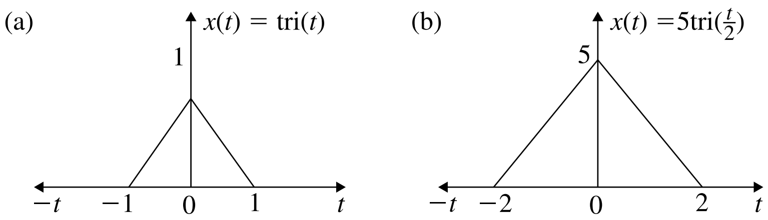


Fig. 1.45 Representation of triangular signals of Example 1.40

c is obtained as 5.

$$x(t) = -\frac{5}{2}t + c \qquad 0 \le t \le 2$$

c is obtained as 5.

Let E_1 be the energy for the time interval $-2 \le t \le 0$ and E_2 energy for the time interval $0 \le t \le 2$.

$$\begin{aligned}
E_1 &= \int_{-2}^{0} \left(\frac{5}{2}t + 5\right)^2 \, dt \\
&= \left[\frac{25}{12}t^3 + 25t + \frac{25}{2}t^2\right]_{-2}^{0} = \frac{50}{3} \\
E_2 &= \int_{0}^{2} \left(-\frac{5}{2}t + 5\right)^2 \, dt \\
&= \int_{0}^{2} \left(\frac{25}{4}t^2 + 25t - 25t\right) dt \\
&= \left[\frac{25}{4}\frac{t^3}{3} + 25t - \frac{25}{2}t^2\right]_{0}^{2} = \frac{50}{3} \\
E &= E_1 + E_2 = \frac{50}{3} + \frac{50}{3}
\end{aligned}$$

$$E = \frac{100}{3} \text{ J}$$

Since energy is finite, the average power $P = 0$.

Example 1.41 Find the energy of the signal:

$$x(t) = \text{ tri}\left(\frac{t-2}{10}\right)$$

Solution

$$\begin{aligned}
x(t) &= \text{tri}\left(\frac{t-2}{10}\right) \\
&= \text{tri}(0.1t - 0.2)
\end{aligned}$$

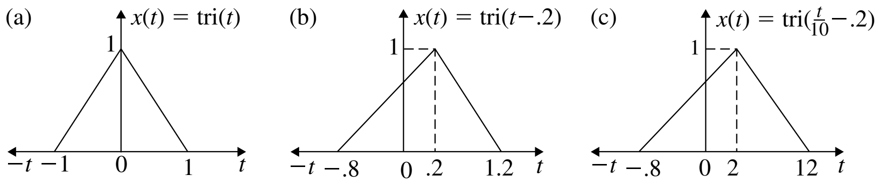


Fig. 1.46 Representation of $x(t) = \text{tri}(\frac{t}{10} - 0.2)$

Figure 1.46a shows $x(t) = \text{tri}(t)$. The time shifted signal $x(t) = \text{tri}(t - 0.2)$ is shown in Fig. 1.46b. The time shift is 0.2 towards right. By time elongation by factor 10, $x(t) = \text{tri}(\frac{t}{10} - 0.2)$ is obtained and is shown in Fig. 1.46c. For Fig. 1.46c the following equations are written:

$$x(t) = \frac{1}{10}t + c \qquad -8 \le t \le 2$$

For $t = 2$, $x(t) = 1$

$$\begin{aligned} 1 &= \frac{2}{10} + c \\ c &= 0.8 \\ x(t) &= 0.1t + 0.8 \\ x(t) &= -\frac{1}{10}t + c \qquad 2 \le t \le 12 \end{aligned}$$

For $t = 2$, $x(t) = 1$

$$\begin{aligned} 1 &= \frac{-2}{10} + c \\ c &= 1.2 \\ x(t) &= -0.1t + 1.2 \end{aligned}$$

Energy of the signal is given as

$$\begin{aligned} E &= \int_{-8}^{2} (0.1t + 0.8)^2 dt + \int_{2}^{12} (-0.1t + 1.2)^2 dt \\ &= E_1 + E_2 \end{aligned}$$

where

$$E_1 = \int_{-8}^{2} (0.1t + 0.8)^2 dt$$

and

$$E_2 = \int_{2}^{12} (-0.1t + 1.2)^2 dt$$

$$\begin{aligned}
E_1 &= \frac{1}{100}\int_{-8}^{2} (t+8)^2 dt = \frac{1}{100}\int_{-8}^{2} (t^2 + 16t + 64) dt \\
&= \frac{1}{100}\left[\frac{t^3}{3} + 8t^2 + 64t\right]_{-8}^{2} = \frac{10}{3} \\
E_2 &= \int_{2}^{12} \frac{1}{100}(12-t)^2 dt \\
&= \frac{1}{100}\int_{2}^{12} (t^2 - 24t + 144) dt \\
&= \frac{1}{100}\left[\frac{t^3}{3} - 12t^2 + 144t\right]_{2}^{12} = \frac{10}{3} \\
E &= E_1 + E_2 = \frac{10}{3} + \frac{10}{3} = \frac{20}{3}
\end{aligned}$$

$$E = \frac{20}{3}\ \text{J}$$

Since the energy is finite, the average power is zero.

Example 1.42 Find the energy of the following signal:

$$x(t) = 2\,\text{rect}\left(\frac{t}{2}\right)$$

Solution The rectangular or unit gate function is represented in Fig. 1.47a. It is defined as

$$\begin{aligned}
x(t) &= 1 \quad -1 \le t \le 1 \\
&= 0 \quad \text{otherwise}
\end{aligned}$$

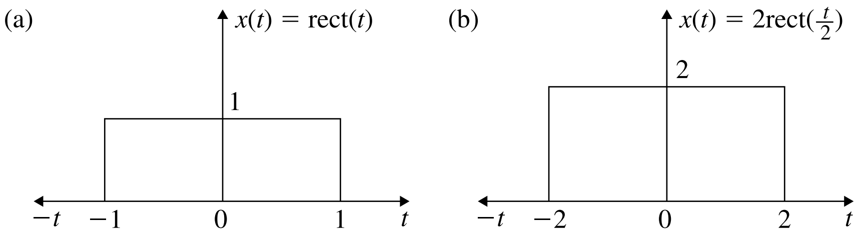


Fig. 1.47 Representation of rectangular function

The rectangular signal with amplitude scaling and time elongation is shown in Fig. 1.47b. From Fig. 1.47b, the equation for energy is written as follows:

$$E = \int_{-2}^{2} (2)^2 dt = 4\big[t\big]_{-2}^{2} = 16$$

$$E = 16\,\mathrm{J}$$

Since the energy is finite, the average power = 0.

Example 1.43 A trapezoidal pulse $x(t)$ is defined by

$$x(t) = \begin{cases} (5-t) & 4 \leq t \leq 5 \\ 1 & -4 \leq t \leq 4 \\ (t+5) & -5 \leq t \leq -4 \end{cases}$$

(a) Determine total energy of $x(t)$.
(b) Sketch $x(2t-3)$.
(c) If $y(t) = \frac{dx(t)}{dt}$, determine the total energy of $y(t)$.

(*Anna University, December 2007*)

Solution

(a) **To determine the total energy of $x(t)$.**
The given trapezoid pulse $x(t)$ is represented in Fig. 1.48a. The total energy of the signal is determined as described below:

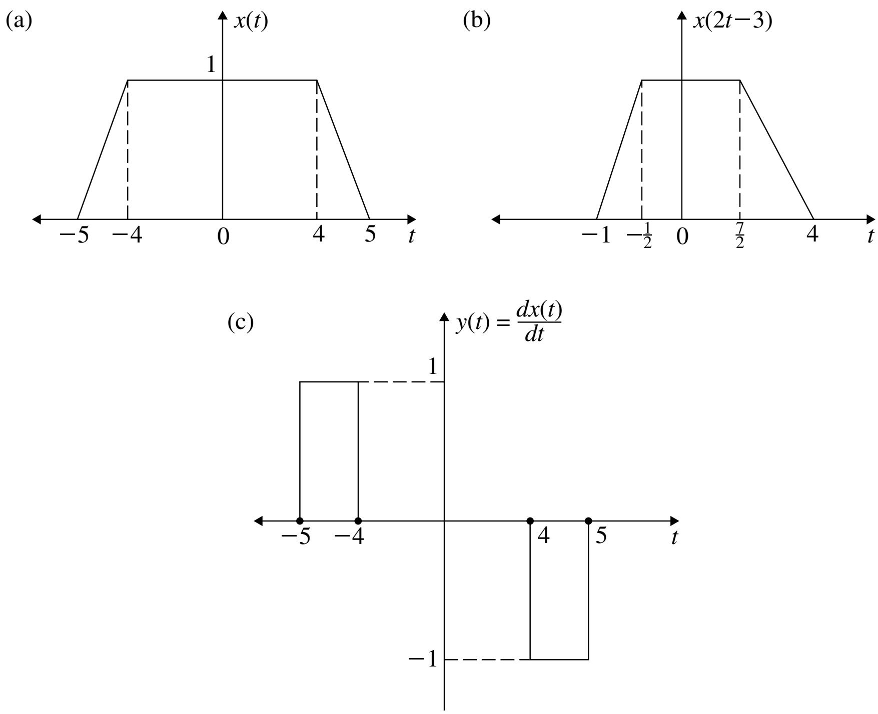


Fig. 1.48 Example 1.53

$$
\begin{aligned}
E &= \int_{-5}^{-4} (t+5)^2 dt + \int_{-4}^{4} (1)^2 dt + \int_{4}^{5} (5-t)^2 dt \\
&= \int_{-5}^{-4} (t^2+10t+25) dt + \int_{-4}^{4} dt + \int_{4}^{5} (t^2-10t+25) dt \\
&= \left[\frac{t^3}{3} + 5t^2 + 25t\right]_{-5}^{-4} + \Big[t\Big]_{-4}^{4} + \left[\frac{t^3}{3} - 5t^2 + 25t\right]_{4}^{5} \\
&= -\frac{64}{3} + 80 - 100 + \frac{125}{3} - 125 + 125 + 8 + \frac{125}{3} - 125 \\
&\quad +125 - \frac{64}{3} + 80 - 100 \\
&= \frac{1}{3} + 8 + \frac{1}{3}
\end{aligned}
$$

$$
E = \frac{26}{3}\ \text{J}
$$

(b) **To sketch $x(2t-3)$**
$x(t)$ in Fig. 1.48a is right shifted by $t_0 = 3$ and time compressed by a factor 2. $x(2t-3)$ is shown in Fig. 1.48b.

(c) **To determine the total energy for $y(t) = \frac{dx}{dt}$.**

$$\begin{aligned}
x(t) &= 5 + t && -5 \le t \le -4 \\
y(t) &= \frac{dx(t)}{dt} = 1 && -5 \le t \le -4 \\
x(t) &= 1 && -4 \le t \le 4 \\
y(t) &= \frac{dx(t)}{dt} = 0 && -4 \le t \le 4 \\
x(t) &= 5 - t && 4 \le t \le 5 \\
y(t) &= \frac{dx(t)}{dt} = -1 && 4 \le t \le 5
\end{aligned}$$

The sketch of the above equations is shown in Fig. 1.48c. From this figure, the total energy is calculated as follows:

$$E = \int_{-5}^{-4} (1)^2 dt + \int_{4}^{5} (-1)^2 dt = \left[t\right]_{-5}^{-4} + \left[t\right]_{4}^{5} = 1 + 1$$

$$E = 2\ \text{J}$$

1.7 System

A system is an interconnection of objects with a definite relationship with the objects and attributes. Consider a simple R, L, C series electric circuit. The components (objects) R, L, and C when connected together form the system. The current flow in the series circuit and the voltages across the elements R, L, and C are the attributes. If i is the current flowing in the circuit, the voltage across the resistor R is iR. Thus, the object R and the attribute i have a definite relationship between them. The voltages across any of these objects R, L, and C can be taken as the output. Thus, the system when excited by a signal, processes and produces signals as outputs in the same form or in a modified form. Electrical motors, communication systems, automotive vehicles, human body, government, stock markets, *etc*. are examples of systems. The block diagram representation of a system is shown in Fig. 1.49.

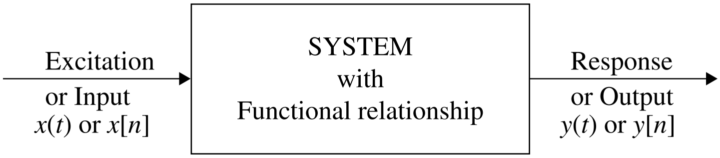


Fig. 1.49 Block diagram representation of system

In Fig. 1.49 the system is excited by the output signal $x(t)$ or $x[n]$. It is being processed by the functional relationship of the system and the response is obtained as $y(t)$ or $y[n]$. The functional relationship includes differential equation or difference equation or the system transfer function which is $H(s)$ for CT system and $H(z)$ for DT system.

1.8 Linear Time Invariant Continuous (LTIC) Time System

The block diagram of a continuous-time system is shown in Fig. 1.50a. $x(t)$ is the input signal which is continuous. The system with the functional relationship $H(s)$ produces the output $y(t)$ which is also continuous. The system dynamics or the functional relationship is written in the form of differential equation connecting $x(t)$ and $y(t)$. If the Laplace transforms of $x(t)$ and $y(t)$ are $X(s)$ and $Y(s)$ respectively, the system functional relationship is written as

$$\frac{Y(s)}{X(s)} = H(s) \tag{1.44}$$

$H(s)$ is called system function or system transfer function.

Consider the electric network shown in Fig. 1.50b. The following dynamic equation is written for Fig. 1.50b:

$$e(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt \tag{1.45}$$

$$e_0(t) = \frac{1}{C}\int i(t)dt \tag{1.46}$$

In the continuous-time system shown in Fig. 1.50b, $e(t)$ is represented by $x(t)$ and $e_0(t)$ is represented by $y(t)$. The system dynamic equations are given in Eqs. (1.45) and (1.46).

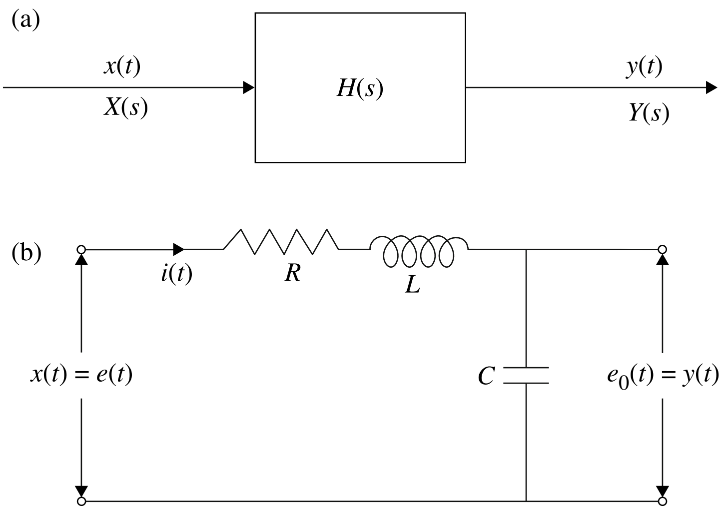


Fig. 1.50 **a** Block diagram of CT system. **b** *R-L-C* series electric circuit

1.9 Properties (Classification) of Continuous-Time System

The continuous-time system possesses the following properties and it is classified accordingly.

1. Linear and non-linear systems.
2. Time invariant and time varying systems.
3. Causal and non-causal systems.
4. Static and dynamic systems (systems without and with memory).
5. Stable and unstable systems.
6. Invertible and non-invertible systems.

The above properties of LTIC time system are defined, described and illustrated with examples below.

1.9.1 Linear and Non-linear Systems

For a linear system if an input $x_1(t)$ produces an output $y_1(t)$ and another input $x_2(t)$ when applied separately produces an output $y_2(t)$, then when both inputs $x(t) = [x_1(t) + x_2(t)]$ are applied to the system simultaneously will produce an output $y(t) = y_1(t) + y_2(t)$. Thus,

$$x_1(t) = y_1(t)$$
$$x_2(t) = y_2(t)$$

$$[x_1(t) + x_2(t)] = [y_1(t) + y_2(t)] \tag{1.47}$$

Equation (1.47) obeys the **Additivity** property of superposition theorem. Further, the linear system should also satisfy the **homogeneity** or **scaling** property of superposition theorem. According to this property, if

$$a_1x_1(t) = a_1y_1(t)$$
$$a_2x_2(t) = a_2y_2(t)$$

then

$$[a_1x_1(t) + a_2x_2(t)] = [a_1y_1(t) + a_2y_2(t)] \tag{1.48}$$

Thus, for a continuous system to be linear, the weighted sum of several inputs produces the weighted sum of outputs. In other words, it should satisfy the homogeneity and additivity properties of superposition theorem. If the above conditions are not satisfied the system is said to be non-linear.

STEP-BY-STEP PROCEDURE TO TEST LINEARITY

1. Let

$$y_1(t) = f(x_1(t))$$
$$y_2(t) = f(x_2(t))$$

   Find the weighted sum of the output

$$y_3(t) = a_1y_1(t) + a_2y_2(t)$$
$$y_3(t) = a_1\ f(x_1(t)) + a_2\ f(x_2(t))$$

2. For the linear combination of input $[a_1x_1(t) + a_2x_2(t)]$ find the output for the weighted sum of the input.

$$y_4(t) = f[a_1x_1(t) + a_2x_2(t)]$$

3. If

$$y_3(t) = y_4(t)$$

   the system is linear, otherwise the system is non-linear. The following examples illustrate the method of testing the linearity of continuous-time systems.

Example 1.44 Consider the following input–output equation of a certain system.

$$y(t) = [2x(t)]^2$$

Determine whether the system is linear or non-linear.

Solution

$$\begin{aligned}
y(t) &= [2x(t)]^2 \\
&= 4x^2(t) \\
y_1(t) &= 4x_1^2(t) \\
y_2(t) &= 4x_2^2(t) \\
y_3(t) &= a_1 y_1(t) + a_2 y_2(t) \\
&= 4a_1 x_1^2(t) + 4a_2 x_2^2(t) \\
y_4(t) &= 4[a_1 x_1(t) + a_2 x_2(t)]^2 \\
&= 4[a_1^2 x_1^2(t) + a_2^2 x_2^2(t) + 2a_1 a_2 x_1(t) x_2(t)] \\
y_3(t) &\neq y_4(t)
\end{aligned}$$

Hence, the system is non-linear.

Example 1.45 Consider the following systems. Determine whether each of them is linear.

$$\begin{aligned}
&\text{(a)} \qquad y(t) = 5x(t)\sin 10t \\
&\text{(b)} \qquad y(t) = 3x(t) + 5 \\
&\text{(c)} \qquad y(t) = t^2 x(t+1) \\
&\text{(d)} \qquad y(t) = E_v x(t) \\
&\text{(e)} \qquad y(t) = x(t^2) \\
&\text{(f)} \qquad y(t) = \int_{-\infty}^{t} 10x(\tau)d\tau \\
&\text{(g)} \qquad y(t) = e^{-2x(t)} \\
&\text{(h)} \qquad y(t) = x(t-7) - x(5-t)
\end{aligned}$$

Solution

(a) $\boldsymbol{y(t) = 5x(t)\sin 10t}$

$$\begin{aligned}
y_1(t) &= 5x_1(t)\sin 10t \\
y_2(t) &= 5x_2(t)\sin 10t \\
y_3(t) &= a_1 y_1(t) + a_2 y_2(t) \\
&= 5\sin 10t(a_1 x_1(t) + a_2 x_2(t)) \\
y_4(t) &= 5\sin 10t(a_1 x_1(t) + a_2 x_2(t)) \\
y_3(t) &= y_4(t)
\end{aligned}$$

The system is Linear.

(b) $\boldsymbol{y(t) = 3x(t) + 5}$

$$\begin{aligned}
y_1(t) &= 3x_1(t) + 5\\
y_2(t) &= 3x_2(t) + 5\\
y_3(t) &= a_1 y_1(t) + a_2 y_2(t)\\
&= 3(a_1x_1(t) + a_2x_2(t)) + 5(a_1 + a_2)\\
y_4(t) &= 3(a_1x_1(t) + a_2x_2(t)) + 5\\
y_3(t) &\neq y_4(t)
\end{aligned}$$

The system is Non-linear.

(c) $\boldsymbol{y(t) = t^2x(t + 1)}$

$$\begin{aligned}
y_1(t) &= t^2x_1(t + 1)\\
y_2(t) &= t^2x_2(t + 1)\\
y_3(t) &= a_1 y_1(t) + a_2 y_2(t)\\
&= t^2[a_1x_1(t + 1) + a_2x_2(t + 1)]\\
y_4(t) &= t^2[a_1x_1(t + 1) + a_2x_2(t + 1)]\\
y_3(t) &= y_4(t)
\end{aligned}$$

The system is Linear.

(d) $\boldsymbol{y(t) = E_v x(t)}$

$$\begin{aligned}
y(t) &= \frac{1}{2}[x(t) + x(-t)]\\
y_1(t) &= \frac{1}{2}[x_1(t) + x_1(-t)]\\
y_2(t) &= \frac{1}{2}[x_2(t) + x_2(-t)]\\
y_3(t) &= a_1 y_1(t) + a_2 y_2(t)\\
&= \frac{1}{2}[a_1x_1(t) + a_2x_2(t) + a_1x_1(-t) + a_2x_2(-t)]\\
y_4(t) &= \frac{1}{2}[a_1(x_1(t) + x_1(-t)) + a_2(x_2(t) + x_2(-t))]\\
&= \frac{1}{2}[a_1x_1(t) + a_2x_2(t) + a_1x_1(-t) + a_2x_2(-t)]\\
y_3(t) &= y_4(t)
\end{aligned}$$

The system is Linear.

(e) $\boldsymbol{y(t) = x(t^2)}$

$$\begin{aligned}
y_1(t) &= x_1(t^2)\\
y_2(t) &= x_2(t^2)\\
y_3(t) &= a_1 y_1(t) + a_2 y_2(t)\\
&= a_1 x_1(t^2) + a_2 x_2(t^2)\\
y_4(t) &= a_1 x_1(t^2) + a_2 x_2(t^2)\\
y_3(t) &= y_4(t)
\end{aligned}$$

The system is Linear.

(f) $\boldsymbol{y(t) = 10\int_{-\infty}^{t} x(\tau)d\tau}$

$$\begin{aligned}
y_1(t) &= 10\int_{-\infty}^{t} x_1(\tau)d\tau\\
y_2(t) &= 10\int_{-\infty}^{t} x_2(\tau)d\tau\\
y_3(t) &= a_1 y_1(t) + a_2 y_2(t)\\
&= 10\left[a_1\int_{-\infty}^{t} x_1(\tau)d\tau + a_2\int_{-\infty}^{t} x_2(\tau)d\tau\right]
\end{aligned}$$

$$\begin{aligned}
y_4(t) &= 10\left[\int_{-\infty}^{t}\left\{a_1 x_1(\tau) + a_2 x_2(\tau)\right\}d\tau\right]\\
&= 10\left[\left\{\int_{-\infty}^{t} a_1 x_1(\tau)d\tau + \int_{-\infty}^{t} a_2 x_2(\tau)d\tau\right\}\right]\\
y_3(t) &= y_4(t)
\end{aligned}$$

The system is Linear.

(g) $\boldsymbol{y(t) = e^{-2x(t)}}$

$$\begin{aligned}
y_1(t) &= e^{-2x_1(t)} \\
y_2(t) &= e^{-2x_2(t)} \\
y_3(t) &= a_1 y_1(t) + a_2 y_2(t) \\
&= a_1 e^{-2x_1(t)} + a_2 e^{-2x_2(t)} \\
y_4(t) &= e^{-2(a_1 x_1(t) + a_2 x_2(t))} \\
&= e^{-2a_1 x_1(t)} e^{-2a_2 x_2(t)} \\
y_3(t) &\neq y_4(t)
\end{aligned}$$

The system is Non-linear.

(h) $\boldsymbol{y(t) = x(t-7) - x(5-t)}$

$$\begin{aligned}
y_1(t) &= x_1(t-7) - x_1(5-t) \\
y_2(t) &= x_2(t-7) - x_2(5-t) \\
y_3(t) &= a_1 y_1(t) + a_2 y_2(t) \\
&= a_1[x_1(t-7) - x_1(5-t)] + a_2[x_2(t-7) - x_2(5-t)] \\
y_4(t) &= a_1[x_1(t-7) - x_1(5-t)] + a_2[x_2(t-7) - x_2(5-t)] \\
y_3(t) &= y_4(t)
\end{aligned}$$

The system is Linear.

Linearity Test for the System Described by Differential Equation

Step 1. Write down the system differential equation with responses $y_1(t)$ and $y_2(t)$ for the inputs $x_1(t)$ and $x_2(t)$ respectively.

Step 2. Multiply the $y_1(t)$ response equation with a_1 and $y_2(t)$ response equation with a_2 and add them.

Step 3. Write down the differential equation for the sum of the inputs $x(t) = a_1 x_1(t) + a_2 x_2(t)$.

Step 4. If $y(t) = a_1 y_1(t) + a_2 y_2(t)$ obtained in Steps 2 and 3 are same, the given differential equation is linear. Otherwise the differential equation is non-linear.

The following examples illustrate the above method.

Example 1.46 Determine whether the following differential equations are linear or non-linear:

$$\text{(a)} \qquad \frac{dy(t)}{dt} + 10y(t) = 2x(t)$$
$$\text{(b)} \qquad \frac{dy(t)}{dt} + 10\sin y(t) = 2x(t)$$
$$\text{(c)} \qquad y(t)\frac{dy(t)}{dt} + 10y(t) = 2x(t)$$

Solution (a) The weighted sum of the response due to each input signal is

$$\frac{d}{dt}[a_1y_1(t)] + 10a_1y_1(t) = 2a_1x_1(t)$$
$$\frac{d}{dt}[a_2y_2(t)] + 10a_2y_2(t) = 2a_2x_2(t)$$

Adding the above two equations, we get

$$\frac{d}{dt}[a_1y_1(t) + a_2y_2(t)] + 10[a_1y_1(t) + a_2y_2(t)] = 2[a_1x_1(t) + a_2x_2(t)] \quad \text{(a)}$$

The response of the system due to weighted sum of input is given as

$$a_1\frac{dy_1(t)}{dt} + a_2\frac{dy_2(t)}{dt} + 10[a_1y_1(t) + a_2y_2(t)] = 2[a_1x_1(t) + a_2x_2(t)]$$

$$\frac{d}{dt}[a_1y_1(t) + a_2y_2(t)] + 10[a_1y_1(t) + a_2y_2(t)] = 2[a_1x_1(t) + a_2x_2(t)] \quad \text{(b)}$$

Equations (a) and (b) are same. Hence, the given system is linear.

The system is Linear.

(b) $\boldsymbol{\frac{dy(t)}{dt} + 10\sin y(t) = 2x(t)}$

$$\frac{dy(t)}{dt} + 10\sin y(t) = 2x(t)$$

The weighted sum of responses due to $a_1x_1(t)$ and $a_2x_2(t)$ are

$$\frac{d}{dt}[a_1y_1(t)] + 10\sin a_1y(t) = 2a_1x_1(t)$$
$$\frac{d}{dt}[a_2y_2(t)] + 10\sin a_2y_2(t) = 2a_2x_2(t)$$

The weighted sum of the responses is obtained by adding the above two equations.

$$\frac{d}{dt}[a_1y_1(t)+a_2y_2(t)]+10\sin a_1y_1(t)+10\sin a_2y_2(t)=2[a_1x_1(t)+a_2x_2(t)] \quad \text{(a)}$$

The output response due to weighted sum of inputs $x(t)=a_1x_1(t)+a_2x_2(t)$ is

$$a_1\frac{d}{dt}y_1(t)+a_2\frac{d}{dt}y_2(t)+10a_1\sin y_1(t)+10a_2\sin y_2(t)=2[a_1x_1(t)+a_2x_2(t)]$$

$$\frac{d}{dt}[a_1y_1(t)+a_2y_2(t)+10[a_1\sin y_1(t)+a_2\sin y_2(t)]=2[a_1x_1(t)+a_2x_2(t)] \quad \text{(b)}$$

Equations (a) and (b) are not the same. Hence, it is not linear.

The system is Non-linear.

(c) $\boldsymbol{y(t)\frac{dy(t)}{dt}+10y(t)=2x(t)}$

$$y(t)\frac{dy(t)}{dt}+10y(t)=2x(t)$$

The weighted responses due to inputs $a_1x_1(t)$ and $a_2x_2(t)$ are

$$a_1y_1(t)\frac{d}{dt}[a_1y_1(t)]+10a_1y_1(t)=2a_1x_1(t)$$
$$a_2y_2(t)\frac{d}{dt}[a_2y_2(t)]+10a_2y_2(t)=2a_2x_2(t)$$

The sum of the weighted response due to $x(t)=a_1x_1(t)+a_2x_2(t)$ is obtained by adding the above two equations.

$$a_1^2y_1(t)\frac{d}{dt}[y_1(t)]+a_2^2y_2(t)\frac{d}{dt}[y_2(t)]+10[a_1y_1(t)+a_2y_2(t)]=2[a_1x_1(t)+a_2x_2(t)] \quad \text{(a)}$$

The response due to weighted sum of inputs $x(t)=a_1x_1(t)+a_2x_2(t)$ is

$$a_1y_1(t)\frac{d}{dt}y_1(t)+10a_1y_1(t)+a_2y_2(t)\frac{d}{dt}y_2(t)+10a_2y_2(t)=2[a_1x_1(t)+a_2x_2(t)]$$

$$a_1y_1(t)\frac{d}{dt}y_1(t)+a_2y_2(t)\frac{d}{dt}y_2(t)+10[a_1y_1(t)+a_2y_2(t)]=2[a_1x_1(t)+a_2x_2(t)] \quad \text{(b)}$$

Equations (a) and (b) are not equal. Hence, the system is not linear.

The system is Non-linear.

1.9.2 Time Invariant and Time Varying Systems

A continuous-time system is said to be time invariant if the parameters of the system do not change with time. The characteristics of such system are fixed over a time. The input–output of a certain continuous-time system is shown in Fig. 1.51a and b respectively. If the input is delayed by t_0 seconds, the characteristic of the output response remains the same but delayed by t_0 seconds. This is illustrated in Fig. 1.51c and d respectively. This property is also illustrated in Fig. 1.51e and f in block diagram form. In Fig. 1.51e the output $y(t)$ of the system H is delayed by t_0 seconds to get $y(t - t_0)$ as the delayed output. The delayed output $y(t - t_0)$ of system H can also be obtained by delaying the input $x(t)$ as $x(t - t_0)$. This is illustrated in Fig. 1.51f. This time delay the system commutes only if the system is time

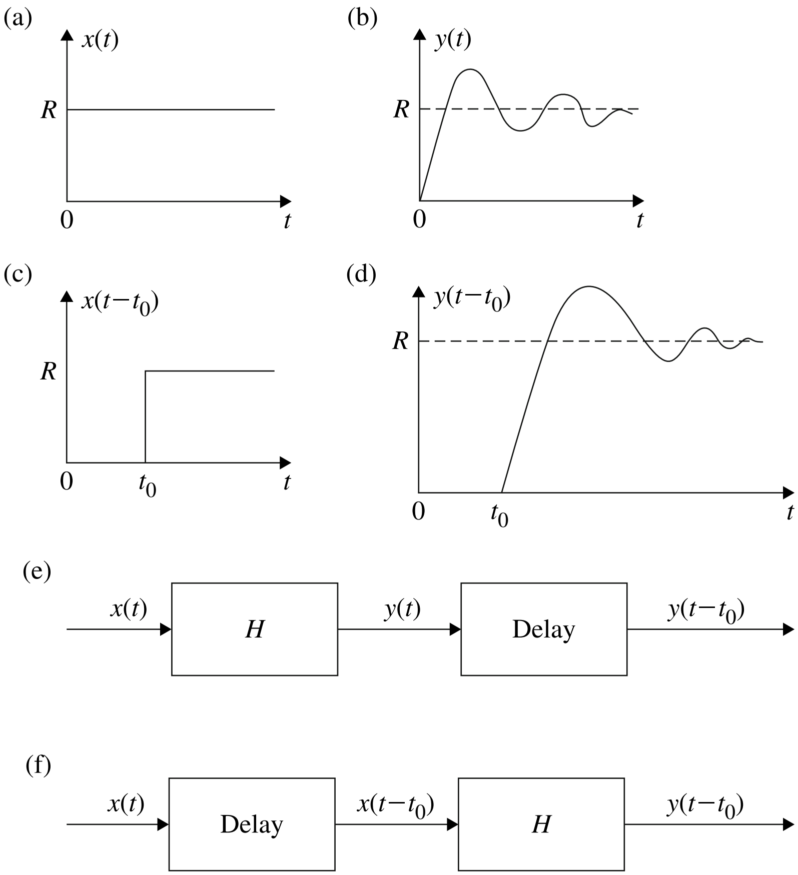


Fig. 1.51 Time invariancy property

invariant. The above property will not apply if the system is time varying which can be easily proved. Thus, to identify the time invariant system, the steps given below are followed:

Step 1. For the delayed input $x(t-t_0)$ obtain the output $y(t,t_0)$.
Step 2. Obtain the expression for the delayed output $y(t-t_0)$ by substituting $t = (t-t_0)$.
Step 3. If $y(t,t_0) = y(t-t_0)$, then the system is time invariant. Otherwise it is a time varying system.

The following examples illustrate the method of identifying time invariancy.

Example 1.47 Check whether the following systems are time invariant or not:

$$\text{(a)} \quad y(t) = tx(t)$$
$$\text{(b)} \quad y(t) = \cos x(t)$$
$$\text{(c)} \quad y(t) = x(t)\cos x(t)$$
$$\text{(d)} \quad y(t) = e^{-2x(t)}$$
$$\text{(e)} \quad \frac{d^2}{dt}y(t) + 2\frac{d}{dt}y(t) + 5y(t) = x(t)$$
$$\text{(f)} \quad \frac{d^2}{dt}y(t) + 2t\frac{d}{dt}y(t) + 5y(t) = x(t)$$
$$\text{(g)} \quad y(t) = \left[\frac{dx(t)}{dt}\right]^2$$

Solution

(a) $\boldsymbol{y(t) = tx(t)}$

1. For the delayed input $x(t-t_0)$, the output $y(t,t_0)$ is obtained as

$$y(t,t_0) = tx(t-t_0)$$

2. The delayed output $y(t-t_0)$ is obtained by substituting $t = t - t_0$ in the given equation

$$y(t-t_0) = (t-t_0)x(t-t_0)$$

3. $y(t-t_0) \neq y(t,t_0)$
4.

The system is Time Varying.

(b) $\boldsymbol{y(t) = \cos x(t)}$

1. $y(t,t_0) = \cos x(t-t_0)$ [Delayed input]
2. $y(t-t_0) = \cos x(t-t_0)$ [Delayed output]

3. $y(t - t_0) = y(t, t_0)$
4.

The system is Time Invariant.

(c) $\boldsymbol{y(t) = x(t)\cos x(t)}$

1. $y(t, t_0) = x(t - t_0)\cos x(t - t_0)$ [Delayed input]
2. $y(t - t_0) = x(t - t_0)\cos x(t - t_0)$ [Delayed output]
3. $y(t - t_0) = y(t, t_0)$
4.

The system is Time Invariant.

(d) $\boldsymbol{y(t) = e^{-2x(t)}}$

1. The output due to delayed input is

$$y(t, t_0) = e^{-2x(t-t_0)}$$

2. The delayed output is obtained by putting $t = t - t_0$

$$y(t - t_0) = e^{-2x(t-t_0)}$$

3. $y(t - t_0) = y(t, t_0)$
4.

The system is Time Invariant.

(e) $\boldsymbol{\frac{d^2}{dt}y(t) + 2\frac{d}{dt}y(t) + 5y(t) = x(t)}$
The coefficients of the given differential equation are 1, 2 and 5 and they are constants. They do not vary with time. Hence

The system is Time Invariant.

(f) $\boldsymbol{\frac{d^2}{dt^2}y(t) + 2t\frac{d}{dt}y(t) + 5y(t) = x(t)}$
The coefficient of $\frac{dy(t)}{dt}$ is $2t$ and it varies with respect to time. Hence

The system is Time Varying.

(g) $\boldsymbol{y(t) = \left[\frac{d}{dt}x(t)\right]^2}$

1. For the delayed input $x(t - t_0)$ the output is obtained as

$$y(t, t_0) = \left[\frac{d}{dt}x(t - t_0)\right]^2$$

2. The delayed output is obtained by putting $t = t - t_0$ in the given equation

$$y(t - t_0) = \left[\frac{d}{dt}x(t - t_0)\right]^2$$

3. $y(t - t_0) = y(t, t_0)$
4.

The system is Time Invariant.

1.9.3 Static and Dynamic Systems (Memoryless and System with Memory)

Consider the R-C series electrical circuit shown in Fig. 1.52a. The charge in the capacitor is determined by the current that has flown through it. By this mechanism the capacitor remembers about something about its past. Similarly consider the mechanical system in Fig. 1.52b. The stored energy in the mechanical spring depends on the past history of the applied force. The present response of these systems which have energy storing elements **depends not only on the present excitation but also on the past excitation** which are remembered by these elements. Such systems are called dynamic systems or systems with memory.

Consider the electrical network shown in Fig. 1.52a in which only a resistor is connected. The current flowing through the resistor depends on the present value of the excitation. The response does not depend on the excitation at any other time.

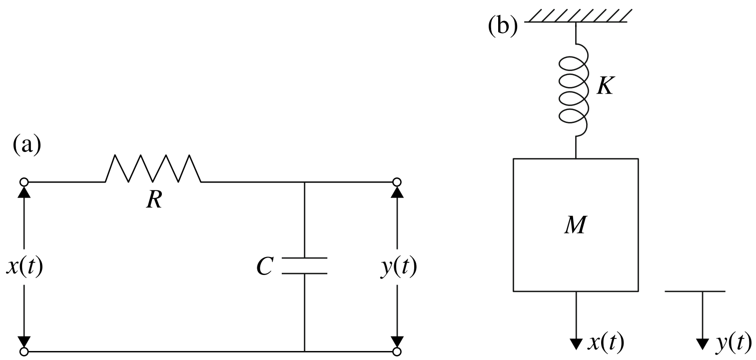


Fig. 1.52 Dynamic systems

Such systems which have no energy storing elements are called static systems or systems without memory.

A dynamic system is, therefore, defined as a system in which the output signal at any specified time depends on the values of the input signals at the specific time at other time also.

A static system is defined as a system in which the output signal at any specified time depends on the present value of the input signal alone.

The following examples illustrate the method of identifying static and dynamic systems.

Example 1.48 Determine whether the following systems are static or dynamic:

(a) $y(t) = x(t+1) + 5$

(b) $y(t) = x(t^2)$

(c) $y(t) = x(t)\sin 2t$

(d) $y(t) = x(t-3) + x(3-t)$

(e) $y(t) = x\left(\frac{t}{4}\right)$

(f) $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

(g) $\frac{dy(t)}{dt} + 5y(t) = 2x(t)$

(h) $y(t) = 2x(t) + 3$

(i) $y(t) = e^{-2x(t)}$

Solution

(a) $\boldsymbol{y(t) = x(t+1) + 5}$

$$y(0) = x(1) + 5$$

The system response depends on the future input $x(t+1)$. Hence

The system is Dynamic.

(b) $\boldsymbol{y(t) = x(t^2)}$

For $t = 1$,

$$y(1) = x(1)$$

For $t = 2$,

$$y(1) = x(4)$$

The response depends on the future input. Hence

The system is Dynamic.

(c) $\boldsymbol{y(t) = x(t)\sin 2t}$
The system response depends on the present value of the input $x(t)$ and due to $\sin 2t$, only its magnitude varies from -1 to $+1$. Hence

The system is Static.

(d) $\boldsymbol{y(t) = x(t-3) + x(3-t)}$
For $t = 0$,

$$y(0) = x(-3) + x(3)$$

For $t = 3$,

$$y(3) = x(0) + x(0)$$

For $t = -3$,

$$y(-3) = x(-6) + x(6)$$

The system response depends on past and future values of input. Hence

The system is Dynamic.

(e) $\boldsymbol{y(t) = x\left(\frac{t}{4}\right)}$

$$y(0) = x(0)$$
$$y(1) = x\left(\frac{1}{4}\right)$$
$$y(-1) = x\left(-\frac{1}{4}\right)$$

The system response depends on present, future, and past values of input. Hence

The system is Dynamic.

(f) $\boldsymbol{y(t) = \int_{-\infty}^{t} x(\tau)d\tau}$
By integrating the input, the output is retained and stored in a memory from time t to the infinite past. Hence

The system is Dynamic.

(g) $\boldsymbol{\frac{dy(t)}{dt} + 5y(t) = 2x(t)}$

The input–output is described by a first-order differential equation. It requires an energy storing element which remembers the past history of the input applied. Hence

The system is Dynamic.

(h) $\boldsymbol{y(t) = 2x(t) + 3}$
The output always depends on the present input. Hence

The system is Static.

(i) $\boldsymbol{y(t) = e^{-2x(t)}}$
The output always depends on the present input only. Hence

The system is Static.

1.9.4 Causal and Non-causal Systems

Consider a continuous-time system excited by the signal $x(t)$. **If the response (output) depends on the present and past values of the input $x(t)$, the system is said to be causal**. In a causal signal, the output cannot start before the input is applied. Hence, the causal system is also called **non-anticipative system**. On the other hand, if the system acts on the knowledge of future input, before it is being applied such systems are called **anticipative or non-causal systems**. Real-time systems are all causal systems.

Consider the system described by the following input–output equation

$$y(t) = x(t-3) + x(t+3) \tag{1.49}$$

For the input shown in Fig. 1.53a, the output $y(t)$ is sketched and shown in Fig. 1.53b. The output $y(t)$ at time t is given by the sum of the input values at $(t-3)$ which is 3 seconds before and at $(t+3)$ which is 3 seconds after. This is illustrated in Fig. 1.53b. Here the system responds to the future input $x(t+3)$ and it is non-causal system and cannot be realizable in real time. The following examples illustrate the method of identifying causal and non-causal systems.

Example 1.49 Consider the continuous-time systems described below by their input–output equations. Identify whether they are causal or non-causal.

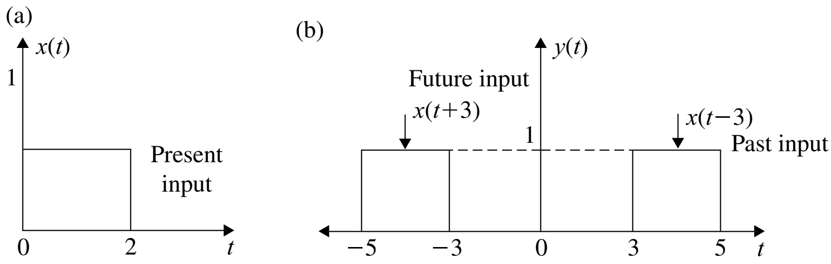


Fig. 1.53 A non-causal system

$$\text{(a)} \quad y(t) = x\left(\frac{t}{4}\right)$$

$$\text{(b)} \quad y(t) = x(t)\sin(1+t)$$

$$\text{(c)} \quad y(t) = x(t^2)$$

$$\text{(d)} \quad y(t) = x(\sqrt{t})$$

$$\text{(e)} \quad y(t) = x(t+1)$$

$$\text{(f)} \quad y(t) = x(t-1)$$

$$\text{(g)} \quad y(t) = \frac{d}{dt}x(t)$$

$$\text{(h)} \quad y(t) = \int_{t-4}^{t+4} x(\tau)d\tau$$

Solution

(a) $\boldsymbol{y(t) = x\left(\frac{t}{4}\right)}$

$$y(0) = x(0)$$
$$y(-4) = x(-1)$$

The output depends on future value of input which is evident from $y(-4) = x(-1)$. Hence

The system is Non-causal.

(b) $\boldsymbol{y(t) = x(t)\sin(1+t)}$

$$y(0) = x(0)\sin(1)$$
$$y(1) = x(1)\sin(2)$$
$$y(-1) = x(-1)\sin(0)$$

Thus, at all time, the output depends on the present input only. Hence

The system is Causal.

(c) $\boldsymbol{y(t) = x(t^2)}$

$$y(0) = x(0)$$
$$y(1) = x(1)$$
$$y(2) = x(4)$$

The system output depends on the present input as seen from $y(0) = x(0)$ and $y(1) = x(1)$. The system output $y(t)$ at $t = 2$, which is $y(2) = x(4)$ depends on the future input $x(t)$. Hence

The system is Non-causal.

(d) $\boldsymbol{y(t) = x(\sqrt{t})}$
At $t = 0.64$

$$y(0.64) = x(0.8)$$

The output depends on the future input. Hence

The system is Non-causal.

(e) $\boldsymbol{y(t) = x(t + 1)}$
For $t = 0$,

$$y(0) = x(1)$$

The system output depends on the future input. Hence

The system is Non-causal.

(f) $\boldsymbol{y(t) = x(t - 1)}$

$$y(0) = x(-1)$$
$$y(1) = x(0)$$
$$y(2) = x(1)$$

The output depends on the past values of the input. Hence

The system is Causal.

(g) $\boldsymbol{y(t) = \frac{d}{dt}x(t)}$

$$y(0) = \frac{d}{dt}x(0)$$
$$y(1) = \frac{d}{dt}x(1)$$

The output depends on the present input. Hence

The system is Causal.

(h) $\boldsymbol{y(t) = \int_{t-4}^{t+4} x(\tau)d\tau}$

$$\begin{aligned} y(t) &= \Big[x(\tau)\Big]_{t-4}^{t+4} \\ &= x(t+4) - x(t-4) \end{aligned}$$

For $t = 0$,

$$y(0) = x(4) - x(-4)$$

The output $y(0)$ depends on future input $x(4)$. Hence

The system is Non-causal.

1.9.5 Stable and Unstable Systems

Consider a cone which is resting on its base as shown in Fig. 1.54a. The cone at this position when given a small disturbance will stay in the same position with a small displacement which is the new equilibrium state. Now this position of the cone is said to be in stable state. On the other hand, consider the cone resting on its tip. When the cone is given a small displacement (say an impulse) the contact of the tip with the resting surface is lost and it rolls over the surface. The output position (resting on the tip) is never reached. This state of the cone is said to be unstable.

Consider a linear time invariant continuous-time system which is excited by an impulse as shown in block diagram of Fig. 1.55a. The output response of the system is shown in Fig. 1.55b and c. In Fig. 1.55b the area under the impulse response curve is finite. It can be mathematically proved that such systems whose area of the impulse response curve is finite are said to be stable. On the other hand, consider Fig. 1.55c. The area under this impulse curve is infinite. Systems which possess such an impulse curve are said to be unstable.

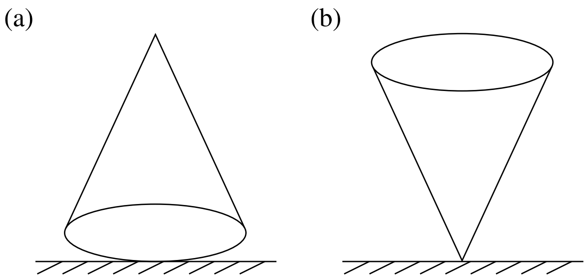


Fig. 1.54 Stable and unstable systems

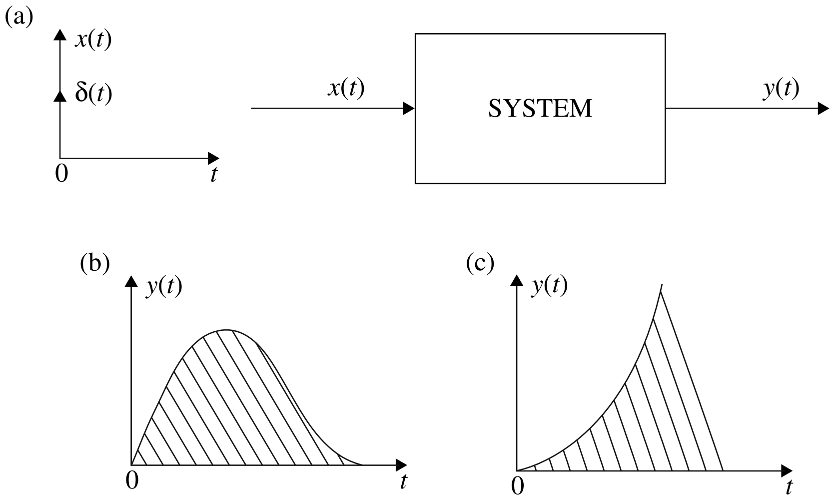


Fig. 1.55 Impulse response of stable and unstable systems

A linear time invariant continuous-time system is said to be Bounded Input, Bounded Output (BIBO) stable, if for any bounded input, it produces bounded output. This also implies that for BIBO stability, the area under the impulse response (output) curve should be finite.

The BIBO stability concept is mathematically expressed as follows. Let the input–output of a linear time invariant system be expressed as

$$y(t) = f[x(t)] \qquad \text{for all } t \tag{1.50}$$

If $|x(t)|$ is bounded, $|y(t)|$ should also be bounded for the system to be stable.

$$|y(t)| \leq M_y < \infty \qquad \text{for all } t \tag{1.51}$$

$$|x(t)| \leq M_x < \infty \qquad \text{for all } t \tag{1.52}$$

where $|M_x|$ and $|M_y|$ represent positive values. It can be easily established that the necessary and sufficient condition for the LTIC time system to be stable is

$$y(t) = \int_{-\infty}^{\infty} |x(t)| dt < \infty \tag{1.53}$$

The following examples illustrate the method of finding the stability of LTIC time system.

Example 1.50 Determine whether the systems described by the following equations are BIBO stable.

(a) $y(t) = tx(t)$
(b) $y(t) = e^{-2|t|}$
(c) $y(t) = x(t) \sin t$
(d) $y(t) = te^{2t} u(t)$
(e) $y(t) = e^{4t} u(t-3)$
(f) $y(t) = e^{-2t} \sin 2t \, u(t)$

Solution

(a) $\boldsymbol{y(t) = tx(t)}$

If $x(t)$ is bounded, $y(t)$ varies with respect to time and becomes unbounded. Hence

The system is BIBO Unstable.

(b) $\boldsymbol{y(t) = e^{-2|t|}}$

Here

$$\begin{aligned} x(t) &= e^{-2t} \qquad 0 \le t < \infty \\ &= e^{2t} \qquad -\infty < t < 0 \\ y(t) &= \int_{-\infty}^{\infty} x(t) \\ &= \int_{-\infty}^{0} e^{2t} dt + \int_{0}^{\infty} e^{-2t} dt \\ &= \left[\frac{1}{2} e^{2t}\right]_{-\infty}^{0} - \left[\frac{1}{2} e^{-2t}\right]_{0}^{\infty} \\ &= \frac{1}{2}[1+1] = 1 < \infty \end{aligned}$$

The output is bounded and the system is stable.

The system is BIBO Stable.

(c) $\boldsymbol{y(t) = x(t)\sin t}$
It $x(t)$ is bounded, $y(t)$ is also bounded because $\sin t$ will take a maximum value of $+1$ and -1. Hence, $y(t)$ is bounded.

The system is BIBO Stable.

(d) $\boldsymbol{y(t) = te^{2t}u(t)}$
Here the output varies linearly as t and also exponentially increasing due to e^{2t}. Hence, $|y(t)| = \infty$ and the system is BIBO unstable. Mathematically this can be proved as follows. For a causal system, $|y(t)|$ can be written as

$$|y(t)| = \int_0^\infty te^{2t}dt$$

The following integration formula is used to evaluate the above integral:

$$\int_0^\infty te^{at}dt = \frac{1}{a^2}\Big[e^{at}\{at - 1\}\Big]_0^\infty$$

$$\begin{aligned}|y(t)| &= \frac{1}{4}\Big[e^{2t}\{2t - 1\}\Big]_0^\infty \\ &= \frac{1}{4}[e^\infty\{2\infty - 1\} + 1] \\ &= \infty\end{aligned}$$

The system is BIBO Unstable.

(e) $\boldsymbol{y(t) = e^{4t}u(t-3)}$
The output response is exponentially increasing as t increases with a time delay of $t = 3$. Hence, the system is unstable. This is mathematically proved as follows:

$$\begin{aligned}|y(t)| &= \int_{-\infty}^\infty |x(t)|dt \\ &= \int_3^\infty e^{4t}dt \\ &= \frac{1}{4}\Big[e^{4t}\Big]_3^\infty \\ &= \infty - \frac{1}{4}e^{12} \\ &= \infty\end{aligned}$$

The system is BIBO Unstable.

(f) $\boldsymbol{y(t) = e^{-2t} \sin 2t\, u(t)}$
The output response is a function of exponential decay and a sinusoid. The sinusoid will have a maximum value of $+1$ and -1. As t increases, $y(t)$ will exponentially decrease and the output is bounded. The result can be mathematically obtained as follows. For a causal signal $u(t)$

$$|y(t)| = \int_0^{\infty} e^{-2t} \sin 2t\, dt$$

Using the formula,

$$\int_0^{\infty} e^{at} \sin bt\, dt = \frac{\left[e^{at}\{a \sin bt - b \cos at\}\right]_0^{\infty}}{a^2 + b^2}$$

we get

$$\begin{aligned}|y(t)| &= \frac{2}{2^2 + 2^2}\left[e^{-2t}\{\sin 2t - \cos 2t\}\right]_0^{\infty} \\ &= \frac{1}{4} < \infty\end{aligned}$$

The system is BIBO Stable.

1.9.6 *Invertibility and Inverse System*

Consider the system H which is excited with $x(t)$. The system produces the output $y(t)$. This signal is applied as the input to the inverse system H^{-1} which produces the output $x(t)$. The block diagram representation of the system and the inverse system is shown in Fig. 1.56a. From Fig. 1.56a, the inverse system is defined as follows.

A system is said to be invertible if the distinct inputs give distinct output.

Consider the system shown in Fig. 1.56b. The input–output relationship of system 1 is described as

$$\frac{d}{dt} y(t) = x(t)$$

Consider system 2, the input–output of this system is described by

$$\frac{d}{dt} y(t) = x(t)$$

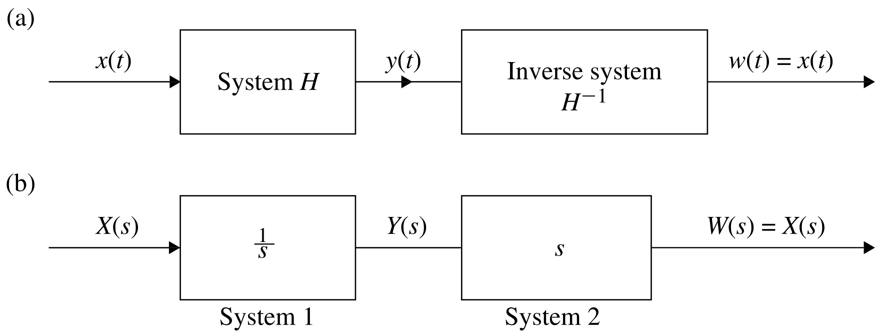


Fig. 1.56 Representation of inverse system

When these two systems are cascaded, the output response of the interconnected system is same as the excitation of the system itself. **The system which makes this possible is called inverse system. Here unique excitation produces unique response**.

Example 1.51 Consider the systems described by the equations given below:

(a) The impulse $h(t)$ is given as

$$h(t) = \delta(t) - 3e^{-3t}u(t) + 4e^{-4t}u(t)$$

(b)

$$\frac{dy(t)}{dt} + 5y(t) = \frac{d^2x(t)}{dt^2} + 2\frac{dx(t)}{dt} - 8x(t)$$

Determine the inverse systems for the above. Are these systems both causal and stable?

Solution

(a) $\boldsymbol{h(t) = \delta(t) - 3e^{-3t}u(t) + 4e^{-4t}u(t)}$
Taking Laplace transform on both sides, we get

$$H(s) = 1 - \frac{3}{s+3} + \frac{4}{s+4} = \frac{(s^2 + 8s + 12)}{(s+3)(s+4)}$$

The inverse of the above system is

$$H^{-1}(s) = \frac{1}{H(s)} = \frac{(s+3)(s+4)}{s^2 + 8s + 12}$$

$$H^{-1}(s) = \frac{(s+3)(s+4)}{(s+2)(s+6)}$$

The poles of H^{-1} are at $s = -2$ and $s = -6$. Hence, the inverse systems is stable. The region of convergence (ROC) is to the right of rightmost pole $s = -2$. Hence, it is causal.

The inverse system is both Causal and Stable.

(b) $\boldsymbol{\frac{dy(t)}{dt} + 5y(t) = \frac{d^2x(t)}{dt^2} + 2\frac{dx(t)}{dt} - 8x(t)}$
Taking Laplace transform on both sides of the above equation, we get

$$(s+5)Y(s) = (s^2 + 2s - 8)X(s)$$
$$H(s) = \frac{Y(s)}{X(s)} = \frac{(s^2 + 2s - 8)}{(s+5)} = \frac{(s-2)(s+4)}{(s+5)}$$

The inverse system is

$$H^{-1}(s) = \frac{1}{H(s)} = \frac{(s+5)}{(s-2)(s+4)}$$

$$H^{-1}(s) = \frac{(s+5)}{(s-2)(s+4)}$$

The poles of the inverse systems are at $s = 2$ and $s = -4$. The pole at $s = 2$ will make the system unstable if the system is causal. For the system to be stable the ROC should form a strip between $s = 2$ and $s = -4$ in which case it includes the $j\omega$ axis. In this case, the system has to be non-causal.

The system is not both Causal and Stable.

Example 1.52 Determine whether the given system is memoryless, time invariant, linear, causal, and stable. Justify your answers.

$$y(t) = (\cos 3t)\,x(t)$$

(*Anna University, December 2006*)

Solution

$$y(0) = x(0)$$
$$y(1) = \cos 3x(1)$$
$$y(-1) = \cos 1x(-1)$$

1. The output depends only on the present input. Hence, the system is memoryless (static). Since the output does not depend on the future input, it is **causal**.
2. The output due to the delayed input is

$$y(t, t_0) = \cos 3t \, x(t - t_0)$$

The delayed output is obtained by substituting $t = (t - t_0)$ in the given equation

$$y(t - t_0) = \cos 3(t - t_0)x(t - t_0)$$
$$y(t - t_0) \neq y(t, t_0)$$

The system is therefore time varying.

3. To test the linearity of the system, consider the given equation

$$y(t) = (\cos 3t)x(t)$$
$$y_1(t) = (\cos 3t)x_1(t)$$
$$y_2(t) = (\cos 3t)x_2(t)$$
$$y_3(t) = a_1y_1(t) + a_2y_2(t) = \cos 3t[a_1x_1(t) + a_2x_2(t)]$$

The output due to the weighted sum of input is

$$y_4(t) = (\cos 3t)[a_1x_1(t) + a_2x_2(t)]$$
$$y_3(t) = y_4(t)$$

The system is Linear.

4.

$$|y(t)| = \cos 3t|x(t)|$$

If $x(t)$ is bounded $|y(t)|$ is also bounded. **Hence, the system is stable.**

The system is

(a) Static, (b) Time Variant, (c) Linear, (d) Causal, and (e) Stable.

Example 1.53 Verify whether the system given by

$$y(t) = x(t^2)$$

is causal, instantaneous, linear, and shift invariant.

(*Anna University, May 2006*)

Solution

1.

$$y(t) = x(t^2)$$
$$y(2) = x(4)$$

The output depends on the future input. Hence, the system is **not causal.**

2. Since the output depends on the present and future inputs, it requires memory. It is, therefore, **not instantaneous.**
3. The response due to the delayed input is

$$y(t, t_0) = x[(t^2 - t_0)]$$

The delayed output is obtained by putting $t = t - t_0$ in the given equation

$$y(t - t_0) = x[(t - t_0)^2]$$
$$y(t, t_0) \neq y(t - t_0)$$

Hence, the system is shift variant.

4.

$$\begin{aligned} y(t) &= x(t^2) \\ y_1(t) &= x_1(t^2) \\ y_2(t) &= x_2(t^2) \\ y_3(t) &= a_1 y_1(t) + a_2 y_2(t) \\ &= a_1 x_1(t^2) + a_2 x_2(t^2) \\ y_4(t) &= f[a_1 x_1(t) + a_2 x_2(t)] \\ &= a_1 x_1(t^2) + a_2 x_2(t^2) \\ y_3(t) &= y_4(t) \end{aligned}$$

The system is linear.

The system is

(a) Non-causal, (b) Not Instantaneous, (c) Shift Variant, and (d) Linear.

Example 1.54 Determine whether the system described by the following equation is static, linear, time variant, and causal.

$$y(t) = E_v[x(t)]$$

Solution

1. $\boldsymbol{y(t) = E_v[x(t)]}$

$$\begin{aligned} y(t) &= E_v[x(t)] \\ &= \frac{1}{2}[x(t) + x(-t)] \\ y(-1) &= \frac{1}{2}[x(-1) + x(1)] \end{aligned}$$

The output depends on the present value of $x(-1)$ and also the future value of $x(1)$. Hence, the system is **non-causal**. Since $x(1)$ requires memory, **the system is dynamic.**

2.

$$y(t) = \frac{1}{2}[x(t) + x(-t)]$$

The output due to the delayed input is

$$y(t, t_0) = \frac{1}{2}[x(t - t_0) + x(-t - t_0)]$$

The delayed output is obtained by putting $t = t - t_0$

$$\begin{aligned} y(t - t_0) &= \frac{1}{2}[x(t - t_0) + x(-t + t_0)] \\ y(t, t_0) &\neq y(t - t_0) \end{aligned}$$

Hence, the system is time variant.

3.

$$\begin{aligned} y(t) &= \frac{1}{2}[x(t) + x(-t)] \\ y_1(t) &= \frac{1}{2}[x_1(t) + x_1(-t)] \\ y_2(t) &= \frac{1}{2}[x_2(t) + x_2(-t)] \end{aligned}$$

$$\begin{aligned} y_3(t) &= a_1 y_1(t) + a_2 y_2(t) \\ &= \frac{1}{2}[a_1 x_1(t) + a_1 x_1(-t) + a_2 x_2(t) + a_2 x_2(-t)] \\ y_4(t) &= f[a_1 x_1(t) + a_2 x_2(t)] \\ &= \frac{1}{2}[a_1\{x_1(t) + x_1(-t)\} + a_2\{x_2(t) + x_2(-t)\}] \\ y_3(t) &= y_4(t) \end{aligned}$$

The system is linear.

The system is

(a) Dynamic, (b) Non-causal, (c) Time Variant, and (d) Linear.

Example 1.55 Determine whether the following system is static, time invariant, linear, causal, and stable.

$$3\frac{dy(t)}{dt} + 5t\,y(t) = x(t)$$

Solution

1. The system is described by differential equation. Hence, it is dynamic.
2. In the given differential equation, the coefficient of $y(t)$ is $5t$ which is a function of time t. Hence, the system is time varying.
3. The differential equations of the input a_1x_1 and a_2x_2 are written as follows:

$$3\frac{d}{dt}[a_1y_1(t)] + 5t\,a_1y_1(t) = a_1x_1(t)$$
$$3\frac{d}{dt}[a_2y_2(t)] + 5t\,a_2y_2(t) = a_2x_2(t)$$

Adding the above two equations, we get

$$3\frac{d}{dt}[a_1y_1(t) + a_2y_2(t)] + 5t[a_1y_1(t) + a_2y_2(t)] = a_1x_1(t) + a_2x_2(t)$$
$$3\frac{d}{dt}y_3(t) + 5t\,y_3(t) = a_1x_1(t) + a_2x_2(t)$$

where

$$y_3(t) = a_1y_1(t) + a_2y_2(t)$$

The differential equation for the weighted sum of input is written as

$$3\frac{d}{dt}[a_1y_1(t) + a_2y_2(t)] + 5t[a_1y_1(t) + a_2y_2(t)] = a_1x_1(t) + a_2x_2(t)$$
$$3\frac{d}{dt}y_4(t) + 5t\,y_4(t) = a_1x_1(t) + a_2x_2(t)$$

where

$$y_4(t) = a_1y_1(t) + a_2y_2(t)$$
$$y_3(t) = y_4(t)$$

Hence, the system is linear.

4. From the given differential equation it is obvious that $y(t)$ depends on the present input only. **Hence, the system is causal.**
   The system is described by first-order differential equation with varying coefficient. As long as $x(t)$ is bounded, $y(t)$ is also bounded. If $x(t)$ is an impulse, $y(t)$ exponentially decays and the area under the impulse response curve becomes finite. Hence, the system is stable.

The system is

(a) Dynamic, (b) Time Varying, (c) Linear, (d) Causal, and (e) Stable.

Example 1.56 Check whether the system having the input–output relation

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

is linear and time invariant.

(*Anna University, April 2004*)

Solution

1. $\boldsymbol{y(t) = \int_{-\infty}^{t} x(\tau)d\tau}$

$$a_1 y_1(t) = \int_{-\infty}^{t} a_1 x_1(\tau)d\tau$$
$$a_2 y_2(t) = \int_{-\infty}^{t} a_2 x_2(\tau)d\tau$$

   The weighted sum of the output is

$$\begin{aligned} y_3(t) &= a_1 y_1(t) + a_2 y_2(t) \\ &= \int_{-\infty}^{t} a_1 x_1(\tau)d\tau + a_2 \int_{-\infty}^{t} a_2 x_2(\tau)d\tau \end{aligned}$$

   The output due to the weighted sum of input is

$$y_4(t) = \int_{-\infty}^{t} [a_1 x_1(\tau) + a_2 x_2(\tau)]d\tau$$
$$y_3(t) = y_4(t)$$

   The system is linear.
2. The output due to the input is

$$y(t, t_0) = \int_{-\infty}^{t} x(\tau - t_0)d\tau$$

The delayed output due to the input is

$$y(t - t_0) = \int_{-\infty}^{t} x(\tau - t_0) d\tau$$
$$y(t, t_0) = y(t - t_0)$$

The system is time invariant.

The system is both

(a) Linear and (b) Time Invariant.

Example 1.57 A certain is described by the following input–output equation

$$y(t) = x(t + 1) + x(t^2)$$

Determine whether the system is static, causal, time invariant, linear, and stable.

Solution

1. $\boldsymbol{y(t) = x(t + 1) + x(t^2)}$

$$y(0) = x(1) + x(0)$$

The output depends on the present input $x(0)$ and also the future input $x(1)$. To store the future input it requires memory, and hence it is **dynamic system**. Since the output depends on future input it is non-causal.

2. If the input is delayed by t_0, the output is

$$y(t, t_0) = x(t - t_0 + 1) + x(t^2 - t_0)$$

The delayed output due to the input is

$$y(t - t_0) = x(t - t_0 + 1) + x(t - t_0)^2$$
$$y(t, t_0) \neq y(t - t_0)$$

The system is time variant.

3. The weighted sum of the output due to input is

$$a_1 y_1(t) = a_1[x_1(t + 1) + x_1(t^2)]$$
$$a_2 y_2(t) = a_2[x_2(t + 1) + x_2(t^2)]$$
$$y_3(t) = a_1 y_1(t) + a_2 y_2(t)$$
$$= a_1[x_1(t + 1) + x_1(t^2)] + a_2[x_2(t + 1) + x_2(t^2)]$$

The output due to the weighted sum of input is

$$y_4(t) = a_1[x_1(t+1) + x_1(t^2)] + a_2[x_2(t+1) + x_2(t^2)]$$
$$y_3(t) = y_4(t)$$

The system is linear.

4. If the system, input $x(t)$ is bounded, then the output $y(t)$ is also bounded. Hence, the system is stable.

The system is

(a) Dynamic, (b) Non-causal, (c) Time Variant, (d) Linear, and (e) Stable.

Example 1.58 The input–output relationship of a certain system is given by the following equation:

$$y(t) = x(t-5) - x(2-t)$$

Determine whether the above system is linear and causal.

Solution

1. $\boldsymbol{y(t) = x(t-5) - x(2-t)}$

$$y(t) = x(t-7) - x(2-t)$$

The weighted sum of the output due to the input is given as

$$y_3(t) = a_1 y_1(t) + a_2 y_2(t)$$
$$a_1 y_1(t) = a_1[x_1(t-7) - x_1(2-t)]$$
$$a_2 y_2(t) = a_2[x_2(t-7) - x_2(2-t)]$$
$$y_3(t) = a_1[x_1(t-7) - x_1(2-t)] + a_2[x_2(t-7) - x_2(2-t)]$$

The output due to the weighted sum of input is

$$y_4(t) = a_1[x_1(t-7) - x_1(2-t)] + a_2[x_2(t-7) - x_2(2-t)]$$
$$y_3(t) = y_4(t)$$

The system is linear.

2.

$$y(t) = x(t-7) - x(2-t)$$
$$y(0) = x(-7) - x(2)$$

The output depends on the past input $x(-7)$ and also depends on the future input $x(2)$. **Hence, it is non-causal.**

The system is

(a) Linear and (b) Non-causal.

1.10 Modeling of Mechanical Systems

Mechanical systems are of two kinds. They are

1. Translational system.
2. Rotational system.

In translational system we have three passive components connected and they are mass, spring, and dash-pot. Lever arrangement is connected to change the power level. Force is the input given to the system and linear displacement or velocity is taken as the output. Mass and spring are energy storage elements and dash-pot dissipates energy. The two energy storage elements are analogous to inductor and capacitor in electrical network. In rotational mechanical system, we have three passive components, namely, inertia, spring, and rotational dash-pot. Torque is the input given to such systems and angular velocity or angular acceleration is taken as the output. Gear arrangement is used to change the power level. We give below the notations used to identify the mechanical systems and the variables.

Mechanical Translational System

(a) $M =$ Mass, (kg)
(b) $B =$ Viscous friction coefficient of dash-pot, (N/m/s.)
(c) $K =$ Spring stiffness constant, (N/m)
(d) $f(t) =$ Applied force, (N)
(e) $x(t) =$ Linear displacement, (m)
(f) $v(t) = \frac{dx(t)}{dt}$ = Linear velocity, (m/s)
(g) $a = \frac{dv(t)}{dt}$ = Linear acceleration, (m/s^2)

Mechanical Rotational System

(a) $J =$ Moment of inertia, (kg-m^2)
(b) $B =$ Rotational friction coefficient of dash-pot, (N-m/(rad/s.))
(c) $K =$ Spring stiffness constant, (N-m/rad.)
(d) $T(t) =$ Applied force, (N-m)
(e) $\theta(t)$ = Angular displacement, (rad.)
(f) $\omega(t) = \frac{d\theta(t)}{dt}$ = Angular velocity, (rad/s.)
(g) $\frac{d\omega(t)}{dt}$ = Angular acceleration, (rad/s^2)

1.10.1 Dynamic Equations of Mechanical Translational System

In mechanical translational system, mass M, spring K, and dash-pot B are the three elements connected. Mass stores kinetic energy, spring stores potential energy, and the dash-pot dissipates energy and provides damping to the system. To write the dynamic equation of mechanical system, the free body diagram is made use of. In the free body diagram, various forces acting on a particular element are represented. The sum of the forces acting in one direction is equated to the sum of the forces acting in the opposite direction. The free body diagram of all the elements is written, and then the simultaneous equations so obtained are solved to get the input/output relationship. We give below the free body diagram of mass M, spring K, and dash-pot B.

Mass M

The mechanical system consisting of mass M is shown in Fig. 1.57a. The applied force $f(t)$ acts towards right. The displacement $x(t)$ is in the direction of the applied force. The opposing force developed by mass M is proportional to acceleration $\frac{d^2x(t)}{dt^2}$ and the proportionality constant is M and hence the opposing force $M\frac{d^2x(t)}{dt^2}$ acts in the direction opposite to the direction of the applied force $f(t)$. This is shown in Fig. 1.57b.

From Fig. 1.57b, equating the sum of the forces acting towards the right to the sum of the forces acting towards the left, we get

$$f(t) = M\frac{d^2x(t)}{dt^2}$$

Spring K,

The spring K connected to a reference frame is shown in Fig. 1.58a and its free body diagram is shown in Fig. 1.58b. The opposing force developed by the spring is proportional to the displacement $x(t)$ and the proportionality constant is K. It acts in the direction opposite to the applied force. Equating the right-hand direction force to the left-hand direction force, we get the following equation:

$$f(t) = Kx(t)$$

Dash-pot B

The dash-pot B connected to the reference frame is shown in Fig. 1.59a and the free body diagram is shown in Fig. 1.59b. The opposing force developed by the dash-pot is proportional to the velocity $\frac{dx(t)}{dt}$ and the proportionality constant is B. From Fig. 1.59b the following equation is written:

$$f(t) = B\frac{dx(t)}{dt}$$

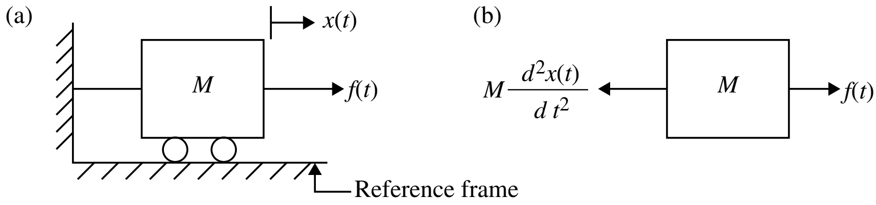


Fig. 1.57 Mechanical system with mass M

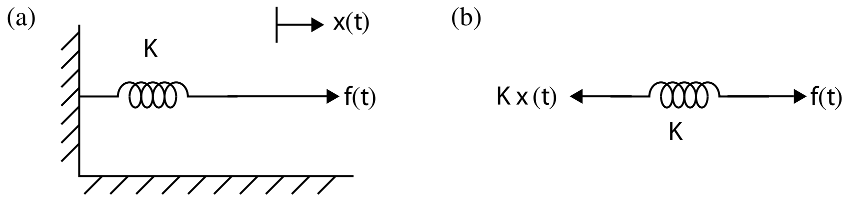


Fig. 1.58 Free body diagram of a spring

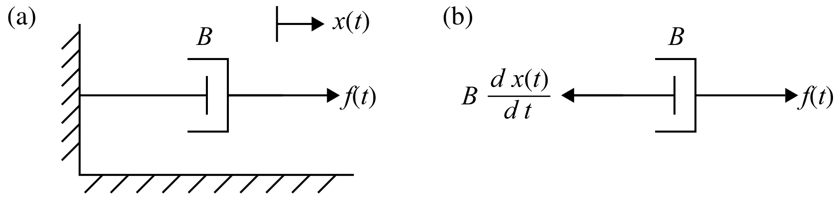


Fig. 1.59 Free body diagram of a dash-pot

While drawing the free body diagram, the following points are to be observed:

1. Each mass is to be given an independent displacement $x(t)$ irrespective of whether one end of it is connected to the reference frame or to the end of any other element. Thus, if there are four masses in a particular mechanical system, the displacement of these masses should be $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$.
2. In the case of the spring and dash-pot, it is necessary to identify the variables with which the two ends of the spring or the dash-pot move. If one end of the spring (say) is connected to the mass M, whose displacement is $x_1(t)$, then that end of the spring moves with a displacement $x_1(t)$. The other end of the spring, if it is connected to the reference frame, its displacement is zero. On the other hand, if it is not connected to the reference frame, then it is necessary to give a new displacement say $x_2(t)$ for that end. Now the opposing force generated by the spring is proportional to the difference in displacement with the proportionality constant K. The same is true for the dash-pot also. This is illustrated in Fig. 1.60a.

In Fig. 1.60a one end of the spring K is connected to the mass M. This end moves with the displacement $x(t)$. Its free body diagram is also shown with various forces acting. Now consider the spring whose one end is connected to the mass M which

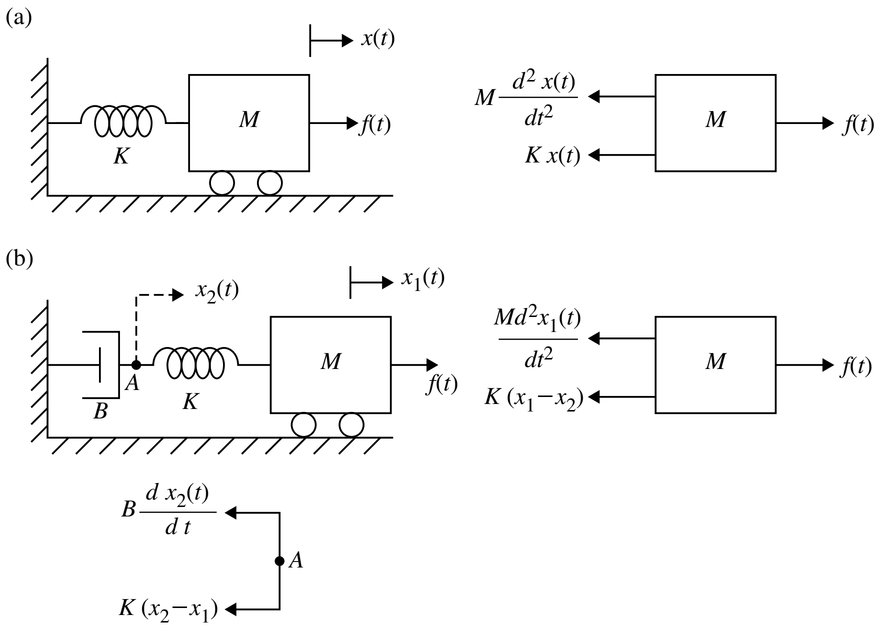


Fig. 1.60 **a** Mechanical system component spring connected to the reference frame; **b** Spring not connected to the reference frame

moves with a displacement of $x_1(t)$. The other end of the spring K is connected to the dash-pot B. This end is neither connected to any mass nor to the reference frame. This end is identified as point A. A new variable $x_2(t)$ is given for the end A. When we write the free body diagram for the mass M, the opposing force due to mass M is $M\frac{d^2x(t)}{dt^2}$. The opposing force generated by the spring is proportional to the final displacement minus initial displacement. Final displacement is the displacement of the point under consideration which is $x_1(t)$ here. The initial displacement is the displacement of the other end of the spring which is denoted as $x_2(t)$.

Now at point A, the opposing forces due to the spring and dash-pot act to the left. At point A, the final displacement is $x_2(t)$. Therefore, the initial displacement for the spring now is $x_1(t)$. For the dash-pot B, the final velocity is $\frac{dx_2(t)}{dt}$. Since its other end is connected to the reference frame, the initial velocity is zero. The opposing forces acting at point A are shown in the free body diagram. For the system represented in Fig. 1.60b following equation is written:

$$M\frac{d^2x(t)}{dt^2} + Kx(t) = f(t)$$

For the system represented in Fig. 1.60b the following equation is written:

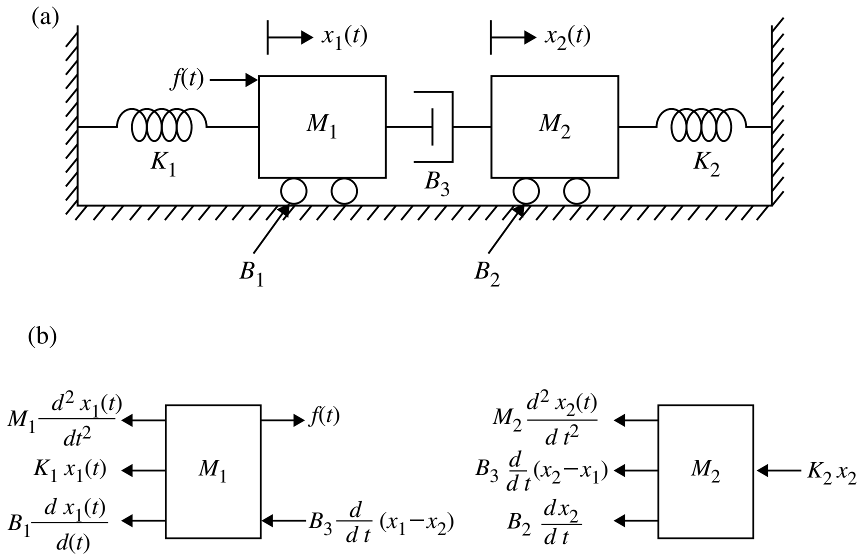


Fig. 1.61 **a** Mechanical system for Example 1.59. **b** Free body diagram for Example 1.59

$$M\frac{d^2x_1(t)}{dt^2} + K(x_1 - x_2) = f(t)$$
$$B\frac{d^2x_2(t)}{dt^2} + K(x_2 - x_1) = 0$$

By taking Laplace transform, one can obtain the transfer function model of the above two mechanical systems. The rotational mechanical system model is obtained exactly by similar approach. The following examples illustrate the method of obtaining transfer function model of mechanical system.

Example 1.59 For the mechanical system shown in Fig. 1.61 obtain the transfer function $\frac{X_1}{F}(s)$.

Solution

1. The displacements of the masses M_1 and M_2 are identified as $x_1(t)$ and $x_2(t)$.
2. The two ends of the elements K_1, K_2, B_1, B_2, and B_3 are either connected to the masses or to the reference frame. This enables us for complete description of the system.
3. The input variable is $f(t)$ and the output variable is $x_1(t)$. By solving the simultaneous equation, the third variable $x_2(t)$ has to be eliminated. By taking Laplace transform, the transfer function $\frac{X_1}{F}(s)$ is obtained.
4. From free body diagram, the following equations are obtained:

$$M_1\frac{d^2x_1(t)}{dt^2} + B_1\frac{dx_1(t)}{dt} + B_3\frac{d}{dt}(x_1 - x_2) + K_1x_1(t) = f(t)$$

Taking Laplace transform on both sides ω we get

$$[M_1s^2 + (B_1 + B_3)s + K_1]X_1(s) - B_3sX_2(s) = F(s)$$
$$M_2\frac{d^2x_2(t)}{dt^2} + B_2\frac{dx_2(t)}{dt} + B_3\frac{d}{dt}(x_2 - x_1) + K_2x_2(t) = 0$$

Taking Laplace transform on both sides, we get

$$[M_2s^2 + (B_2 + B_3)s + K_2]X_2(s) = B_3sX_1(s)$$
$$X_2(s) = \frac{B_3sX_1(s)}{[M_2s^2 + (B_2 + B_3)s + K_2]}$$

Substituting for $X_2(s)$, we obtain

$$[M_1s^2 + (B_1 + B_3)s + K_1]X_1(s) - \frac{B_3^2s^2X_1(s)}{M_2s^2 + (B_2 + B_3)s + K_2} = F(s)$$

$$\begin{aligned}\frac{X_1(s)}{F(s)} =&[M_2s^2 + (B_2 + B_3)s + K_2]\Big/\\ &[M_1M_2s^4 + (M_2(B_1 + B_3) + M_1(B_2 + B_3))s^3\\ &+ (M_2K_1 + B_1B_2 + B_1B_3 + B_2B_3 + +M_1K_2)s^2\\ &+ (B_2 + B_3)K_1 + (B_1 + B_3)K_2)s + K_1K_2]\end{aligned}$$

Example 1.60 Consider the mechanical system shown in Fig. 1.62a. Determine $\frac{X_2}{F}(s)$.

Solution Consider the free body diagram shown in Fig. 1.62b. The following equations are written:

$$M_1\frac{d^2x_1(t)}{dt^2} + B\frac{d}{dt}(x_1 - x_2) + K_1x_1 = f(t)$$

Taking Laplace transform on both sides, we get

$$[M_1s^2 + Bs + K_1]X_1(s) - BsX_2(s) = F(s)$$
$$M_2\frac{d^2x_2(t)}{dt^2} + B\frac{d}{dt}(x_2 - x_1) + K_2x_2 = 0$$
$$(M_2s^2 + Bs + K_2)X_2(s) = BsX_1(s)$$

Substituting for $X_1(s)$, we get

$$\left[(M_1s^2 + Bs + K_1)\frac{(M_2s^2 + Bs + K_2)}{Bs} - Bs\right]X_2(s) = F(s)$$

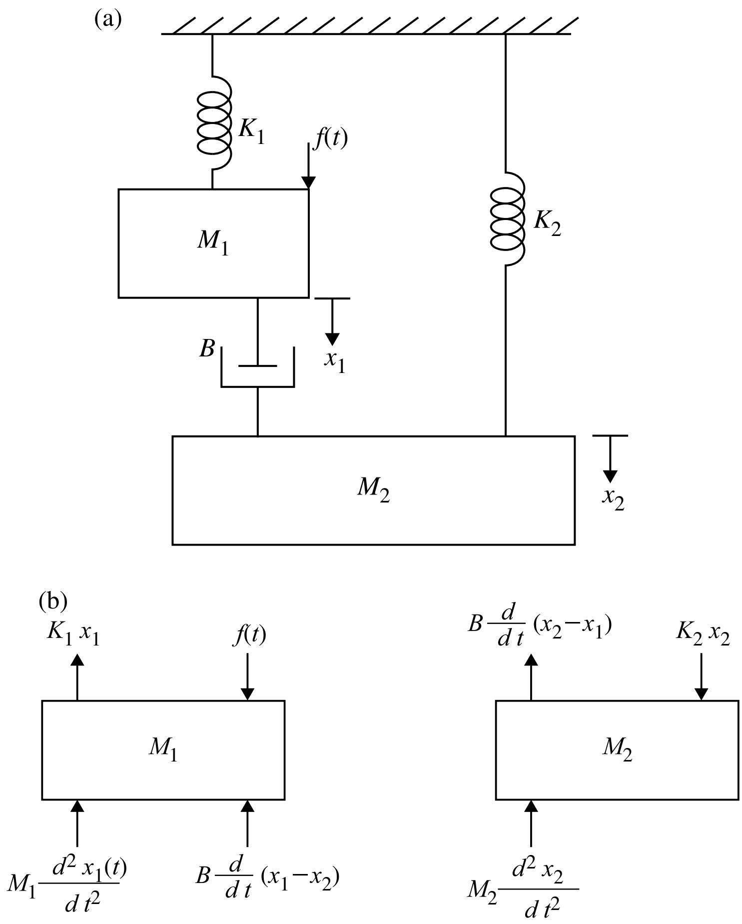


Fig. 1.62 **a** Mechanical system for Example 1.60. **b** Free body diagram for Example 1.60

$$\frac{X_2}{F}(s) = Bs \Big/ [M_1M_2s^4 + (M_1 + M_2)Bs^3 + (M_1K_2 + M_2K_1)s^2 + (K_1 + K_2)Bs + K_1K_2]$$

Example 1.61 For the electromechanical system shown in Fig. 1.63a determine $\frac{X}{E}(s)$. The solenoid parameters are

- Assume back emf effect is negligible.
- Force constant K_s N/amp.
- Coil inductance L Henry.
- Coil resistance R ohms.

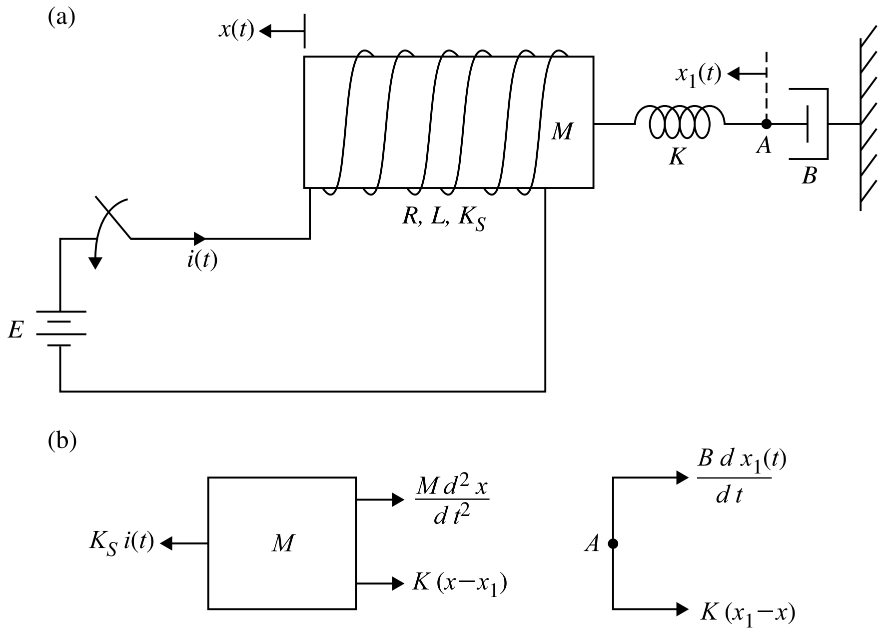


Fig. 1.63 **a** Electromechanical system for Example 1.61. **b** Free body diagram for Example 1.61

Solution The free body diagram for the given mechanical system is shown in Fig. 1.63b and the following equations are written from there.

$$E = L\frac{di}{dt} + Ri$$

Taking Laplace transform on both sides, we get

$$E(s) = (R + Ls)I(s)$$
$$I(s) = \frac{E(s)}{(R + Ls)}$$

The electromechanical force generated by the solenoid is $K_s i(t)$. Thus

$$K_s i(t) = M\frac{d^2x}{dt^2} + K(x - x_1)$$
$$K_s I(s) = (Ms^2 + K)X(s) - KX_1(s)$$

At point A, the following equation is written:

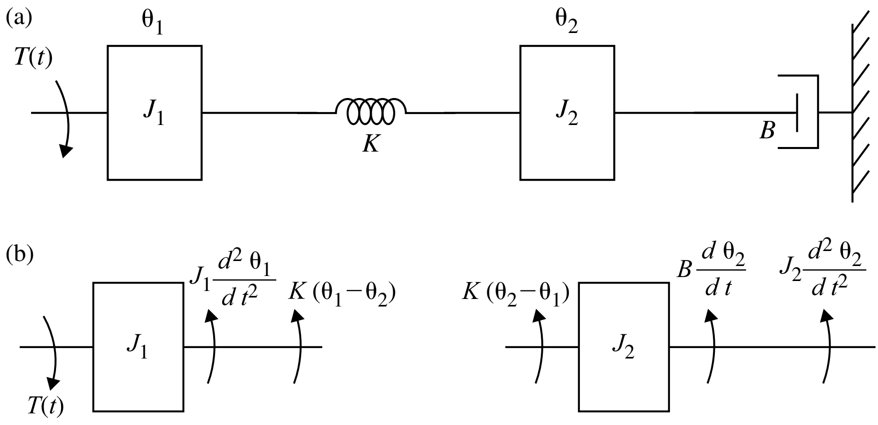


Fig. 1.64 **a** Mechanical Rotational System. **b** Free body diagram for Example 1.62

$$B\frac{d}{dt}(x_1) + K(x_1 - x) = 0$$
$$(Bs + K)X_1(s) = KX(s)$$

Substituting for $X_1(s)$ and $I(s)$, we get

$$\frac{K_s E(s)}{(R + Ls)} = (Ms^2 + K)X(s) - \frac{K}{Bs + K}X(s)$$

$$\frac{X(s)}{E(s)} = \frac{K_s(Bs + K)}{s(R + Ls)(MBs^2 + MKs + BK)}$$

Example 1.62 For the mechanical rotational system shown in Fig. 1.64a derive the T.F. $\frac{\theta_2}{T}(s)$.

Solution The free body diagram for the rotational system is drawn exactly in similar way as was done for translational system. Here each inertia is to be identified with an angular displacement. From the free body diagram the following equations are written:

$$J_1\frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta_2) = T(t)$$

Taking Laplace transform on both sides, we get

$$(J_1 s^2 + K)\theta_1(s) - K\theta_2(s) = T(s)$$
$$J_2\frac{d^2\theta_2}{dt^2} + K(\theta_2 - \theta_1) + B\frac{d\theta_2}{dt} = 0$$

Taking Laplace transform on both sides, we get

$$(J_2 s^2 + Bs + K)\theta_2(s) = K\theta_1(s)$$

Substituting for $\theta_1(s)$, we get

$$\left[(J_1 s^2 + K)\frac{(J_2 s^2 + Bs + K)}{K} - K\right]\theta_2(s) = T(s)$$

Re-arranging the terms and simplifying, we get

$$\frac{\theta_2}{T}(s) = \frac{K}{s(J_1 J_2 s^3 + J_1 B s^2 + K(J_1 + J_2)s + BK)}$$

1.11 Electrical Analogue

An electric circuit which is analogous to a system from another discipline is called electric circuit analogs. Thus, the mechanical systems discussed above can be conveniently converted into its electric circuit equivalent and different variables in the mechanical system can be analyzed in terms of electric circuit variables. Analogs can be obtained by comparing the equations describing the mechanical system with those describing the electric circuit. When the equations of motion of mechanical system are compared with the mesh equations of electric circuit, the analogy is called force–voltage analogy or series analog. Similarly when the equations of motion of mechanical system are compared with the nodal equations of the electric circuit, the analogy is called force–current (torque–current for rotational system) or parallel analog.

1.11.1 Force–Voltage Analogy (F–V analogy)

Consider the mechanical system shown in Fig. 1.65a. For Fig. 1.65a, the following equation is written:

$$M\frac{dv(t)}{dt} + Bv(t) + K\int v(t)dt = f(t) \tag{1.54}$$

where $v(t) = \frac{dx(t)}{dt}$ = velocity of mass M.

Now consider the electric circuit shown in Fig. 1.65b. For this the following equation is written:

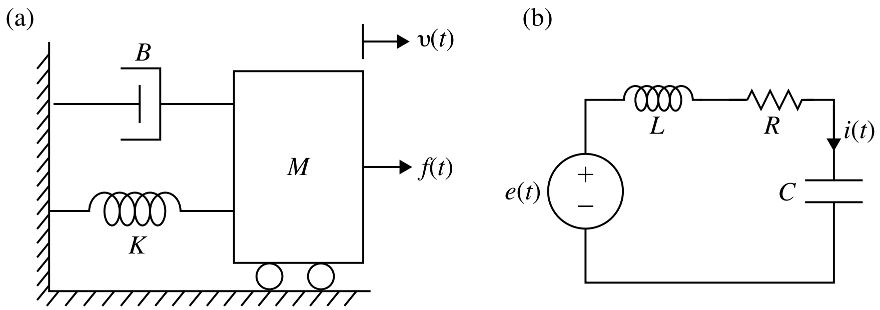


Fig. 1.65 Force–voltage analogy of mechanical system

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)dt = e(t) \tag{1.55}$$

Equations (1.54) and (1.55) are identical and therefore the following analog is derived: Using the above table, the steps given below are followed to obtain the

Mechanical system	Electric circuit (Series analog)
1. Applied force, $f(t)$	Voltage source $e(t)$
2. Velocity $v(t)$	Mesh current $i(t)$
3. Mass M	Inductance L
4. Dash-pot B	Resistance R
5. Spring K	Reciprocal of capacitance C

force–voltage analogous electric circuit (loop or series circuit):

1. For the given mechanical system, each mass is identified with its velocity. This corresponds to a single current flowing through the inductor. For example, if there are five masses in a mechanical system, they move with velocity $v_1(t)$, $v_2(t)$, $v_3(t)$, $v_4(t)$, and $v_5(t)$. Correspondingly, in the electric circuit, there will be five inductances and single current $i_1(t)$, $i_2(t)$, $i_3(t)$, $i_4(t)$, and $i_5(t)$ will flow through these inductances.
2. Identify the source voltages which are equivalent to applied forces.
3. Corresponding to velocity differences applied across the mechanical elements, current differences will flow through the corresponding electrical components which are identified as per the table given above.
4. Thus, by inspection of mechanical system, its electrical analog is drawn.
5. Just to verify whether the analog circuit drawn is correct, write down the equation describing the motion of the given mechanical system. Also write down the mesh equation of the electric circuit drawn. See both are identical.

1.11.2 *Force–Current Analogy (F–I Analogy)*

Consider the electric circuit with the current source $i(t)$, inductance L, resistance R, and capacitance C connected to the node which is at a potential $e(t)$. For Fig. 1.66 the following equation is written:

$$i_1 + i_2 + i_3 = i(t)$$
$$\frac{1}{L}\int e(t)dt + \frac{e(t)}{R} + C\frac{de(t)}{dt} = i(t) \tag{1.56}$$

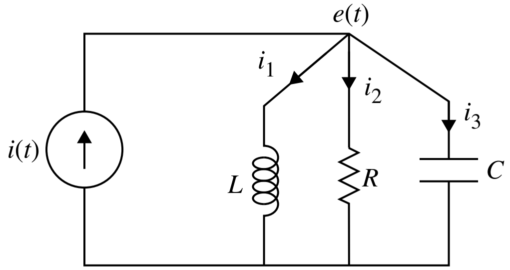
Equations (1.54) and (1.56) are identical and therefore the following analogy is derived.

Mechanical system	Electric circuit (Parallel analog)
1. Applied force $f(t)$	Source current $i(t)$
2. Velocity $v(t)$	Nodal voltage $e(t)$
3. Mass M	Capacitance C
4. Dash-pot B	Reciprocal of resistance R
5. Spring K	Reciprocal of inductance L

Using the above table, the following steps are followed to obtain the force–current analogous electric circuit (nodal or parallel circuit).

1. For the given mechanical system, each mass is identified with its velocity. This is equivalent to a nodal voltage to which one end of the capacitance is connected. There should be as many nodes in the electric circuit as there are masses in mechanical system. Further, in addition, other nodes are created if any of the elements in mechanical system are neither connected to any mass nor to the reference frame.
2. Identify source currents which are equivalent to applied forces.
3. Corresponding to velocity differences applied across the mechanical elements, the electrical components are connected between the two nodes representing these velocity differences.
4. By inspection, the electrical analogous circuit is drawn.

Fig. 1.66 Electric circuit with a current source



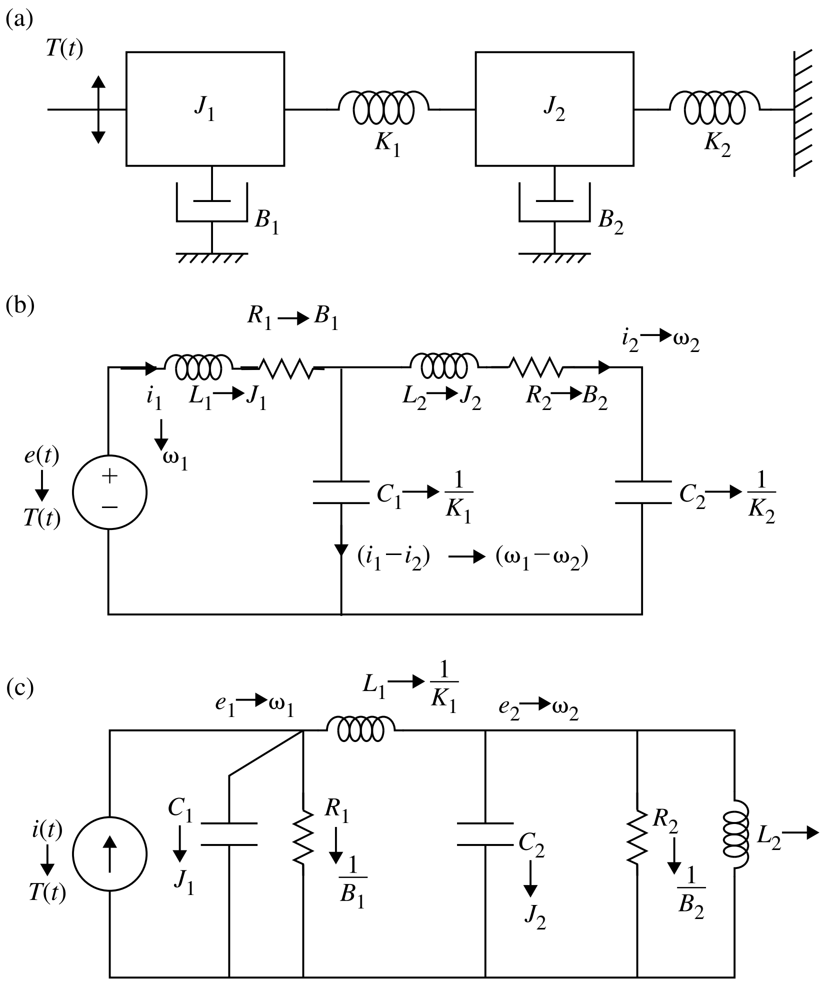


Fig. 1.67 **a** Mechanical notational system. **b** and **c** Electrical analogous circuit for Example 1.63

5. Just by writing the equations of motion of mechanical system and the nodal electric circuit, it is verified they are identical. Thus the analog is verified.

The following examples illustrate the method of obtaining electrical analogous circuits.

Example 1.63 Consider the mechanical rotational system shown in Fig. 1.67. Draw the torque–voltage and torque–current electrical analogous circuits and verify by writing mesh and nodal equations.

(*Anna University, December 2009*)

Solution Torque–Voltage Analogy

1. There are two inertias in the mechanical rotational system. They are given angular velocity ω_1 and ω_2. Corresponding to these displacements single branch currents i_1 and i_2 are opened.
2. To the source voltage $e(t) \to T(t)$, the inductance $L_1 \to J_1$, resistance $R_1 \to B_1$ are connected in series. The current flow is $i_1 \to \omega_1$.
3. In the branch where the current flow is $i_2 \to \omega_2$ the inductance $L_2 \to J_2$, resistance $R_2 \to B_2$, and the capacitance $C_2 \to \frac{1}{K_2}$ are connected in series.
4. A branch current $(i_1 - i_2) \to (\omega_1 - \omega_2)$ is opened and a capacitor $C_1 \to \frac{1}{K_1}$ is connected in this branch since the velocity difference across K_1 is $(\omega_1 - \omega_2)$. The torque–voltage analogous electric circuit is shown in Fig. 1.67b.

Verification

For the mechanical circuit, for the elements connected to the inertia J_1, the following equation is written:

$$J_1 \frac{d\omega_1}{dt} + B_1\omega_1 + K_1 \int (\omega_1 - \omega_2)dt = T(t) \tag{1.57}$$

For the elements connected to the inertia J_2, the following equation is written:

$$J_2 \frac{d\omega_2}{dt} + B_2\omega_2 + K_1 \int \omega_2 dt + K_1 \int (\omega_2 - \omega_1)dt = 0 \tag{1.58}$$

Now consider T–V analogous electric circuit. For the mesh where i_1 current is flowing the following equation is written:

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1} \int (i_1 - i_2)dt = e(t) \tag{1.59}$$

For the mesh where i_2 current is flowing the following equation is written:

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2} \int i_2 dt + \frac{1}{C_2} \int (i_2 - i_1)dt = 0 \tag{1.60}$$

Equation (1.57) is identical to Eq. (1.59) and Eq. (1.58) is identical to Eq. (1.60). Hence the analogous circuit drawn just by inspection of the given mechanical system is correct.

Torque–Current Analogy

1. There are two velocities ω_1 and ω_2 with which the inertias J_1 and J_2 rotate. Corresponding to these angular velocities nodes with e_1 and e_2 voltages are marked.
2. To the e_1 node, current source $i(t) \to T(t)$, capacitance $C_1 \to J_1$, resistance $R_1 \to 1/B_1$ are connected.

3. Inertia J_2, dash-pot B_2 and inductance K_2 all rotate at ω_2. Therefore, $C_2 \rightarrow J_2$, $R_2 \rightarrow 1/B_2$ and $L_2 \rightarrow 1/K_2$ are connected to the e_2 node. Their other ends are connected to the common point.
4. The spring K_1 rotates with a velocity $(\omega_1 - \omega_2)$. Therefore, the inductance $L_1 \rightarrow 1/K_1$ is connected in between e_1 and e_2 nodes. The complete T–I analogous circuit is shown in Fig. 1.67c.

Verification

1. At e_1 node the following equation is written:

$$C_1\frac{de_1}{dt} + \frac{e_1}{R_1} + \frac{1}{L_1}\int(e_1 - e_2)dt = i(t) \tag{1.61}$$

2. At e_2 node the following equation is written:

$$C_2\frac{de_2}{dt} + \frac{e_2}{R_2} + \frac{1}{L_2}\int e_2 dt + \frac{1}{L_1}\int(e_2 - e_1)dt = 0 \tag{1.62}$$

Equations (1.57) and (1.61) are identical. Similarly Eqs. (1.58) and (1.62) are identical. Therefore the T–I diagram shown in Fig. 1.67c is correct.

Example 1.64 Draw the force–voltage and force–current electrical analogous circuits and verify by writing mesh and node equation for the mechanical system shown in Fig. 1.68a.

(*Anna University, December 2009*)

Solution Force–Voltage Analogy

1. Mass M_1 moves with velocity v_1 and mass M_2 moves with velocity v_2. Single currents $i_1 \rightarrow v_1$ and $i_2 \rightarrow v_2$ are opened out. i_1 flows through the series combination of $L_1 \rightarrow M_1$, $R_1 \rightarrow B_1$ and the source voltage $e(t) \rightarrow f(t)$.
2. Mass M_2, dash-pot B_2, and spring K_2 all move with velocity v_2. Hence, the mesh current $i_2 \rightarrow v_2$ flows through the series combination of $L_2 \rightarrow M_2$, $R_2 \rightarrow B_2$, and $C_2 \rightarrow 1/K_2$.
3. The velocity difference across K_1 and B_{12} is $(v_1 - v_2)$. Hence, from mesh currents i_1 and i_2, a branch is created in which the current $(i_1 - i_2)$ flows and the elements $C_1 \rightarrow 1/K_1$ and $R_{12} \rightarrow B_{12}$ are connected in series. This completes the F–V analogous circuit and is shown in Fig. 1.68b.

Verification
For the mechanical system, for mass M_1, the following equation is written:

$$M_1\frac{dv_1}{dt} + B_1 v_1 + B_{12}(v_1 - v_2) + K_1\int(v_1 - v_2)dt = f(t) \tag{1.63}$$

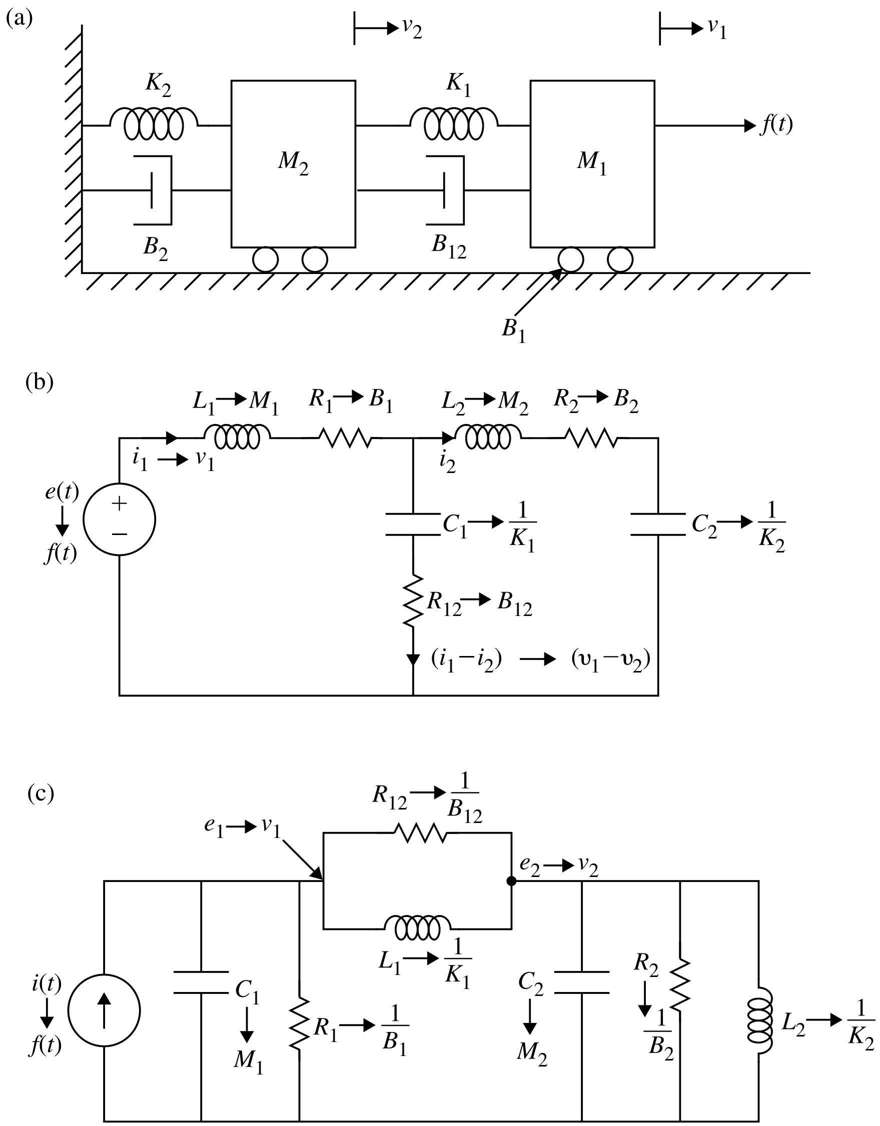


Fig. 1.68 **a** Electrical analogous circuits for Example 1.64. **b** F–V analogy. **c** F–I analogy

For mass M_2, the following equation is written:

$$M_2\frac{dv_2}{dt}+B_2v_2+\ K_2\int v_2dt+B_{12}(v_2-v_1)$$
$$+\ K_1\int(v_2-v_1)dt=0 \tag{1.64}$$

For the F–V electric circuit, the following equation is written for the mesh where i_1 current flows:

$$L_1\frac{di_1}{dt}+R_1i_1+R_{12}(i_1-i_2)+\frac{1}{C_1}\int(i_1-i_2)dt=e(t) \tag{1.65}$$

For the mesh where i_2 current flows, the following equation is written:

$$L_2\frac{di_2}{dt}+R_2i_2+\ \frac{1}{C_2}\int i_2dt+R_{12}(i_2-i_1)$$
$$+\frac{1}{C_1}\int(i_2-i_1)dt=0 \tag{1.66}$$

Equation (1.63) is identical to Eq. (1.65) and Eq. (1.64) is identical to Eq. (1.66). Hence F–V electric circuit drawn just by inspection of the mechanical system is correct. Now let us consider F–I analogous circuit shown in Fig. 1.68c. At node e_1, the following equation is written:

$$C_1\frac{de_1}{dt}+\frac{e_1}{R_1}+\frac{e_1-e_2}{R_{12}}+\frac{1}{L_1}\int(e_1-e_2)dt=i(t) \tag{1.67}$$

At node e_2, the following equation is written:

$$C_2\frac{de_2}{dt}+\frac{1}{R_2}e_2+\frac{1}{L_2}\int e_2dt+\frac{1}{R_{12}}(e_2-e_1)+\frac{1}{L_1}\int(e_2-e_1)dt=0 \tag{1.68}$$

Equations (1.63) and (1.64) are identical to Eqs. (1.67) and (1.68) respectively. Hence the F–I analog circuit represented in Fig. 1.68c is correct.

Example 1.65 Derive the transfer functions of the systems shown in Fig. 1.69 and show these systems are analogous.

(*Anna University, December 2004*)

Solution For the mechanical system, at point A the following equation is written:

$$B\frac{d}{dt}(x_0-x_i)+Kx_0=0$$

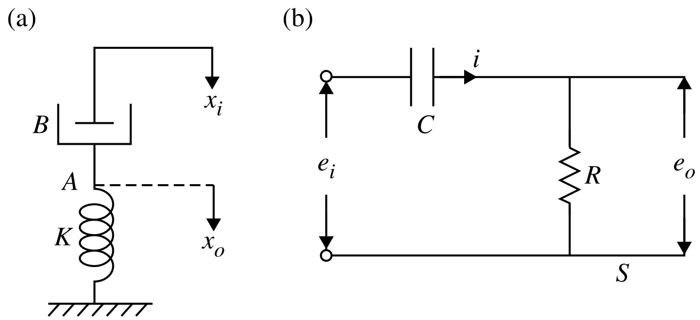


Fig. 1.69 Mechanical system and electric circuit for Example 1.65

Taking Laplace transform on both sides and arranging the like terms, we get the TF as

$$(Bs + K)x_0(s) = BsX_i(s)$$

$$\frac{X_i}{X_0}(s) = \frac{Bs}{Bs + K} \tag{1.69}$$

For the electric circuit, the following equations are written:

$$\frac{1}{C}\int idt + iR = e_0$$
$$iR = e_0$$

Taking Laplace transform on both sides of the above equations and dividing one by the other, we get the TF of the network as

$$\frac{E_0}{E_i}(s) = \frac{R}{(R + \frac{1}{Cs})}$$

$$\frac{E_0}{E_i}(s) = \frac{Rs}{(Rs + \frac{1}{C})} \tag{1.70}$$

Comparison of Eqs. (1.69) and (1.70) shows the electric circuit drawn is F–V analogous circuit. Here E_0 which is proportional to i is again proportional to velocity $v_0 \cdot R \to B$ and $C \to 1/K$. E_i is the source voltage which is equivalent to $x_i \to$ force. Hence the electric circuit shown is analogous to mechanical system.

Example 1.66 Obtain the analogous electrical network for the mechanical system shown in Fig. 1.70a.

(*Anna University, December 2007*)

Solution Force–Voltage Analogy

1. Single currents corresponding to the velocities v_1, v_2, and v_3 are created as i_1, i_2, and i_3 respectively. In the mesh where i_1 current flows, $L_1 \rightarrow M_1$ and $e(t) \rightarrow f(t)$ are connected in series.
2. A branch is created in which the current flow is $(i_1 - i_2) \rightarrow (v_1 - v_2)$. In this branch a resistor $R_1 \rightarrow B_1$ is connected.
3. In the branch where $i_2 \rightarrow v_2$ flows, $L_2 \rightarrow M_2$ is connected.
4. From i_2 and i_3 current branches, a branch with current $(i_2 - i_3) \rightarrow (v_2 - v_3)$ is created. In this branch the series combination of $R_2 \rightarrow B_2$ and $C_1 \rightarrow 1/K_1$ is connected.
5. In the branch where $i_3 \rightarrow v_3$ flows, $L_3 \rightarrow M_3$ and $C_2 \rightarrow 1/K_2$ series combination is connected. This completes the F–V analogous electric circuit.

Verification by equations, F–V Analogy
For the mechanical system, for mass M_1, the following equation is written:

$$M_1 \frac{dv_1}{dt} + B_1(v_1 - v_2) = f(t) \tag{1.71}$$

For mass M_2, the following equation is written:

$$M_2 \frac{dv_2}{dt} + B_2(v_2 - v_3) + B_1(v_2 - v_1) + K_1 \int (v_2 - v_3)dt = 0 \tag{1.72}$$

For mass M_3, the following equation is written:.

$$M_3 \frac{dv_3}{dt} + B_2(v_3 - v_2) + K_1 \int (v_3 - v_2)dt + K_2 \int v_3 dt = 0 \tag{1.73}$$

Now consider the F–V analogous circuit shown in Fig. 1.70b. For the mesh where i_1 current flows, the following equation is written:

$$L_1 \frac{di_1}{dt} + R_1(i_1 - i_2) = e(t) \tag{1.74}$$

For the mesh where i_2 current flows, the following equation is written:

$$L_2 \frac{di_2}{dt} + R_2(i_2 - i_3) + R_1(i_2 - i_1) + \frac{1}{C_1} \int (i_2 - i_3)dt = 0 \tag{1.75}$$

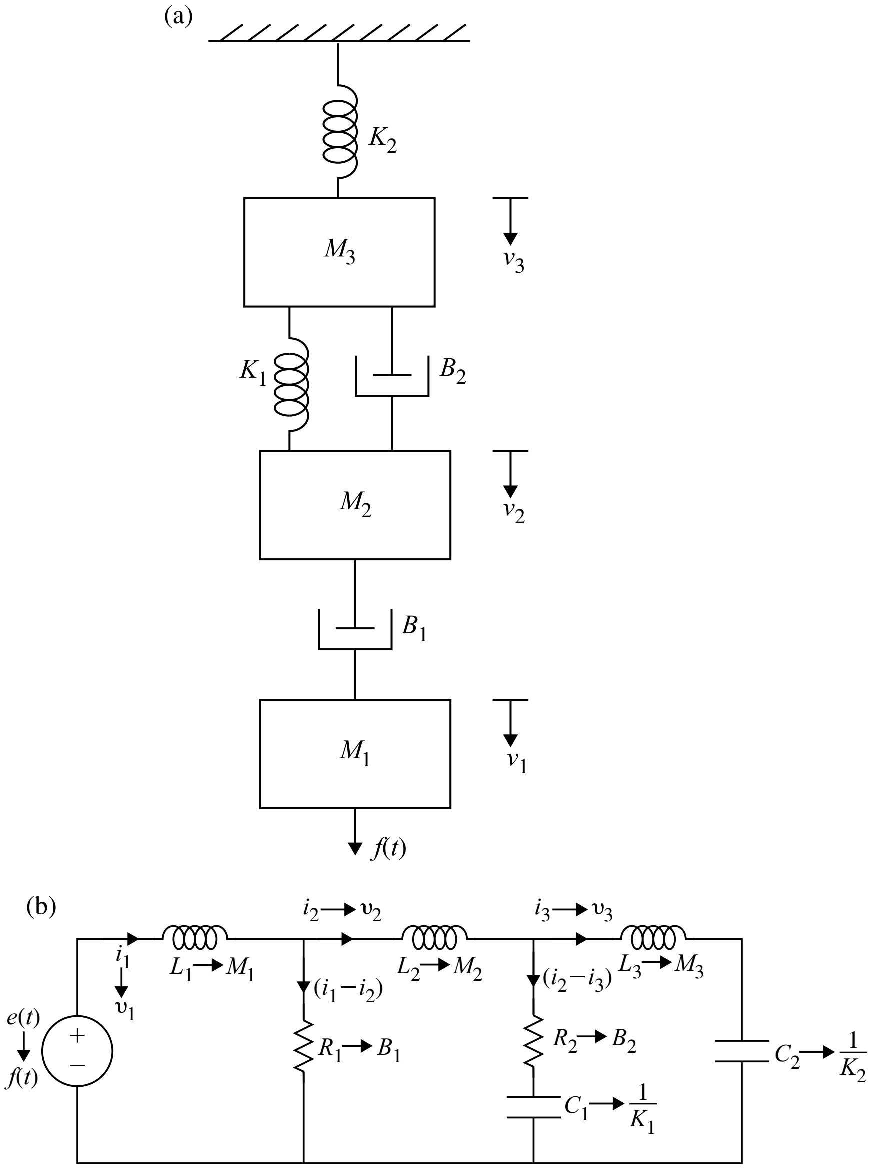


Fig. 1.70 **a** Mechanical System for Example 1.67. **b** F–V analogous circuit for Example 1.66. **c** F–I analogous circuit for Example 1.66

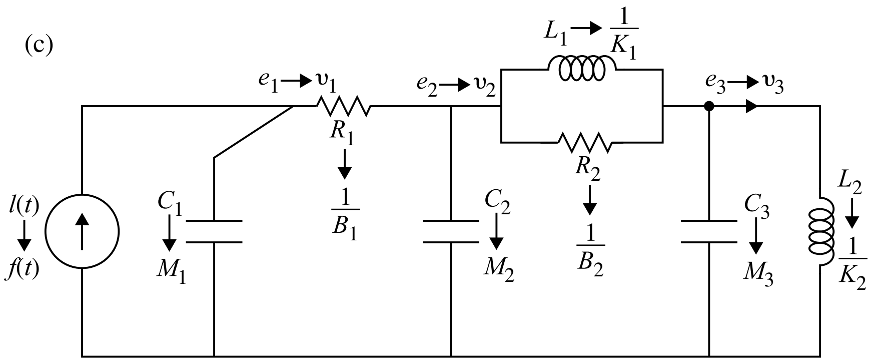


Fig. 1.70 (continued)

For the mesh where i_3 current flows, the following equation is written:

$$L_3\frac{di_3}{dt} + R_2(i_3 - i_2) + \frac{1}{C_1}\int(i_3 - i_2)dt + \frac{1}{C_2}\int i_3 dt = 0 \tag{1.76}$$

Equations (1.71), (1.72), and (1.73) are respectively analogous to Eqs. (1.74), (1.75), and (1.76) and hence the circuit shown in Fig. 1.70b represents the F–V analogy of the given mechanical system.

Verification by equations, F–I Analogy

Now consider the F–I analogous circuit shown in Fig. 1.70c. At node e_1, the following equation is written:

$$C_1\frac{de_1}{dt} + \frac{1}{R_1}(e_1 - e_2) = i(t) \tag{1.77}$$

At node e_2, the following equation is written:

$$C_2\frac{de_2}{dt} + \frac{(e_2 - e_3)}{R_2} + \frac{(e_2 - e_1)}{R_1} + \frac{1}{L_1}\int(v_2 - v_3)dt = 0 \tag{1.78}$$

At node e_3, the following equation is written:

$$C_3\frac{de_3}{dt} + \frac{(e_3 - e_2)}{R_2} + \frac{1}{L_1}\int(e_3 - e_2)dt + \frac{1}{L_2}\int e_3 dt = 0 \tag{1.79}$$

Equations (1.71), (1.72), and (1.73) of the given mechanical system are analogous to the Eqs. (1.77), (1.78), and (1.79) of F–I analogous electric circuit.

1.12 Analysis of First- and Second-Order Linear Systems

When the control system is excited by the input $r(t)$, the output $c(t)$ which is expressed as a function of time t is known as time response of the system. When time t tends to infinity, the output response reaches the steady state. Such a response is called the steady-state response. The dynamic behavior of continuous-time system is described by differential equation. The functional relationship between the output and the input of a linear time invariant system is described by transfer function which is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable with all initial conditions being zero. On the other hand the dynamic behavior of a discrete time system is described by the difference equation. The system function here is represented by means of z-transform whereas for a continuous system it is represented by Laplace transform. The system function of linear continuous-time system as well as discrete-time system is expressed in terms of poles and zeros. The values of s at which the transfer function becomes infinity are called poles of Linear Time Invariant Continuous system (LTIC). The poles are also known as the factors of the denominator polynomial. Similarly the values of s at which the transfer function becomes zero are called zeros of the transfer function of the systems. They are also the factors of the numerator polynomial of the transfer function. On similar line the transfer function and poles and zeros of discrete-time system are defined in the z-plane. For system performance analysis, the transfer function is represented in the form of a block. A complex control system when interconnected by numerous blocks can be ultimately reduced to a single block with the system required output and the input. This is done by what is known as block diagram reduction technique.

1.13 First-Order Continuous-Time System

1.13.1 System Modeling

Consider the following first-order differential equation with input $r(t)$ and output $c(t)$:

$$\frac{dc(t)}{dt} + ac(t) = Kr(t) \tag{1.80}$$

Taking Laplace transform on both sides, we get

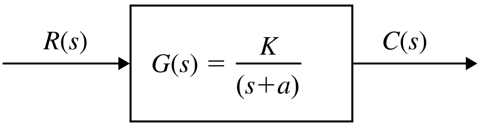


Fig. 1.71 Block diagram representation of first-order system

$$\begin{aligned}(s+a)C(s) &= kR(s)\\ \frac{C(s)}{R(s)} &= G(s)\\ &= \frac{K}{(s+a)}\end{aligned} \tag{1.81}$$

Equation (1.81) gives the transfer function of a first-order system. This is represented in block diagram and is shown in Fig. 1.71.

1.13.2 *Time Response of First-Order System*

1.13.2.1 Impulse Response of First-Order System

The impulse input is defined as

$$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise} \end{cases}$$

Here $r(t) = \delta(t)$

$$R(s) = 1$$

From Eq. (1.81), for impulse input it is written as

$$C(s) = \frac{K}{(s+a)} \tag{1.82}$$

The impulse response of a first-order system is obtained by taking inverse Laplace transform. Thus

$$c(t) = Ke^{-at}.$$

The impulse response curve is plotted and is shown in Fig. 1.72.

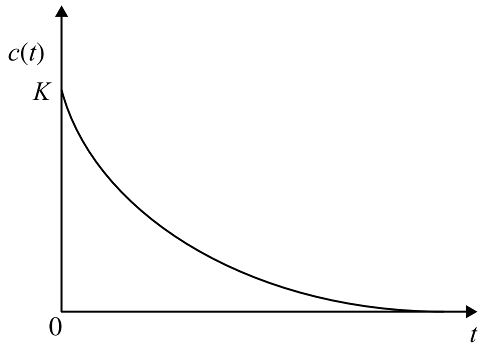


Fig. 1.72 Impulse response of a first-order system

1.13.2.2 Step Response of First-Order System

The unit step input is defined as follows:

$$r(t) = \begin{cases} u(t), & \\ 1, & t \geq 0 \\ 0, & t \leq 0 \end{cases}$$

The Laplace transform of unit step input is

$$R(s) = \frac{1}{s}$$

Substituting the above in Eq. (1.81), we get

$$C(s) = \frac{K}{s(s+a)} \tag{1.83}$$

The above equation is put into partial fraction as given below:

$$\begin{aligned} C(s) &= \frac{A_1}{s} + \frac{A_2}{s+a} \\ &= \frac{K}{a}\left[\frac{1}{s} - \frac{1}{s+a}\right] \end{aligned}$$

Taking inverse Laplace transform, we get

$$c(t) = \frac{K}{a}[1 - e^{-at}] \tag{1.84}$$

The transient response curve of Eq. (1.84) is shown in Fig. 1.73.

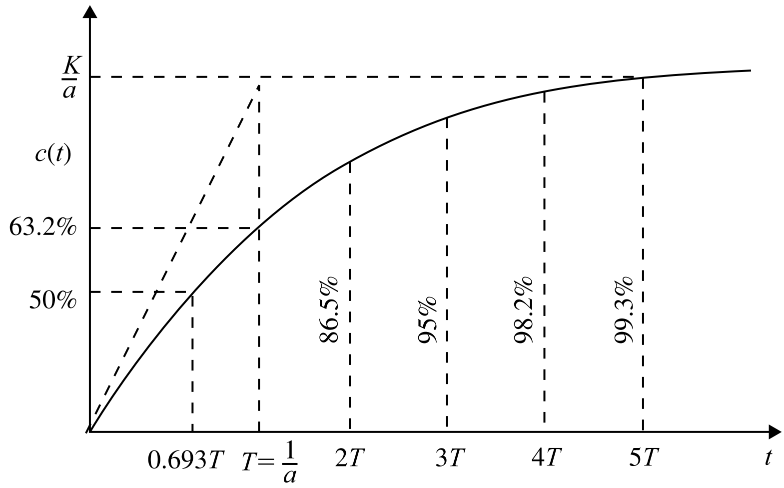


Fig. 1.73 Step response of first-order system

Note: The time response for ramp input $r(t) = t$ can be obtained by integrating Eq. (1.84) and is given below:

$$c(t) = \frac{K}{a^2}\left[-1 + at + e^{-at}\right] \tag{1.85}$$

1.13.3 Time Domain Specifications

The following time domain specifications are defined for a first-order system:

(a) Time constant T.
(b) Rise time t_r.
(c) Settling time t_s.
(d) Time delay t_d.

1.13.3.1 Time Constant

The time constant T is defined as the time taken for the step response to reach 63.2% of its final value for the first time. From Fig. 1.73, the time constant T is obtained as

$$T = \frac{1}{a} \tag{1.86}$$

It is to be noted here that the tangential line of the exponential curve has the slope at $t = 0$ as K. Thus, if the time constant T is small, the response is fast and *vice versa*.

1.13.3.2 Rise Time t_r

The rise time t_r is defined as the time taken for the response to go from 10% to 90% of its final value KT. From Eq. (1.84), at $t = T_1$, let the output be 10% and $t = T_2$ the output be 90%. Thus

$$0.1\frac{K}{a} = \frac{K}{a}[1 - e^{-aT_1}]$$
$$0.9\frac{K}{a} = \frac{K}{a}[1 - e^{-aT_2}]$$

Dividing the second equation by the first equation, we get

$$a = \frac{(1 - e^{-aT_2})}{(1 - e^{-aT_1})}$$

Taking $\log_e$ on both sides, we get

$$2.2 = a(T_2 - T_1)$$

Substituting $t_r = (T_2 - T_1)$, the rise time is obtained as

$$t_r = 2.2T \tag{1.87}$$

1.13.3.3 Time Delay t_d

Time delay t_d is defined as the time taken for the response $c(t)$ to reach 50% of its final value for the first time. From Eq. (1.84), for 50% output, we may write the following equation with $t = t_d$:

$$0.5 = 1 - e^{-\frac{t_d}{T}}$$
$$e^{-\frac{t_d}{T}} = 0.5$$

Taking l_n on both sides, we get

$$t_d = 0.693T \tag{1.88}$$

1.13.3.4 Settling Time t_s

Settling time t_s is defined as the time taken for the response $c(t)$ to reach and stay within 2% of its final value for the first time. From Eq. (1.84) for 98% output, we may write the following equation with $t = t_s$:

$$0.98 = 1 - e^{\frac{-t_s}{T}}$$

Taking l_n on both sides, we get

$$t_s = 3.91T \tag{1.89}$$

For 5% error tolerance

$$t_s = 3T$$

For 7% error tolerance

$$t_s = 2.66T$$

It is to be noted that error tolerance may be given as 5%, 7%, etc., and the corresponding settling time is determined. Unless otherwise the error tolerance is specified it is always taken as 2% error.

Summary of Time Domain Specifications of First-Order System

1.	Time constant	$T = \frac{1}{a}$
2.	Rise time	$t_r = 2.2T$
3.	Time delay	$t_d = 0.693T$
4.	Settling time	$t_s = 3.91T$

Example 1.67 Consider the following T.F. of a certain first-order system.

$$G(s) = \frac{10}{(s + 10)}$$

Derive an expression for the response of the system for $r(t) = 5u(t)$. Find the time constant, settling time, time delay, and rise time.

Solution

$$\frac{C(s)}{R(s)} = G(s) = \frac{10}{(s+10)}$$
$$R(s) = \frac{5}{s}$$
$$C(s) = \frac{50}{s(s+10)} = 5\left[\frac{1}{s} - \frac{1}{s+10}\right]$$

Taking inverse Laplace transform, we get

$$c(t) = 5[1 - e^{-10t}]$$

$$\text{Time constant } T = \frac{1}{10} = 0.1\,\text{s}$$
$$\text{Settling time } t_s = 3.91T = 0.391\,\text{s}$$
$$\text{Rise time } t_r = 2.2T = 0.22\,\text{s}$$
$$\text{Time delay } t_d = 0.693T = 0.0693\,\text{s}$$

$$T = 0.1\,\text{s}$$
$$t_s = 0.391\,\text{s}$$
$$t_r = 0.22\,\text{s}$$
$$t_d = 0.0693\,\text{s}$$

Example 1.68 A glass bulb thermometer reads 98.2% of its final value of temperature 1 min after immersing it in hot water. Determine the time constant, rise time, time delay, and settling time for 5% error tolerance.

Solution The mercury thermometer is a first-order system. From Eq. (1.84), for 98.2% output, the time taken is 60 s.

$$\frac{K}{a}\left[1 - e^{\frac{-60}{T}}\right] = 0.982\frac{K}{a}$$
$$4T = 60\,\text{s}$$
$$T = 15\,\text{s}$$

Rise time

$$t_r = 2.2T$$
$$= 2.2 \times 15$$
$$= 33\,\text{s}$$

Time delay

$$\begin{aligned} t_d &= 0.693T \\ &= 0.693 \times 15 \\ &= 10.4\,\text{s} \end{aligned}$$

For 5% error tolerance, the settling time is

$$\begin{aligned} t_s &= 3T \\ &= 3 \times 15 \\ &= 45\,\text{s} \end{aligned}$$

$$\begin{aligned} T &= 15\,\text{s} \\ t_r &= 33\,\text{s} \\ t_d &= 10.4\,\text{s} \\ t_s &= 45\,\text{s} \end{aligned}$$

1.14 Second-Order System Modeling

For model development of a general second-order system, consider the electrical motor represented in block diagram form as shown in Fig. 1.74a. The system parameters are as follows:

$$\begin{aligned} J &= \text{Moment of inertia of motor not or in Kg.m}^2 \\ B &= \text{Motor frictional coefficient in N-m/rad/s} \\ K &= \text{Error detector constant in N-m/rad error} \end{aligned}$$

Torque developed by the motor is

$$T_d(t) = Ke(t) = K(r(t) - c(t))$$

This torque is to overcome the torque opposed by J and B under no load condition. Thus

$$\begin{aligned} T_u(t) &= J\frac{d^2c(t)}{dt^2} + B\frac{dc(t)}{dt} \\ T_u(t) &= T_d(t) \\ J\frac{d^2c(t)}{dt^2} + B\frac{dc(t)}{dt} &= K(r(t) - c(t)) \end{aligned} \tag{1.90}$$

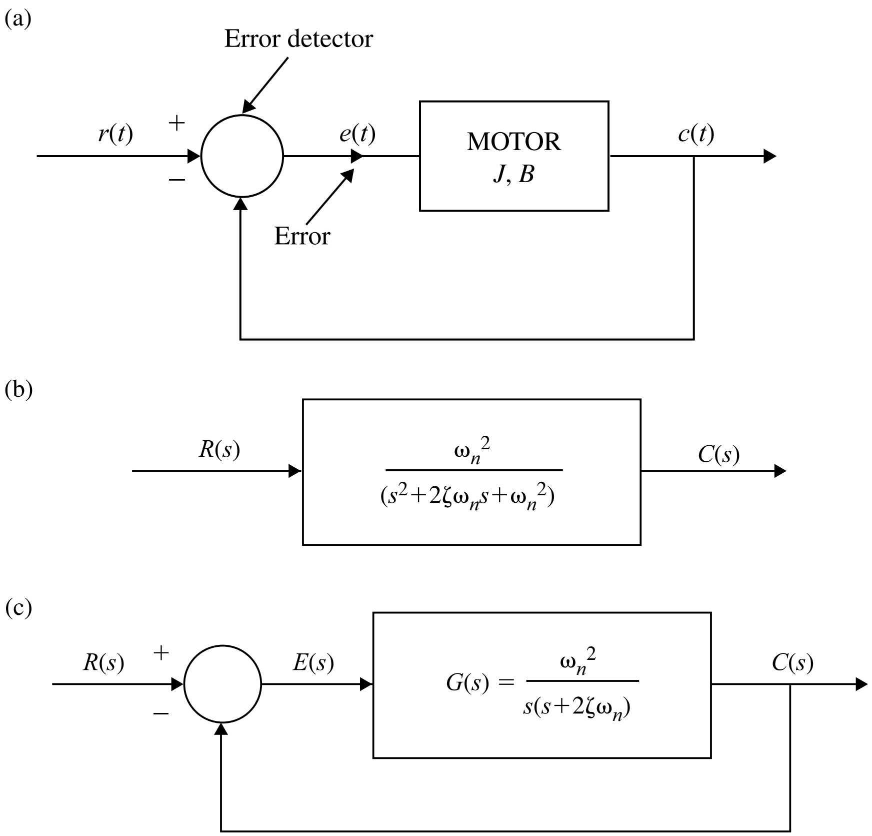


Fig. 1.74 **a** Block diagram representation of a second-order systems. Second-order system representation. **b** open-loop form, **c** closed-loop form

Taking Laplace transform on both sides and expressing the ratio of the output variable to the input variable, we get the following equation:

$$\frac{C(s)}{R(s)} = \frac{K}{(Js^2 + Bs + K)} \tag{1.91}$$

The denominator of Eq. (1.91) is a second-degree polynomial in s and therefore Eq. (1.91) describes the dynamics of a second-order system. Equation (1.91) can be written as

$$\frac{C(s)}{R(s)} = \frac{K/J}{(s^2 + \frac{B}{J}s + \frac{K}{J})} \tag{1.92}$$

Equation (1.92) can be written in a generalized form as given below:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \tag{1.93}$$

where

$$\omega_n = \sqrt{\frac{K}{J}} = \text{Natural frequency of oscillation}$$
$$\zeta = \frac{B}{2\sqrt{KJ}} = \text{Damping factor}$$

Thus, natural frequency of oscillation and damping factor are the two parameters of a generalized second-order system. The systems may be electrical, mechanical, thermal, hydraulic, biological, or in any form. Equation (1.93) is represented in block diagram form as shown in Fig. 1.74b and c which are in open-loop and closed-loop forms respectively. Equation (1.93) is also called standard equation for a second-order system.

The natural frequency ω_n is defined as the frequency of oscillation of a second-order system without damping. If the damping is provided to the system, the system time response contains damped oscillations with exponential decay. Now consider Eq. (1.93) which can be written as follows:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + a_1)(s + a_2)} \tag{1.94}$$

where a_1 and a_2 are the pole locations of Eq. (1.93) and they are expressed in terms of ζ and ω_n. For $0 \leq \zeta \leq 1$.

$$\begin{aligned} a_1 &= -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \\ a_2 &= -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2} \end{aligned} \tag{1.95}$$

In the above case the system is said to be under-damped and the pole locations are as shown in Fig. 1.75.

From Fig. 1.75, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is called damped frequency of oscillation. For $\zeta = 1$, the system is said to be critically damped and the pole locations are shown in Fig. 1.76, which are repeated poles at $s = -\omega_n$.

If $\zeta > 1$, the system is said to be over-damped. In this case, the poles are at

$$\begin{aligned} a_1 &= -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} \\ a_2 &= -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} \end{aligned} \tag{1.96}$$

The pole locations are shown in Fig. 1.77.

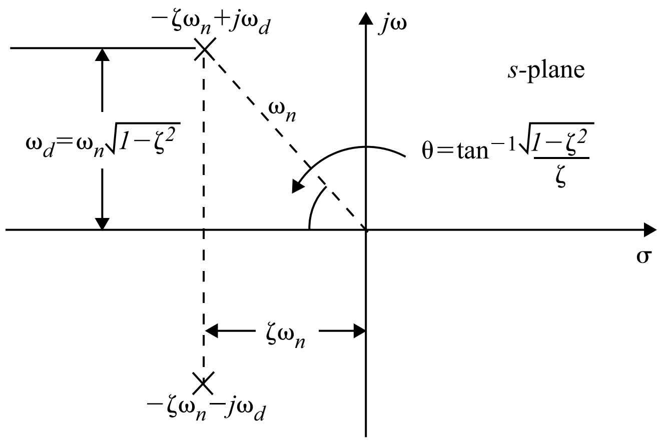


Fig. 1.75 Pole location in the s-plane for complex conjugate poles for $0 \leq \zeta \leq 1$ (under-damped)

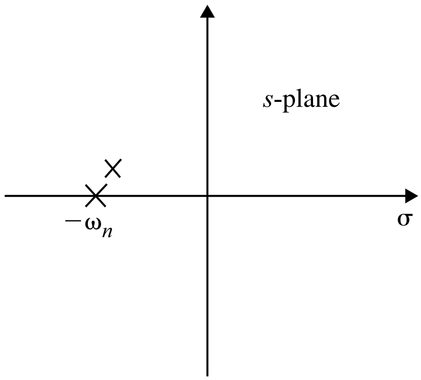


Fig. 1.76 Pole location of a critically damped system ($\zeta = 1$)

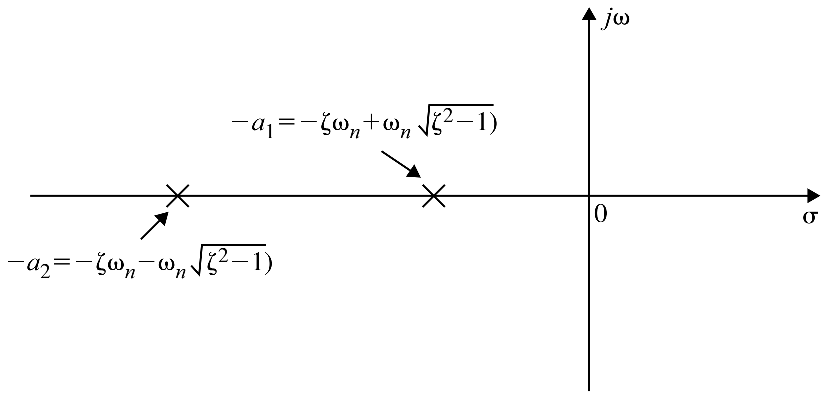


Fig. 1.77 Pole location of an over-damped system ($\zeta > 1$)

1.15 Time Response of a Second-Order System

1.15.1 Impulse Response

Consider Eq. (1.93),

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

For an impulse input,

$$R(s) = 1$$

The above equation is written as

$$C(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \tag{1.97}$$

The impulse response for the following cases are determined:

1. Under-damped case ($\zeta < 1$).
2. Over-damped case ($\zeta > 1$).
3. Critically damped case ($\zeta = 1$).

1.15.1.1 Under-damped Case ($\zeta < 1$)

For $\zeta < 1$, the second-degree denominator polynomial of equation (1.97) is written as follows:

$$C(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

where

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = \text{Damped frequency of oscillation}$$

Putting the above equation into partial fraction, we get

$$\begin{aligned} C(s) &= \frac{A_1}{s + \zeta\omega_n + j\omega_d} + \frac{A_2}{s + \zeta\omega_n - j\omega_d} \\ &= \frac{A_1(s + \zeta\omega_n - j\omega_d) + A_2(s + \zeta\omega_n + j\omega_d)}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \\ \omega_n^2 &= A_1(s + \zeta\omega_n - j\omega_d) + A_2(s + \zeta\omega_n + j\omega_d) \end{aligned}$$

Put

$$\begin{aligned}
s &= -\zeta\omega_n + j\omega_d \\
\omega_n^2 &= A_2(2j\omega_d) \\
A_2 &= \frac{\omega_n}{j2\sqrt{1-\zeta^2}} \\
A_1 &= \text{Conjugate of } A_2 \\
&= \frac{-\omega_n}{j2\sqrt{1-\zeta^2}} \\
C(s) &= \frac{\omega_n}{j2\sqrt{1-\zeta^2}}\left[\frac{-1}{(s+\zeta\omega_n+j\omega_d)} + \frac{1}{(s+\zeta\omega_n-j\omega_d)}\right]\frac{1}{j2\sqrt{5}}
\end{aligned}$$

Taking inverse Laplace transform, we get

$$\begin{aligned}
c(t) &= \frac{\omega_n}{j2\sqrt{1-\zeta^2}}\left[-e^{-(\zeta\omega_n+j\omega_d)t} + e^{-(\zeta\omega_n-j\omega_d)t}\right] \\
&= \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left[\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{2j}\right]
\end{aligned}$$

$$c(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}t \tag{1.98}$$

1.15.1.2 Critically Damped Case ($\zeta = 1$)

Equation (1.97) for $\zeta = 1$ is written as

$$\begin{aligned}
C(s) &= \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} \\
&= \frac{\omega_n^2}{(s+\omega_n)^2}
\end{aligned}$$

Taking inverse Laplace transform, we get

$$c(t) = \omega_n^2 t e^{-\omega_n t} \tag{1.99}$$

1.15.1.3 Over-damped Case ($\zeta > 1$)

Equation (1.97) for $\zeta > 1$ is written as

$$C(s) = \frac{\omega_n^2}{(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})}$$
$$= \frac{A_1}{(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})} + \frac{A_2}{(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})}$$
$$\omega_n^2 = A_1(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}) + A_2(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})$$

Put $s = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$

$$\omega_n^2 = A_1(-2\omega_n\sqrt{\zeta^2-1}); \quad A_1 = \frac{-\omega_n}{2\sqrt{\zeta^2-1}}$$

Put $s = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$

$$\omega_n^2 = A_2(2\omega_n\sqrt{\zeta^2-1}); \quad A_2 = \frac{\omega_n}{2\sqrt{\zeta^2-1}}$$

$$C(s) = \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left[\frac{-1}{(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})} + \frac{1}{(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})}\right]$$

Taking inverse Laplace transform, we get

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2-1}}e^{-\zeta\omega_n t}\left[-e^{-\omega_n\sqrt{\zeta^2-1}t} + e^{\omega_n\sqrt{\zeta^2-1}t}\right] \tag{1.100}$$

The time response curves of Eqs. (1.98), (1.99), and (1.100) are shown in Fig. 1.78.

Example 1.69 Find the unit impulse response of the second-order system whose T.F. is

$$G(s) = \frac{9}{(s^2+4s+9)}$$

(*Anna University, May 2005*)

Solution

$$(s^2+4s+9) = (s+2+j\sqrt{5})(s+2-j\sqrt{5})$$

For an impulse input $R(s) = 1$. Hence

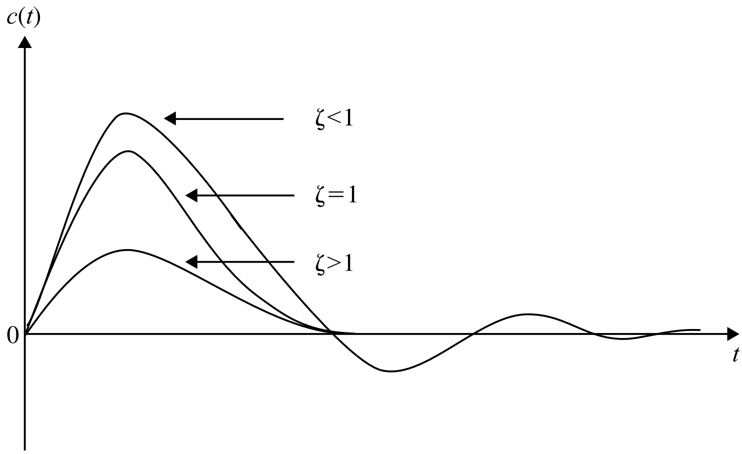


Fig. 1.78 Impulse response curves of a second-order system

$$C(s) = \frac{9}{(s+2+j\sqrt{5})(s+2-j\sqrt{5})} = \frac{A_1}{(s+2+j\sqrt{5})} + \frac{A_2}{(s+2-j\sqrt{5})}$$
$$9 = A_1(s+2-j\sqrt{5}) + A_2(s+2+j\sqrt{5})$$

Put $s = -2 - j\sqrt{5}$

$$9 = A_1(-2 - j\sqrt{5} - j\sqrt{5}) + 0$$
$$A_1 = \frac{-9}{j2\sqrt{5}}$$
$$A_2 = \frac{9}{j2\sqrt{5}}$$
$$C(s) = 9\left[\frac{-1}{(s+2+j\sqrt{5})} + \frac{1}{(s+2-j\sqrt{5})}\right]\frac{1}{j2\sqrt{5}}$$

Taking inverse Laplace transform, we get

$$c(t) = \frac{9}{\sqrt{5}} \frac{[-e^{-(2+j\sqrt{5})t} + e^{-(2-j\sqrt{5})t}]}{2j}$$

$$c(t) = 4e^{-2t} \sin\sqrt{5}t$$

1.15.1.4 Importance of Impulse Response

1. If the area under the impulse response curve is finite, then the system is said to be Bounded Input, Bounded Output (BIBO) stable.
2. From impulse response, by taking inverse Laplace transform the system transfer function is obtained.
3. If impulse response is known, step response can be obtained by integrating it.

1.15.2 Step Response

Step response of a system is important for the following reasons:

1. It is easy to generate step signal and test the system in the laboratory.
2. The step signal is sufficiently drastic and if satisfactory step response is obtained, then the system is likely to give satisfactory performance for other types of inputs.
3. From step response impulse response can be obtained by differentiating it and useful information may be derived. Similarly from step response, ramp response can be obtained by integrating it.
4. The application of step input is equivalent to the application of numerous sinusoidal signals with a wide range of frequencies.

1.15.3 Step Response of a Second-Order System

Consider Eq. (1.83).

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

For a step input of height R, the Laplace transform

$$R(s) = \frac{R}{s}$$

Substituting this in the above equation, we get

$$C(s) = \frac{R\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \tag{1.101}$$

1.15.3.1 Under-damped Response ($\zeta < 1$)

For $\zeta < 1$, Eq. (1.101) is written in the following form:

$$C(s) = \frac{R\omega_n^2}{s(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

$$C(s) = \frac{A_1}{s} + \frac{A_2}{(s + \zeta\omega_n + j\omega_d)} + \frac{A_3}{(s + \zeta\omega_n - j\omega_d)} \tag{1.102a}$$

Analytical Method of Determining the Residues A_1, A_2, and A_3

$$R\omega_n^2 = A_1(s^2 + 2\zeta\omega_n s + \omega_n^2) + A_2 s(s + \zeta\omega_n - j\omega_d) + A_3 s(s + \zeta\omega_n + j\omega_d)$$

Putting $s = 0$ in the above equation, we get

$$R\omega_n^2 = A_1\omega_n^2$$
$$A_1 = R$$

Putting $s = (-\zeta\omega_n - j\omega_d)$, we get

$$R\omega_n^2 = A_2(-\zeta\omega_n - j\omega_d)(-j\omega_d - j\omega_d)$$
$$A_2 = \frac{R}{j2\sqrt{1-\zeta^2}(\zeta + j\sqrt{1-\zeta^2})} = \frac{R\angle -\phi - \frac{\pi}{2}}{2\sqrt{1-\zeta^2}}$$

where $\tan\phi = \frac{\sqrt{1-\zeta^2}}{\zeta}$

$$A_3 = A_2^* = \frac{R\angle\phi + \frac{\pi}{2}}{2\sqrt{1-\zeta^2}}$$

Graphical Method of Determining the Residues

The residues of Eq. (1.102a) can also be determined as explained below. The poles are located along with their residues as shown in Fig. 1.79.

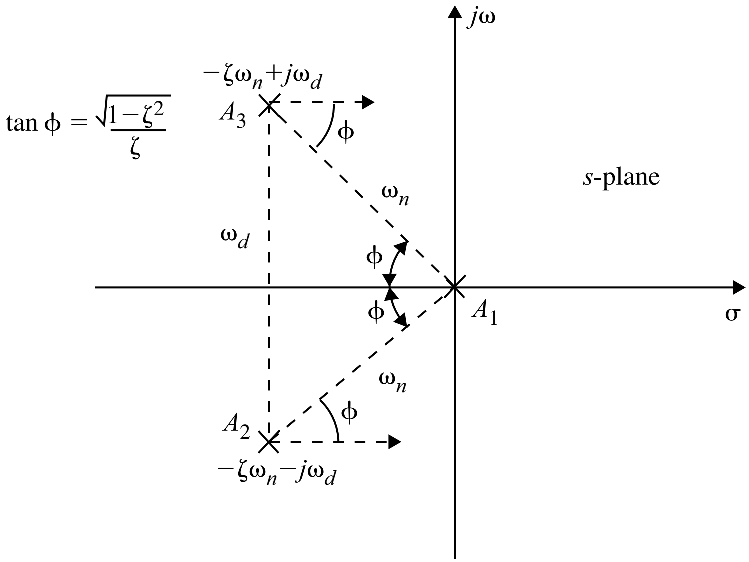


Fig. 1.79 Poles and residue locations of Eq. (1.102a)

$$A_1 = \frac{R\omega_n^2}{\omega_n\angle\phi\omega_n\angle-\phi} = R$$

$$A_3 = \frac{R\omega_n^2}{\omega_n\angle\pi - \phi 2\omega_d\angle\frac{\pi}{2}}$$

$$= \frac{R\angle\frac{\pi}{2}+\phi}{2\sqrt{1-\zeta^2}}$$

$$A_2 = A_3^* = \frac{R\angle-(\frac{\pi}{2}+\phi)}{2\sqrt{1-\zeta^2}}$$

The residues determined by analytical method are the same as obtained by graphical method. However, the graphical method is simpler and quicker. Substituting the above residues in Eq. (1.102a), we get

$$C(s) = R\left[1 + \frac{1}{2\sqrt{1-\zeta^2}}\left\{\frac{e^{-j(\frac{\pi}{2}+\phi)}}{(s+\zeta\omega_n+j\omega_d)} + \frac{e^{j(\frac{\pi}{2}+\phi)}}{(s+\zeta\omega_n-j\omega_d)}\right\}\right]$$

Taking inverse Laplace transform, we get

$$c(t) = R\left[1 + \frac{e^{-\zeta\omega_n t}}{2\sqrt{1-\zeta^2}}\left\{e^{-j(\frac{\pi}{2}+\phi+\omega_d t)} + e^{j(\frac{\pi}{2}+\phi+\omega_d t)}\right\}\right]$$
$$= R\left[1 + \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos\left(\frac{\pi}{2}+\phi+\omega_d t\right)\right]$$

$$c(t) = R\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t + \phi)\right] \tag{1.102}$$

where ϕ is in radians and $\omega_n =$ rad/s.

1.15.3.2 Critically Damped Response ($\zeta = 1$)

For $\zeta = 1$,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s+\omega_n)^2$$

Equation (1.21) can be written as

$$C(s) = \frac{R}{s(s+\omega_n)^2}$$
$$= \frac{A_1}{s} + \frac{A_2}{(s+\omega_n)^2} + \frac{A_3}{(s+\omega_n)}$$
$$R = A_1(s+\omega_n)^2 + A_2 s + A_3 s(s+\omega_n)$$

Put $s = 0$

$$A_1 = \frac{R}{\omega_n^2}$$

Put $s = -\omega_n$

$$A_2 = \frac{-R}{\omega_n}$$

Equating the coefficients of s^2 terms, we get

$$0 = A_1 + A_3$$
$$A_3 = \frac{-R}{\omega_n^2}$$
$$C(s) = \frac{R}{\omega_n^2}\left[\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right]$$

Taking inverse Laplace transform, we get

$$c(t) = \frac{R}{\omega_n^2}\left[1 - t\omega_n e^{-\omega_n t} - e^{-\omega_n t}\right] \tag{1.103}$$

1.15.3.3 Over-damped Response ($\zeta > 1$)

For over-damped case, the time response of a second system for step input can be derived following the method described above. The time response is given below:

$$c(t) = R\left[1 + \frac{e^{-\omega_n(\zeta+\sqrt{\zeta^2-1})t}}{2\{\zeta^2 - 1 + \zeta\sqrt{\zeta^2-1}\}} + \frac{e^{-\omega_n(\zeta-\sqrt{\zeta^2-1})t}}{2\{\zeta^2 - 1 - \zeta\sqrt{\zeta^2-1}\}}\right] \tag{1.104}$$

Using Eqs. (1.103), (1.104), and (1.105), the transient response curves are plotted as shown in Fig. 1.80.

The pole locations in the s-plane and the transient response curves are shown in Fig. 1.81.

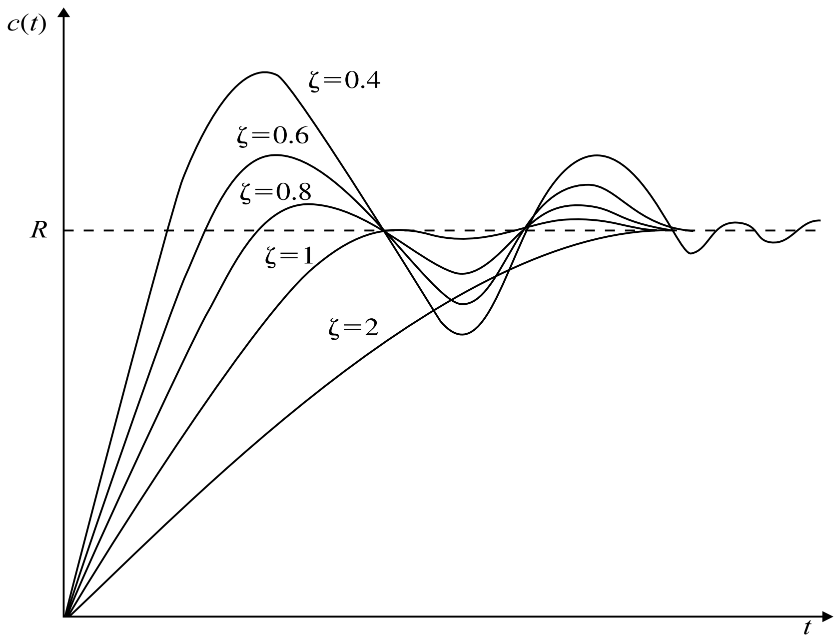


Fig. 1.80 Transient response curves of a second-order system

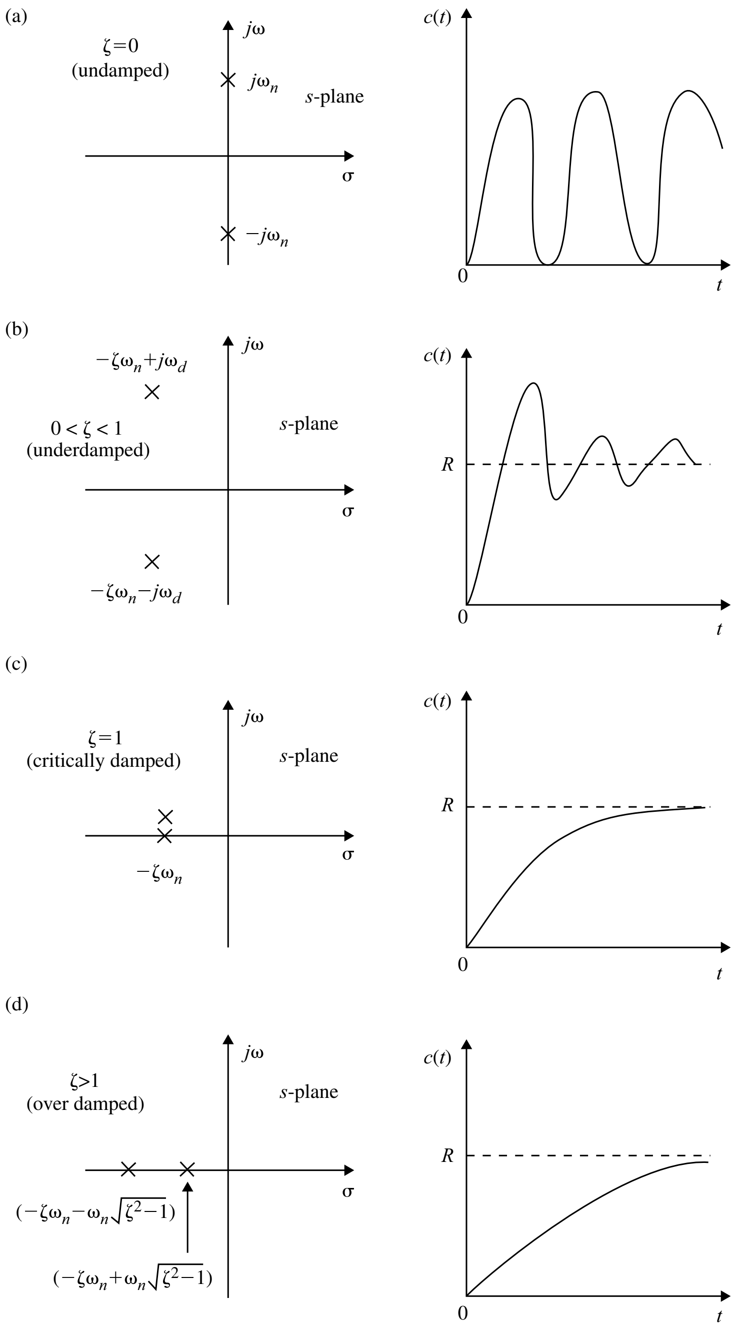


Fig. 1.81 Pole locations and transient response of a second-order system

1.15.4 Time Domain Specifications of a Second-Order System

The performance of a second-order system is measured by the following specifications.

1. Peak over-shoot M_p and % peak over-shoot % M_p.
2. Time at which the peak over-shoot occurs is peak time t_p.
3. Time constant T.
4. Rise time t_r.
5. Settling time t_s.
6. Time delay t_d.

Expressions for the above specifications are derived in terms of the second-order system parameters ζ and ω_n. The transient response curve is shown in Fig. 1.82.

1.15.4.1 Peak Over-Shoot M_p

Peak over-shoot is defined as the amount by which the transient response waveform over-shoots the steady value or the final value.

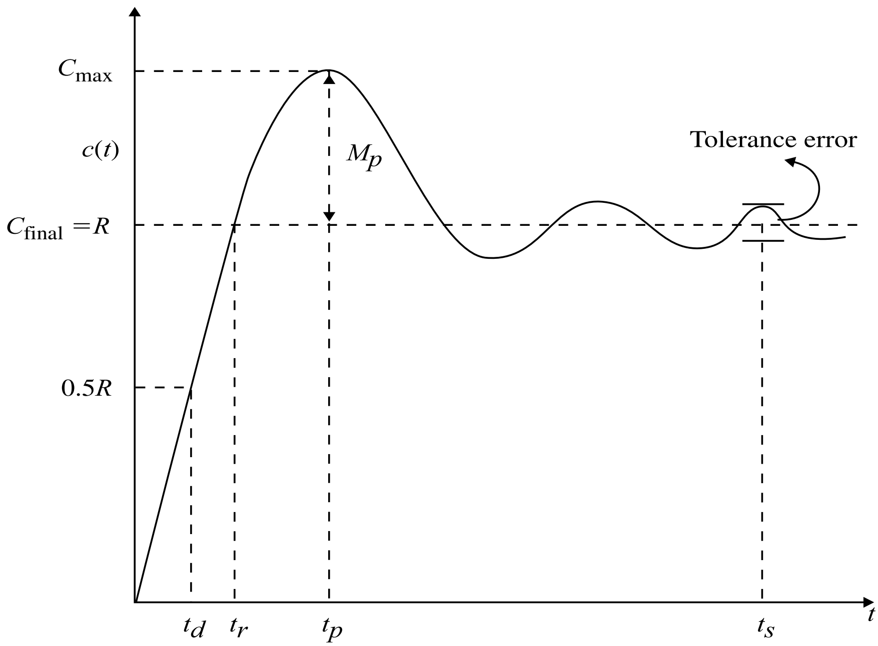


Fig. 1.82 Time domain specifications of a second-order system

The peak over-shoot is denoted by M_P. It is expressed as

$$M_P = C_{\max} - C_{\text{final}}$$

The percentage over-shoot is expressed as

$$\%\ M_P = \frac{(C_{\max} - C_{\text{final}})}{C_{\text{final}}} \times 100$$

Peak over-shoot can be expressed in terms of the system parameters ζ and ω_n. This occurs for the under-damped system response. Consider Eq. (1.103).

$$c(t) = R\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t + \phi)\right]$$

$C_{\max}$ is obtained by differentiating $c(t)$ with respect to t which gives t_p, the time at which the maxima occurs and substituting in Eq. (1.103).

$$\begin{aligned}
\frac{dc(t)}{dt} &= R\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}\right.\\
&\qquad \times \cos(\omega_n\sqrt{1-\zeta^2}t + \phi)\\
&\qquad \left. + \frac{e^{-\zeta\omega_n t}(\zeta\omega_n)\sin(\omega_n\sqrt{1-\zeta^2}t + \phi)}{\sqrt{1-\zeta^2}}\right]\\
&= 0\\
\sqrt{1-\zeta^2}\cos(\omega_n\sqrt{1-\zeta^2}t + \phi) &= \zeta\sin(\omega_n\sqrt{1-\zeta^2}t + \phi)\\
\tan\phi &= \tan(\omega_n\sqrt{1-\zeta^2}t + \phi)\\
\omega_n\sqrt{1-\zeta^2}t &= n\pi \quad \text{where } n = 0, 1, 2, \ldots
\end{aligned}$$

For $n = 1$ maxima occurs and $t = t_p$

$$t_p = \frac{\pi}{\omega_d}$$

$$C_{\max} = R\left[1 - \frac{e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\sin(\pi + \phi)\right] = R\left[1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\right]$$

$$M_P = C_{\max} - C_{\text{final}}$$
$$= R + Re^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} - R$$

$$M_P = Re^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \tag{1.105}$$

$$\% \, M_P = \frac{M_P}{R} \times 100$$

$$\% \, M_P = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 \tag{1.106}$$

1.15.4.2 Peak Time t_p

The peak time t_p is defined as the time at which the first maximum of transient response waveform occurs. It is expressed as

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{1.107}$$

1.15.4.3 Settling Time t_s

The settling time t_s is defined as the time required for the transient response to reach and stay within the prescribed percentage error. The expression for the settling time of a second-order system is derived as follows.

Let

$$\pm \, e^{-m} \times 100 \quad \text{where } m = 1, 2, 3 \ldots \tag{1.108}$$

be the prescribed error within which the system transient response settles down. Equation (1.105) represents the error and can be written as follows:

$$\% \, M_P = e\frac{-\zeta\pi n}{\sqrt{1-\zeta^2}} \times 100 \tag{1.109}$$

where $n = 0, 1, 2, \ldots$. For odd values of n over-shoots occur and for even values of n under-shoots occur. For any error, Eqs. (1.107) and (1.108) are same. Thus,

$$
\begin{aligned}
e^{-m} &= e^{-\frac{\zeta\pi n}{\sqrt{1-\zeta^2}}} \\
n &= \frac{m\sqrt{1-\zeta^2}}{\zeta\pi}
\end{aligned} \tag{1.110}
$$

From Eq. (1.106), for any peak value of the response t_p can be written as follows:

$$
t_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{1.111}
$$

For the given error e^{-m}, $t_p = t_s$. Substituting this in Eq. (1.111) and also for n from Eq. (1.110), we get the following expression for the settling time:

$$
\begin{aligned}
t_s &= \frac{m\sqrt{1-\zeta^2}\pi}{\zeta\pi\omega_n\sqrt{1-\zeta^2}} \\
t_s &= \frac{m}{\zeta\omega_n}
\end{aligned}
$$

$$
t_s = mT \tag{1.112}
$$

In Eq. (1.112), $T = \frac{1}{\zeta\omega_n}$ is called the time constant of the system.

Normally the permissible error prescribed for the settling time is 2%. In this case

$$
\begin{aligned}
e^{-m} &= 0.02 \\
\log_e e^{-m} &= \log_e(0.02) \\
&= -3.91 \\
-m &= -3.91
\end{aligned}
$$

$$
\begin{aligned}
m &= 3.91 \\
t_s &= 3.91T
\end{aligned} \tag{1.113}
$$

For 5% error, $m = 3$ and

$$
t_s = 3T
$$

For 7% error, $m = 2.66$

$$
t_s = 2.66T
$$

1.15.4.4 Time Constant *T*

Time constant is defined as the time taken for the transient response to reach 62.3% of its final value for the first time. From Eq. (1.112) it can be obtained as

$$T = \frac{1}{\zeta \omega_n} \tag{1.114}$$

1.15.4.5 Rise Time t_r

Rise time t_r is defined as the time taken for the transient response to go from 10% to 90% of the final value. Sometimes, the rise time is also defined as the time taken for the transient response to reach the final value for the first time. The expression for the rise time is derived as follows using the second definition stated above. For $c(t) = R$, Eq. (1.102) is written as follows:

$$R = R\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)\right] \tag{1.115}$$

Equation (1.115) gives the solution as

$$\begin{aligned} \sin(\omega_d t + \phi) &= 0 \\ &= \sin n\pi \quad \text{where} \;\; n = 0, 1, 2\ldots \\ \omega_d t + \phi &= n\pi = \pi \quad \text{for } n = 1 \end{aligned}$$

Substituting $t = t_r$ in the above equation, we get

$$t_r = \frac{(\pi - \phi)}{\omega_n\sqrt{1-\zeta^2}} \tag{1.116}$$

In Eq. (1.116) it is to be noted that ϕ is in radians.

1.15.4.6 Time Delay t_d

Time delay is defined as the time taken for the transient response to reach 50% of the final value for the first time. The expression for the time delay is given by the following empirical formula:

$$t_d = \frac{(1 + 0.7\zeta)}{\omega_n} \tag{1.117}$$

Summary of Time Domain Specifications of Second-Order System

1.	Time constant	$T = \frac{1}{\zeta\omega_n}$
2.	Rise time	$t_r = \frac{(\pi-\phi)}{\omega_n\sqrt{1-\zeta^2}}$
3.	Time delay	$t_d = \frac{(1+0.7\zeta)}{\omega_n}$
4.	Settling time	$t_s = mT$
5.	% Peak over-shoot	$\% M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$
6.	Time at which the peak over-shoot occurs	$t_p = \frac{\pi}{\omega_d}$

Example 1.70 Obtain the impulse and step responses of the following unity feedback control system with open-loop transfer function

$$G(s) = \frac{6}{s(s+5)}.$$

(*Anna University, December 2009*)

Solution The given system is represented in block diagram form as shown in Fig. 1.83. From Fig. 1.83, the following equation is derived:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{6}{s^2+5s+6}$$

$$\frac{C(s)}{R(s)} = \frac{6}{(s+2)(s+3)}$$

For unit step input $R(s) = \frac{1}{s}$.

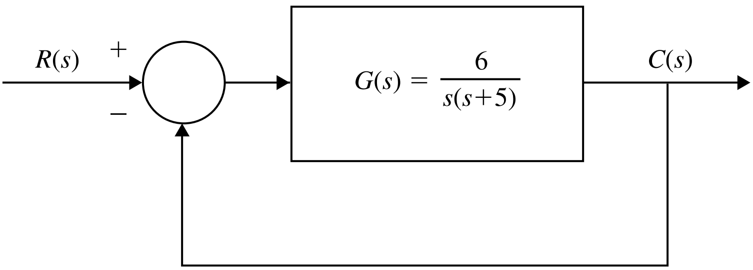


Fig. 1.83 System block diagram for Example 1.70

$$C(s) = \frac{6}{s(s+2)(s+3)} = \frac{A_1}{s} + \frac{A_2}{s+2} + \frac{A_3}{s+3}$$
$$6 = A_1(s+2)(s+3) + A_2 s(s+3) + A_3 s(s+2)$$

For $s = 0$

$$A_1 = 1$$

For $s = -2$

$$A_2 = -3$$

For $s = -3$

$$A_3 = +2$$

$$C(s) = \frac{1}{s} - \frac{3}{s+2} + \frac{2}{s+3}$$

Taking inverse Laplace transform, we get

$$c(t) = 1 - 3e^{-2t} + 2e^{-3t}$$

The impulse response is obtained by differentiating the step response. If we denote impulse response as $h(t)$, then

$$h(t) = \frac{dc(t)}{dt}$$

$$h(t) = 6[e^{-2t} - e^{-3t}]$$

Alternatively, the impulse response is obtained from first principle.

$$\frac{C(s)}{R(s)} = \frac{6}{(s+2)(s+3)}$$

For an impulse $R(s) = 1$

$$C(s) = H(s) = \frac{6}{(s+2)(s+3)}$$
$$= 6\left[\frac{1}{s+2} - \frac{1}{s+3}\right]$$

$$h(t) = 6[e^{-2t} - e^{-3t}]$$

It is to be noted that the poles of the closed-loop transfer function lie on the −ve real axis of the s-plane and hence the system is over-damped.

Example 1.71 A certain unity negative feedback control system has an open-loop transfer function.

$$G(s) = \frac{10}{s(s+2)}$$

Find the rise time, percentage over-shoot, peak time, and settling time for a step input of 12 units.

(*Anna University, December 2009*)

Solution

$$G(s) = \frac{10}{s(s+2)}$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{10}{s^2+2s+10}$$

Comparing the above equation with the following standard second-order equation.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\omega_n = \sqrt{10}$$

$$2\zeta\omega_n = 2$$

$$\zeta = 0.3162$$

Given $R = 12$ units. Using Eq. (1.105),

$$\text{over-shoot } M_p = \text{Re}\frac{-\zeta\pi}{\sqrt{1-\zeta^2}} = 12e^{\frac{-0.3162\pi}{\sqrt{1-0.3162^2}}}$$

$$M_p = 4.2 \text{ units}$$

$$\% \text{ over-shoot} = \frac{M_p}{R} \times 100$$

$$= \frac{4.2}{12} \times 100$$

$$\% M_p = 35\%$$

Using Eq. (1.111), the peak time is obtained.

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\sqrt{10}\sqrt{(1-0.3162^2)}}$$

$$t_p = 1.05\,\text{s}$$

The rise time is obtained using Eq. (1.116)

$$\begin{aligned}
t_r &= \frac{(\pi - \phi)}{\omega_n\sqrt{1-\zeta^2}} \\
\omega_n\sqrt{1-\zeta^2} &= \sqrt{10}\sqrt{1-0.3162^2} \\
&= 3 \\
\cos\phi &= \zeta \\
\phi &= \cos^{-1} 0.3162 \\
&= 1.25\,\text{rad} \\
t_r &= \frac{(\pi - 1.25)}{3}
\end{aligned}$$

$$t_r = 0.63\,\text{s}$$

The settling time is obtained using Eq. (1.113). Here 2% error tolerance is taken. Hence, $m = 3.91$.

$$\text{Time constant, } T = \frac{1}{\zeta\omega_n} = \frac{1}{0.3162\sqrt{10}}$$

$$T = 1\,\text{s}$$

$$t_s = mT$$

$$t_s = 3.91\,\text{s}$$

Answers:

$$M_p = 4.2\,\text{units}$$
$$\%\ M_p = 35\%$$
$$t_p = 1.05\,\text{s}$$
$$t_r = 0.63\,\text{s}$$
$$T = 1\,\text{s}$$
$$t_s = 3.91\,\text{s}$$

Example 1.72 Figure 1.84 shows a unity feedback system. Calculate ζ and ω_n when $K = 0$. Also calculate K when $\zeta = 0.6$.

(*Anna University, December 2009*)

Solution The given block diagram is reduced and represented as shown in Fig. 1.85. From Fig. 1.85, the following equation is written:

$$\frac{C(s)}{R(s)} = \frac{64}{s^2 + (K+4)s + 64}$$

Compare this with the following standard second order by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

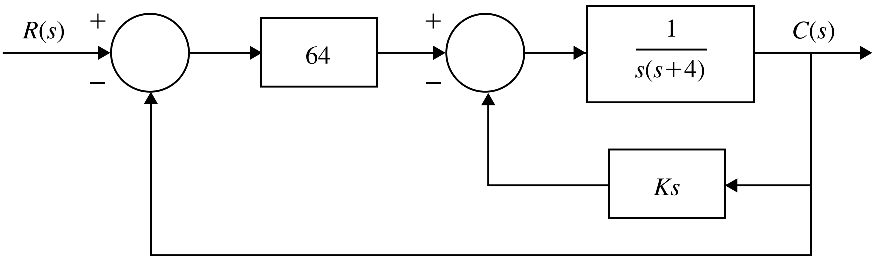


Fig. 1.84 Block diagram of Example 1.72

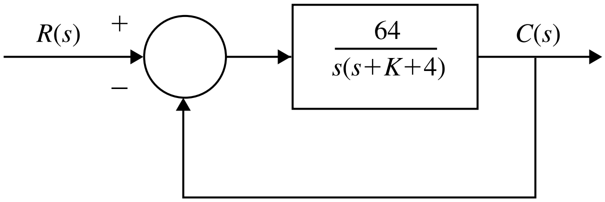


Fig. 1.85 Reduced block diagram of Fig. 1.84

For $K = 0$;

$$\omega_n = \sqrt{64} = 8$$
$$2\zeta\omega_n = 4$$
$$\zeta = 0.25$$

For $\zeta = 0.6$;

$$2\zeta\omega_n = K + 4$$
$$K = 2 \times 0.6 \times 8 - 4$$
$$K = 5.6$$

Answers:

$$\text{For } K = 0; \quad \zeta = 0.25$$
$$\omega_n = 8$$
$$\text{For } \zeta = 0.6; \quad K = 5.6$$

Summary

1. Signals are broadly classified as continuous-time (CT) and discrete-time (DT) signals. They are further classified as deterministic and stochastic, periodic and non-periodic, odd and even, and energy and power signals.
2. Basic CT signal includes impulse, step, ramp, parabolic, rectangular pulse, triangular pulse, signum function, sinc function, sinusoid, and real and complex exponentials.
3. Basic operations on CT signals include addition, multiplication, amplitude scaling, time scaling, time shifting, reflection or folding, and amplitude inverted signals.
4. In time shifting of CT signal, for $x(t + t_0)$ and $x(-t - t_0)$ the time shift is made to the left of $x(t)$ and $x(-t)$ respectively by t_0. For $x(t - t_0)$ and $x(-t + t_0)$ the time shift is made to the right of the $x(t)$ and $x(-t)$ respectively by t_0.
5. To plot CT signal, the operation performed is in the following sequence. The signal is folded (if necessary), time shifted, time scaled, amplitude scaled and inverted.
6. Signals are classified as even signals and odd signals. Even signals are **symmetric** about the vertical axis whereas odd signals are **anti-symmetric** about the time origin. Odd signals pass through the origin. The product of two even signals or two odd signals is an even signal. The product of an even and an odd signal is an odd signal.
7. A CT signal which repeats itself for every T seconds or a DT signal for every N sequence is called a periodic signal. If the signal is not periodic it is called an

aperiodic or non-periodic signal. The necessary condition for the composite of two or more signals to be periodic is that the individual signal should be periodic.
8. A signal is an energy signal iff the total energy of the signal satisfies the condition $0 < E < \infty$. A signal is called a power signal iff the average power of the signal satisfies the condition $0 < P < \infty$. If the energy of a signal is finite, the average power is zero. If the power of the signal is finite, the signal has infinite energy. All periodic signals are power signals. However, all power signals need not be periodic. Signals which are deterministic and non-periodic are usually energy signals. Some signals are neither energy signal nor power signal.
9. The system is broadly classified as continuous- and discrete-time system.
10. The CT and DT systems are further classified based on the property of causality, linearity, time invariancy, invertibility, memory, and stability.
11. The definitions of the above properties are given which are same for both CT and DT systems. Illustrative examples are given to explain these properties.
12. For the first-order and second-order system, the transfer functions are derived and their impulse and step responses are determined.
13. From step responses of first- and second-order systems, the time domain specifications are defined and analytical expressions for these specifications are derived in terms of system parameters.
14. Poles and zeros of continuous- and discrete-time systems are defined and they are located in the complex s-plane and z-plane respectively.

Exercise

I. Short Answer Type Questions

1. **How are signals classified?**
   Signals are generally classified as CT and DT signals. They are further classified as deterministic and non-deterministic, odd and even, periodic and non-periodic, and power and energy signals.
2. **What are odd and even signals?**
   A continuous CT signal is said to be an even signal if it satisfies the condition $x(-t) = x(t)$ for all t. It is said to be an odd signal if $x(-t) = -x(t)$ for all t.
3. **How even and odd components of a signal are mathematically expressed for CT and DT signals?**

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$
$$x_0(t) = \frac{1}{2}[x(t) - x(-t)]$$

4. **What are periodic and non-periodic signals?**
A continuous-time signal is said to be a periodic signal if it repeats itself for every T sec. It satisfies the condition $x(t) = x(t + T)$ for all t. A signal which is not periodic is said to be non-periodic.
5. **What is the fundamental period of a periodic signal? What is fundamental frequency?**
A CT signal is said to be periodic if it satisfies the condition $x(t) = x(t + T)$. If this condition is satisfied for $T = T_0$, it is also satisfied for $T = 2T_0,\ 3T_0, \ldots$. The smallest value of T that satisfies the above condition is called fundamental period. The fundamental frequency $f_0 = \frac{1}{T_0}$ Hz. It is also expressed as $\omega_0 = \frac{2\pi}{T_0}$ rad/s.
6. **What are power and energy signals?**
For a CT signal, the total energy is defined as

$$E = \underset{T\to\infty}{Lt} \int_{-T}^{T} |x(t)|^2 dt$$

and the average power is defined as

$$P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

The square root of P is called root mean square (RMS) value of $x(t)$.
7. **What is the condition that the signal $x(t) = e^{at}u(t)$ to be energy signal?**
For the signal $x(t) = e^{at}u(t)$ to be energy signal $a < 0$.
8. **Is the unit step signal an energy signal?**
The unit step has an average power $P = \frac{1}{2}$. It is a power signal.
9. **Determine the power and RMS value of the signal $x(t) = e^{jat}\cos\omega_0 t$?**
Average power $P = \frac{1}{2}$ and RMS power $P_{\text{RMS}} = \frac{1}{\sqrt{2}}$.
10. **What is the periodicity of $x(t) = e^{j100\pi t + 30^\circ}$?**
The periodicity of the signal $x(t)$ is $T = \frac{1}{50}$ s.
11. **Find the equivalence of the following functions (a) $\delta(at)$; (b) $\delta(-t)$; (c) $t\delta(t)$; (d) $\sin t\delta(t)$; (e) $\cos\delta(t)$; (f) $x(t)\delta(t - t_0)$?**

$$\begin{aligned}
&\text{(a)} && \delta(at) = \frac{1}{a}\delta(t)\\
&\text{(b)} && \delta(-t) = \delta(t)\\
&\text{(c)} && t\delta(t) = 0\\
&\text{(d)} && \sin t\delta(t) = 0\\
&\text{(e)} && \cos t\delta(t) = \delta(t)\\
&\text{(f)} && x(t)\delta(t - t_0) = x(t_0)
\end{aligned}$$

12. **How do you represent an everlasting exponential e^{-at} for $t \geq 0$ and $t < 0$?**
The everlasting exponential e^{-at} is expressed as $e^{-at}u(t)$ for $t \geq 0$ and $e^{-at}u(-t)$ for $t < 0$.
13. **Find the value of $\frac{t^2+5}{t^2+6}\delta(t-2)$.**

$$\frac{(t^2+5)}{(t^2+6)}\delta(t-2) = 0.9\,\delta(t-2)$$

14. **Find the odd and even components of e^{j2t}.**

$$x_e(t) = \cos 2t$$
$$x_0(t) = \sin 2t$$

15. **What are the properties of systems?**
Systems are generally classified as continuous- and discrete-time systems. Further classifications of these systems are done based on their properties which include (a) linear and non-linear, (b) time invariant and time variant, (c) static and dynamic, (d) causal and non-causal, (e) stable and unstable and (f) Invertible and non-invertible.
16. **Define system. What is linear system?**
A system is defined as the interconnection of objects with a definite relationship between objects and attributes.
A system is said to be linear if the weighted sum of several inputs produce weighted sum of outputs. In other words, the system should satisfy the homogeneity and additivity of super position theorem if it is to be linear. Otherwise it is a non-linear system.
17. **What is time invariant and time varying system?**
A system is said to be time invariant if the output due to the delayed input is same as the delayed output due to the input. If the continuous-time system is described by the differential equation its coefficients should be time independent for the system to be time invariant. In the case of discrete-time system, the coefficients of the difference equation describing the system should be time independent (constant) for the system to be time invariants. If the above conditions are not satisfied the system (CT as well as DT) is said to be time variant.
18. **What are static and dynamic systems?**
If the output of the system depends only on the present input, the system is said to be static or instantaneous. If the output of the system depends on the past and future input, the system is not static and it is called dynamic system. Static system does not require memory and so it is called memoryless system. Dynamic system requires memory, and hence it is called system with memory. Systems which are described by differential and difference equations are dynamic systems.

19. **What are causal and non-causal systems?**
If the system output depends on present and on past inputs, it is called causal system. If the system output depends on future input it is called non-causal system.
20. **What are stable and unstable systems?**
If the input is bounded and output is also bounded, the system is called BIBO stable system. If the input is bounded and the output is unbounded, the system is unstable. System whose impulse response curve has finite area is also called stable systems.
21. **What are invertible and non-invertible systems?**
A system is said to be invertible if the distinct inputs give distinct outputs.
22. **State the condition for a discrete-time LTI system to be causal and stable.** (*Anna University, 2008*)
A discrete-time LTI system is said to be causal and stable if the poles of the transfer function all lie in the left half s-plane and the Region of Convergence (ROC) is to the right of the rightmost pole.
23. **Check whether the system having the input–output relation**

$$\boldsymbol{y(t) = \int_{-\infty}^{t} x(\tau)d\tau}$$

is linear and time invariant. (*Anna University, 2004*)
The system is linear. (*See* Example 2.2(f)) By differentiating the above equation, we get

$$\frac{dy(t)}{dt} = x(t)$$

The coefficient of the differential equation is time independent and is constant. Hence, it is a time invariant system.
24. **Check whether the system classified by**

$$\boldsymbol{y(y) = e^{x(t)}}$$

is time invariant or not? (*Anna University, 2007)*

$$\begin{aligned} y(t, t_0) &= e^{x(t-t_0)} \\ y(t-t_0) &= e^{x(t-t_0)} \\ y(t, t_0) &= y(t-t_0) \end{aligned}$$

The system is time invariant.

25. **Determine whether the system described by the following input–output relationship is linear and causal.**

$$\boldsymbol{y(t) = x(-t)}$$

(*Anna University, 2007*)

The system is linear and non-causal.

26. **Is the system $\boldsymbol{y(t) = \cos t\, x(t-5)}$ time invariant?**

$$\begin{aligned} y(t, t_0) &= \cos t x(t - t_0 - 5) \\ y(t - t_0) &= \cos(t - t_0)x(t - t_0 - 5) \\ y(t, t_0) &\neq y(t - t_0) \end{aligned}$$

The system is not time invariant.

27. **What do you understand by transient response of a system?**
The study of different variables in the system as a function of time when the input/disturbance is applied is called transient response of the system.

28. **What are first- and second-order systems?**
A system described by the first-order differential equation is called a first-order system. The system dynamics when described by a second-order differential equation is called a second-order system.

29. **What is a standard second-order system equation?**
Any second-order system whether it is mechanical, electrical, hydraulic, pneumatic, or chemical process can be modeled by a second-order dynamic equation in terms of damping factor ζ and natural frequency of oscillation ω_n. Such a system has the following T.F.:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

30. **What are the time domain specifications of a first-order system?**
The time domain specifications of a first-order system are

    (a) Time constant T,
    (b) Rise time t_r,
    (c) Time delay t_d, and
    (d) Settling time t_s.

31. **What are the time domain specifications of a second-order system?**
The time domain specifications of a second-order system are

    (a) Time constant T,
    (b) Rise time t_r,
    (c) Time delay t_d,
    (d) Settling time t_s,

(e) Peak over-shoot M_p, and
(f) The time at which the peak over-shoot occurs t_p (peak time).

32. **How second-order system is identified according to the nature of damping?**
The second-order system is identified according to the damping as follows:

    (a) Under-damped for $\zeta < 1$.
    (b) Critically damped for $\zeta = 1$.
    (c) Over-damped for $\zeta > 1$.

33. **How location of poles are identified in the s-plane according to the +ve damping?**
For under-damped system, the poles are complex conjugate in the left half s-plane (LHP). For critical damping, the two poles are identical in magnitude and located on the negative real axis. For over-damped system, the two poles are located on the negative real axis at two different points.
34. **What do you understand by negative damping?**
Negative damping makes the system unstable. For a second-order system, the one pole lies in RHP of the s-plane.
35. **What is damped frequency of oscillation of a second-order system?**
The natural frequency and damped frequency of oscillation are related as

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

The damped frequency of oscillation for a stable system will always be less than the natural frequency of oscillation ω_n.
36. **How system function is defined for a CT system?**
For a continuous-time system, the system function is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable.

II. Long Answer Type Questions

1. A triangular pulse signal $x(t)$ is shown in Fig. 1.77a. Sketch the following signals. (a) $x(4t)$; (b) $x(4t+3)$; (c) $x(-3t+2)$; (d) $x(\frac{t}{3}+2)$; (e) $x(3t-2)$; (f) $x(4t+3)+x(2t)$.
2. Sketch the following CT functions. (a) $x(t) = 8u(5-t)$; (b) $x(t) = 3\delta(t+2)$; (c) $x(t) = \text{ramp}(t+1)$; (d) $x(t) = 5\text{rect}\frac{(t+1)}{4}$; (e) $x(t) = -\text{tri}\frac{t-1}{4}$; (f) $x(t) = u(t) - u(t-5)$; (g) $x(t) = u(t) - u(t+5)$; (h) $x(t) = -\text{ramp}(t)u(t-3)$; (i) $x(t) = u(t)(t+\frac{1}{3})\text{ramp}(\frac{1}{3}-t)$; (j) $x(t) = \text{rect}(t+2) - \text{rect}(t-2)$.
3. Determine whether each of the following CT signals are periodic. If periodic determine the fundamental period (Figs. 1.86 and 1.87).

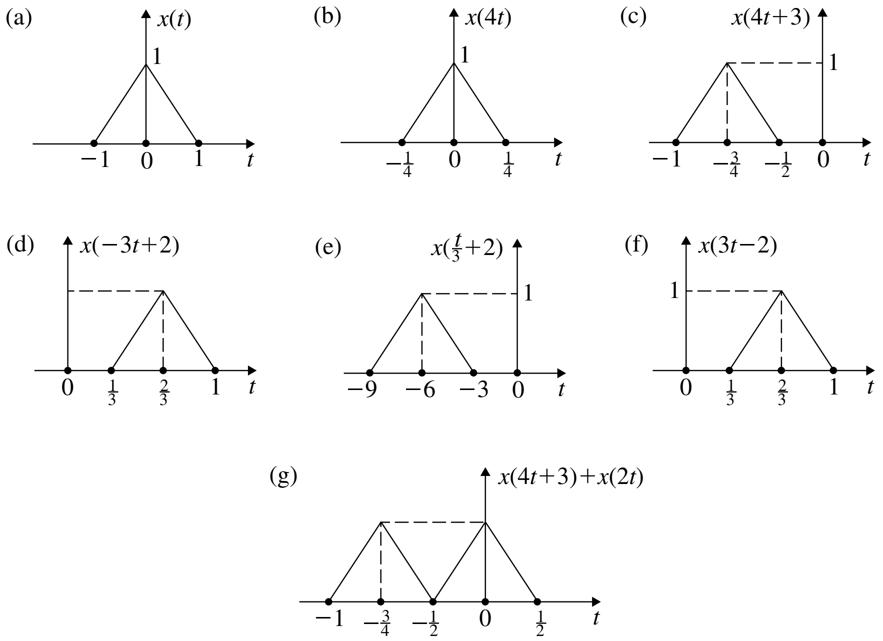


Fig. 1.86 Basic signal operations as applied to a triangular CT signal

(a) $x(t) = e^{j2t}$

(b) $x(t) = e^{(-2+j3)t}$

(c) $x(t) = \sin\left(60\pi t + \dfrac{\pi}{4}\right)$

(d) $x(t) = \cos\left(60\pi t - \dfrac{\pi}{4}\right) - \sin 20\pi t$

(e) $x(t) = \sin\left(8\pi t + \dfrac{\pi}{3}\right) + 5\cos\left(\dfrac{\pi t}{3} + \dfrac{\pi}{2}\right) + 6\cos\left(7\pi t - \dfrac{\pi}{2}\right)$

(f) $x(t) = 30\sin\left(8\pi t + \dfrac{\pi}{3}\right)\cos\left(2\pi t + \dfrac{\pi}{2}\right)\sin\left(5\pi t - \dfrac{\pi}{2}\right)$

(a) Periodic with period $T_0 = \pi$ sec. (b) Not periodic. (c) Periodic. $T_0 = \frac{1}{30}$ s. (d) Periodic $T_0 = \frac{1}{10}$ s. (e) Periodic $T_0 = 6$ s. (f) Periodic $T_0 = 2$ s.

4. Sketch the even and odd parts of the following signals shown in Fig. 1.88a and b.
5. Consider the CT signal $x(t) = \delta(t+4) - \delta(t-4)$. Sketch $\int x(t)dt$ and find the energy of the signal (Fig. 1.89).
   Energy $E = 8$.
6. Find the energy of the following CT signal. (a) $x(t) = \text{tri}3t$; (b) $x(t) = 2\text{tri}(\frac{t}{2})$; (c) $x(t) = \text{rect}10t$; (d) $2\,\text{rect}(\frac{t}{10})$; (e) $\sin(2\pi t)$.
   (a) $E = \frac{2}{9}$; (b) $E = \frac{16}{3}$; (c) $E = \frac{1}{5}$; (d) $E = 80$; (e) $E = \frac{1}{2}$.

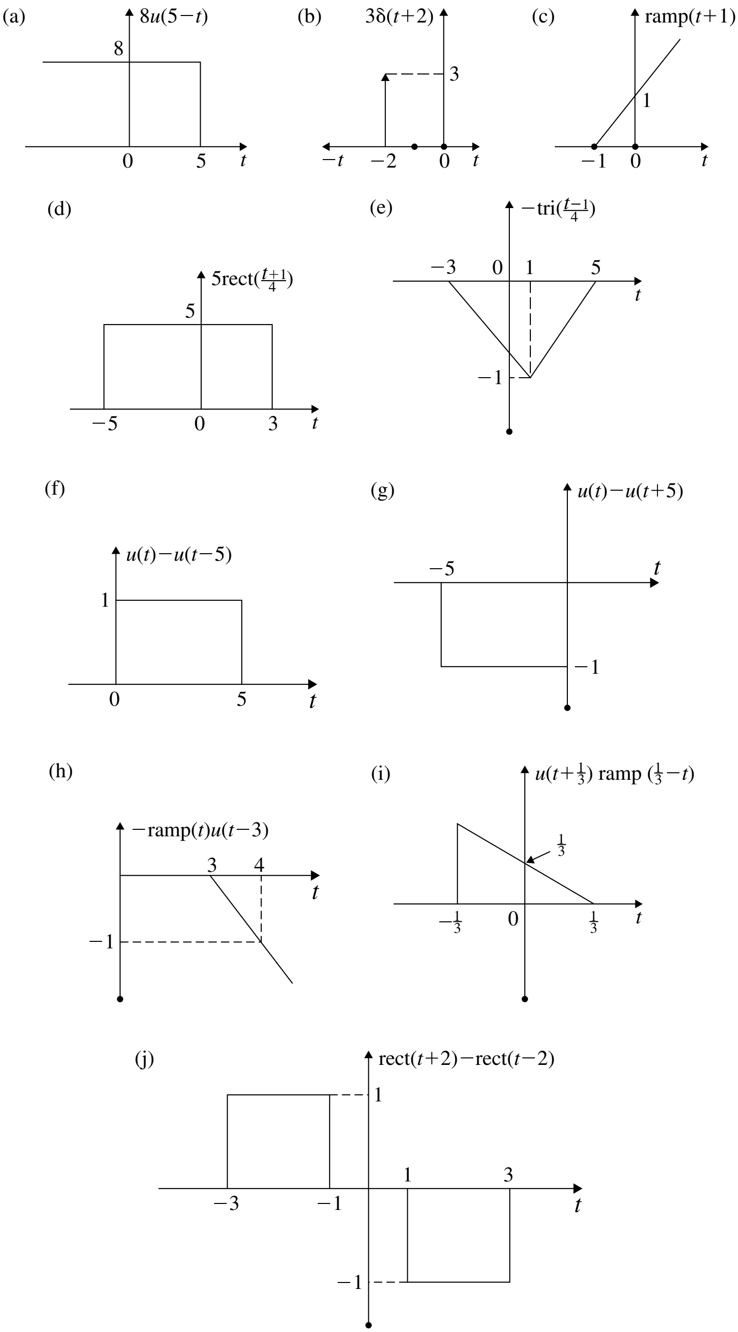


Fig. 1.87 Basic signal operations as applied to a CT signal $x(t) = 8u(5 - t)$

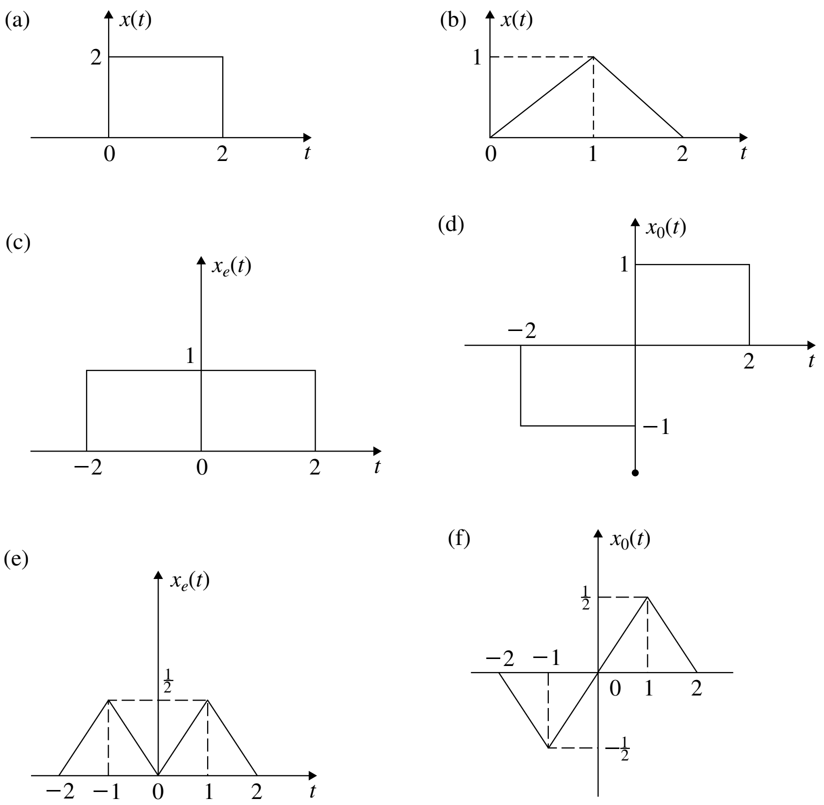


Fig. 1.88 Even and odd signals of CT signals

Fig. 1.89 Basic signal operation as applied to CT rectangular signal

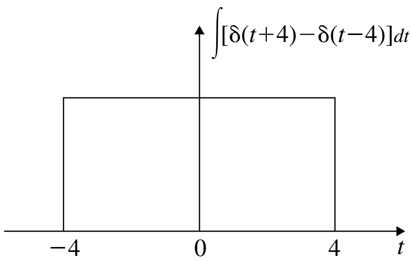
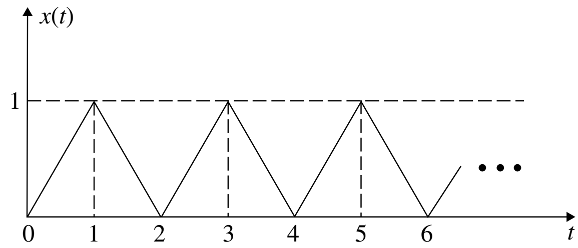


Fig. 1.90 Triangular wave



7. What is the average power of the triangular wave shown in Fig. 1.90?
   Average power $P = \frac{1}{3}$ W.
8.
$$\boldsymbol{y(t) = \frac{d}{dt}[e^{-2t}x(t)]}$$

   (a) The system response requires memory. Hence, it is dynamic.
   (b) The output depends on the present input only. Hence, it is causal.
   (c) The output due to the delayed input is not the same as the delayed output. Hence, it is time variant.
   (d) The weighted sum of the output is the same as output due to weighted sum of the input. Hence, the system is linear.
   (e) Since $\frac{d}{dt}(e^{-2t}x(t))$ is bounded $y(t)$ is also bounded, and hence the system is stable.

9.
$$\boldsymbol{y(t) = x(t) + 10x(t-5) \qquad t \geq 0}$$

   (a) The output response depends on present and past inputs. Hence, it is dynamic.
   (b) The output does not depend on the future input. Hence, it is causal.
   (c) The output due to the delayed input is same as the delayed output. Hence, the system is time invariant.
   (d) The weighted sum of the output is the same as output due to the weighted sum of the input. Hence, it is linear.
   (e) As long as the input $x(t)$ is bounded, $x(t-5)$ is also bounded. Hence, $y(t)$ is bounded. The system is stable.

10.
$$\boldsymbol{y(t) = x(10t)}$$

   (a) The system response depends on present, past, and future inputs. Hence, it is dynamic.
   (b) Since the output depends on the future input, it is non-causal.
   (c) The output due to the delayed input is not the same as the delayed output. Hence, the system is time variant.

(d) The weighted sum of the output is the same as output due to the weighted sum of the input. Hence, it is linear.
(e) If the input is bounded, the output is also bounded. The system is stable.

11.
$$\boldsymbol{y(t) = x\left(\frac{t}{10}\right)}$$

The output depends on present, past, and future inputs.

(a) The system is dynamic.
(b) The system is non-causal.
(c) The output due to the delayed input is not the same as the delayed output. The system is time variant.
(d) The weighted sum of the output will be the same as output due to the weighted sum of the input. The system is linear.
(e) If the input $x(\frac{t}{10})$ is bounded, the output is also bounded. The system is stable.

12.
$$\boldsymbol{y(t) = \frac{d}{dt}x(t-4)}$$

(a) The system requires memory and so it is dynamic.
(b) The output depends on present and past inputs. Hence, it is causal.
(c) The output due to the delayed input is same as the delayed output. The system is time invariant.
(d) The weighted sum of the output is the same as output due to the weighted sum of the input. The system is linear.
(e) If the input is bounded, the output is also bounded. The system is stable.

13. **Consider the system shown in Fig. 1.91. Derive expressions for the impulse response and unit step response of the system. Also determine T, t_r, t_d, t_s for step input.**
Ans:
$$\frac{C(s)}{R(s)} = \frac{10}{s+12}$$

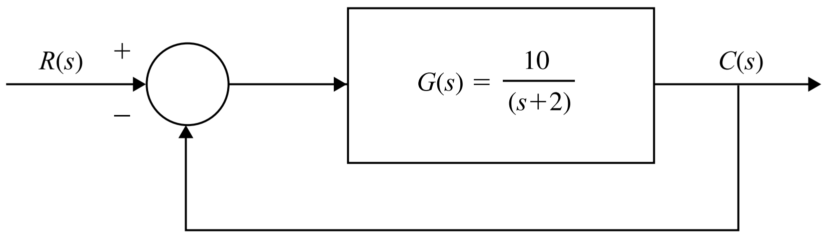


Fig. 1.91 First-order system for Question 13

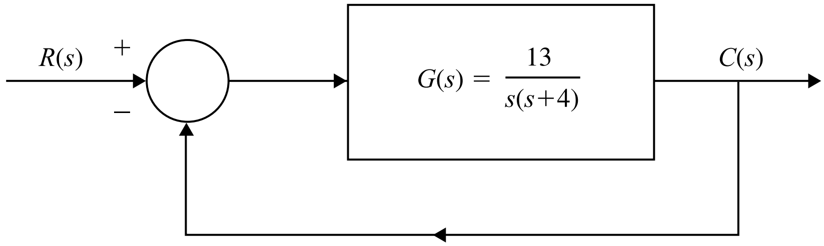


Fig. 1.92 Second-order system for question 14

$$\text{Impulse response, } h(t) = 10e^{-2t}u(t)$$

$$\text{Unit step response, } c(t) = \frac{10}{12}\left[1 - e^{-12t}\right]u(t)$$

$$\text{Time constant, } T = \frac{1}{12}\text{ s.}$$

$$\text{Rise time, } t_r = \frac{2.2}{12} = 0.1833\text{ s.}$$

$$\text{Time delay, } t_d = \frac{0.693}{12} = 0.05775\text{ s.}$$

$$\text{Settling time for 2\% error tolerance, } t_s = \frac{3.91}{12} = 0.3258\text{ s.}$$

14. **Consider the second-order system shown in Fig. 1.92. The system is subjected to unit step input. Derive the expression for the output variable. Determine the time domain specifications T, t_s, t_d, M_p, and t_p. What is the resonant peak M_r and resonant frequency ω_r in the frequency domain?**
Ans:

$$c(t) = \left[1 - 1.2e^{-2t}\sin(3t + 0.98)\right]$$
$$\text{Rise time, } t_t = 0.72\text{ s}$$
$$\text{Time constant, } T = 0.5\text{ s}$$
$$\text{Settling time, } t_s = 1.955\text{ s}$$
$$\text{Time delay, } t_d = 0.3857\text{ s}$$
$$\text{Peak time, } t_p = 1.05\text{ s}$$
$$\text{\% over-shoot \%, } M_p = 12.3\%$$
$$\text{Resonant peak, } M_r = 1.0828$$
$$\text{Resonant frequency, } \omega_r = 2.23\text{ rad/s}$$

15. **Explain why system is tested for impulse and step inputs.**

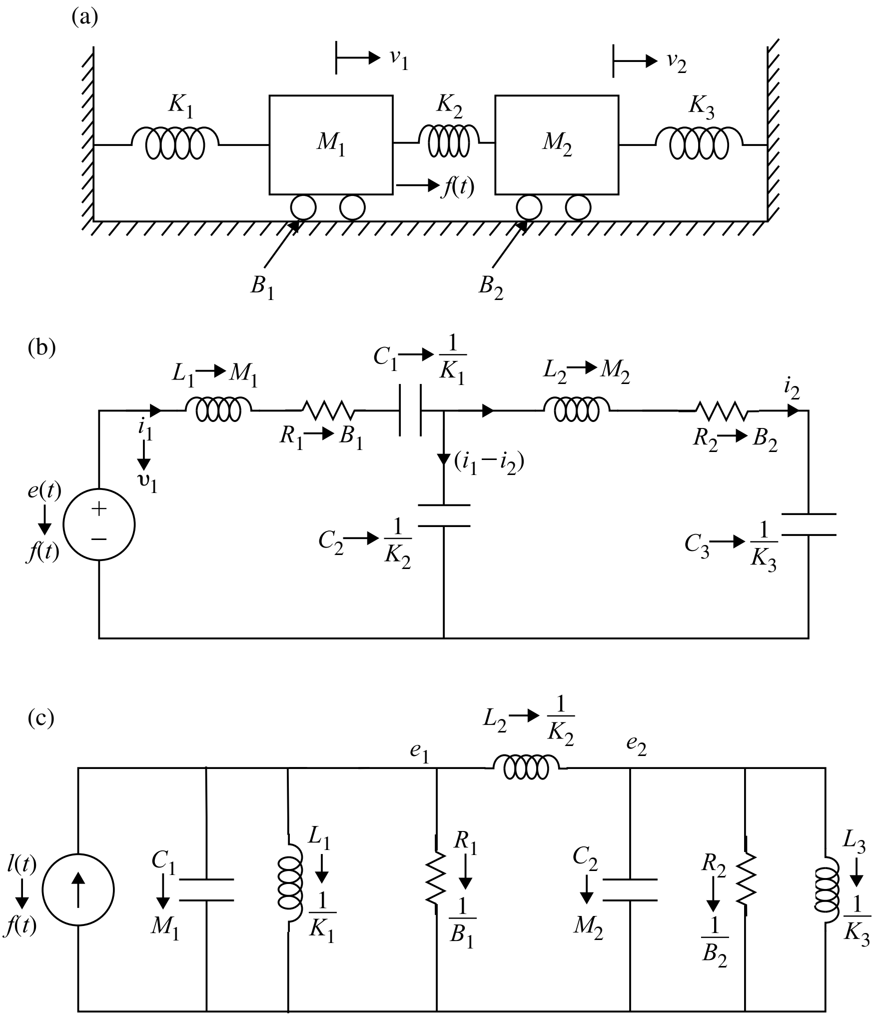


Fig. 1.93 **a** Mechanical translational system. **b** F–V analogous circuit. **c** F–I analogous circuit

16. **Consider the mechanical system shown in Fig. 1.93a. Draw the F–V and F–I analogous circuits and verify by writing down the dynamic equations describing the given system and the electric circuit so drawn.**
For the mechanical system, the following equations are written:

$$M_1 \frac{dv_1}{dt} + B_1 v_1 + K_2 \int (v_1 - v_2)dt + K_1 \int v_1(t) = f(t)$$

$$M_2 \frac{dv_2}{dt} + B_2 v_2 + K_3 \int v_2 dt + K_2 \int (v_2 - v_1)d(t) = 0$$

For F–V analog circuit, the equations are

$$L_1\frac{di_1}{dt} + i_1 R_1 + \frac{1}{C_2}\int (i_1 - i_2)dt + \frac{1}{C_1}\int i_1 dt = e(t)$$
$$L_2\frac{di_2}{dt} + i_2 R_2 + \frac{1}{C_3}\int i_2 dt + \frac{1}{C_2}\int (i_2 - i_1)dt = 0$$

For F–I analog circuit, the equations are

$$C_1\frac{de_1}{dt} + \frac{e_1}{R_1} + \frac{1}{L_2}\int (e_1 - e_2)dt + \frac{1}{L_1}\int e_1 dt = e(t)$$
$$C_2\frac{de_2}{dt} + \frac{e_2}{R_2} + \frac{1}{L_3}\int e_2 dt + \frac{1}{L_2}\int (e_2 - e_1)dt = 0$$

Chapter 2
Fourier Series Analysis of Continuous-Time Signals

Chapter Objectives

- To represent the periodic continuous-time signal by trigonometric Fourier series.
- To represent the CT signal by polar Fourier series.
- To determine the exponential Fourier series and Fourier spectra.
- To establish the properties of Fourier series.
- To establish Parseval's theorem and Dirichlet conditions.

2.1 Introduction

Sinusoidal input signals are often used to study the response of the system which gives useful informations. If a linear time invariant system is excited by a complex sinusoid, then the output response is also a complex sinusoid of the same frequency as the input. However, the amplitude of such a sinusoid is different from the input amplitude and also has a phase shift. The study of input–output if the input frequency is varied in the range $0 \leq \omega \leq \infty$ is termed as the frequency response of the system. The frequency response gives the steady-state response of the system which is the function of sinusoid's frequency. The frequency response is usually represented in graph by its magnitude and phase as a function of frequency. Several methods have been suggested in literature such as polar plot, Bode plot, and Nichols plot. Each method has its own merits. If the system is excited by the signal which is a weighted superposition of the complex sinusoids, then the system output is also a weighted superposition of the system response to each complex sinusoid. Thus, any arbitrary excitation signal $x(t)$ can be expressed as a linear combination of complex sinusoids. The output is obtained by summing up the responses to the individual

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complex sinusoids using superposition. However, expressing any arbitrary real function as a linear combination of complex sinusoids is a matter of concern. **Baron Jean Baptiste Joseph Fourier (1768–1830)**, a French mathematician, represented an arbitrary signal $x(t)$ in the form of a linear combination of complex sinusoids and is called as **Fourier Series**. In a Fourier series representation of a periodical signal, the higher frequency sines and cosines have frequencies that are integer multiples of the fundamental frequency. These multiples are called **harmonic numbers**. The study of signals using sinusoids has widespread applications in every branch of science and engineering. This great mathematical poem which finds wide applications in modern communication, signal processing, antenna design, and several other fields was not shown much enthusiasm by the scientific world during its inception. Fourier could not get the results published for the lack of mathematical rigor. The vehement opposition came from his fellow country men and great mathematical wizards Lagrange and Laplace. However, 15 years later, after several tireless attempts, Fourier successfully published the results in the form of text which is a classic now.

Fourier, born on 21-03-1768 in France, was the son of a tailor. Being orphaned at the age of eight, Fourier was educated in a local military college where he showed his brilliance in mathematics. When the French revolution broke out, many intellectuals decided to leave France to save themselves from the growing barbarism. Fourier escaped guillotine twice. Napoleon Bonaparte, a soldier scientist, captured power in France after the historical French revolution and stopped prosecution of intellectuals. The French ruler, who himself was a great scientist, appointed Fourier chair of mathematics academy in which he served with distinction when he was just 26 years of age. He was honored as the Baron of the empire by Napoleon in 1809. When Napoleon was exiled by King Louis XVIII, Fourier was identified as a Bonapartist and was treated with all disgrace. Napoleon came back to power within a year of his exile from Elba. However he was defeated by the English captain Nelson in the Battle of Waterloo and the great warrior scientist died in 1821 at St. Helena Island where he was exiled for the second time. Fourier should have again become an orphan but for the help of his former student who was now a prefect of Paris. He was appointed as the statistical bureau of the Seine, and subsequently in 1827 elected to the powerful position of Secretary of the Paris Academy of Science.

While carrying out investigations on propagation of heat in solid bodies, Fourier was able to establish the Fourier series and Fourier integral. In 1807, when he was 40 years of age, Fourier published his results. He claimed that any arbitrary function can always be expressed as a sum of sinusoids. For the lack of rigor and generality, the judges who included the great French mathematicians Lagrange, Laplace, Legendre, Monge, and Lacroix criticized Fourier's work for the lack of rigor but appreciated the novelty and importance of the work. Fourier could not defend the criticisms since the necessary tools were not available to him at that time. However in the year 1829, Dirichlet proved most of the claims of Fourier by putting a few restrictions (Dirichlet conditions).

Fifteen years after the paper was rejected mainly due to the vehement opposition given by Lagrange and to some extent by Laplace, Fourier published his results in expanded form as a text which has now become a classic in the area of mathematics, science, and engineering applications. The great mathematician who laid the foundation for the signal representation and analysis died on 16-05-1830, when he was 63 years old.

2.2 Periodic Signal Representation by Fourier Series

A continuous-time signal $x(t)$ is said to be periodic if there is a positive non-zero value of T for which

$$x(t+T) = x(t) \qquad \text{for all } t \tag{2.1}$$

The fundamental period T_0 of $x(t)$ is the smallest positive value of T for which Eq. (2.1) is satisfied. $\frac{1}{T_0}$ is called fundamental frequency f_0 and $\omega_0 = \frac{2\pi}{T_0}$ is called fundamental radian frequency. The real sinusoidal signal

$$x(t) = \cos(\omega_0 t + \phi) \tag{2.2}$$

and the complex exponential signal

$$x(t) = e^{j\omega_0 t} \tag{2.3}$$

have been proved in Chap. 1 as periodic signals as Eq. (2.1) is applicable in the above cases. **The prerequisite for the representation of any arbitrary continuous signal $x(t)$ in Fourier series is that it should be periodic. Non-periodic signals cannot be represented by Fourier series but can be represented by Fourier transform which is discussed later**.

2.3 Different Forms of Fourier Series Representation

Any arbitrary real or complex $x(t)$ signal which is periodic with fundamental period T_0 can be expressed as a sum of a sinusoid of period T_0 and its harmonics. They are represented in the following forms of Fourier series:

1. Trigonometric Fourier series.
2. Complex exponential Fourier series.
3. Polar or harmonic form Fourier series.

The above Fourier series representations are described below with illustrated examples.

2.3.1 Trigonometric Fourier Series

Consider any arbitrary continuous-time signal $x(t)$. This arbitrary signal can be split up as sines and cosines of fundamental frequency ω_0 and all of its harmonics are expressed as given below:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \tag{2.4}$$

Equation (2.4) is the Fourier series representation of an arbitrary signal $x(t)$ in trigonometric form.

In Eq. (2.4), a_0 corresponds to the zeroth harmonic or DC. The expression for the constant term a_0 and the amplitudes of the harmonic can be derived as

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt \tag{2.5}$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos n\omega_0 t\, dt \tag{2.6}$$

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin n\omega_0 t\, dt \tag{2.7}$$

In Eqs. (2.5), (2.6) and (2.7)

$$T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$$

$T_0 =$ Fundamental period of $x(t)$ in seconds;
$f_0 =$ Fundamental frequency in Hz;
$\omega_0 =$ Radian frequency in rad/second.

For the detailed derivation of the above equations, one may refer to standard textbooks. **Equation (2.4) is valid iff $x(t)$ is periodic**.

To Prove the periodicity of $x(t)$

The periodicity $x(t)$ is proved if $x(t) = x(t + T_0)$. Substituting $(t + T_0)$ in place of t in Eq. (2.4), the following equation is obtained:

$$\begin{aligned}x(t+T_0) &= a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0(t+T_0) + \sum_{n=1}^{\infty} b_n \sin n\omega_0(t+T_0) \\ &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t + n\omega_0 T_0) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t + n\omega_0 T_0)\end{aligned}$$

$$\begin{aligned}x(t+T_0) &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t + 2\pi n) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t + 2\pi n) \\ &= a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t\end{aligned}$$

$$x(t+T_0) = x(t) \tag{2.8}$$

Thus, it is established, if $x(t)$ is periodic, at $t = T_0$ every sinusoid starts and repeats the same over the next T_0 seconds and so on. The following points are to be noted while the coefficients a_0, a_n, and b_n are determined. It can be proved that

1. If the periodical signal $x(t)$ is symmetrical with respect to the time axis, then the coefficient $a_0 = 0$.
2. If the periodical signal $x(t)$ represents an even function, only cosine terms in FS exist and therefore $b_n = 0$.
3. If the periodical signal $x(t)$ represents an odd function, only sine terms in FS exist and therefore $a_n = 0$.

2.3.2 Complex Exponential Fourier Series

By using Euler's identity, the complex sinusoids can always be expressed in terms of exponentials. Thus, the trigonometric Fourier series of Eq. (2.4) can be represented as

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt} \tag{2.9}$$

where

$$D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-j\omega_0 nt} dt \tag{2.10}$$

Equation (2.9) represents exponential Fourier series and D_n is the coefficient of the exponential Fourier series. For detailed derivation of Eq. (2.10) one may refer to standard textbooks. It is to be noticed here that Eq. (2.9) is in a compact form and it is much more convenient to handle compared to trigonometric Fourier series. Further, determination of the coefficients D_n using Eq. (2.10) is much easier compared to a_0, a_n, and b_n in Eq. (2.4). For these reasons many authors prefer exponential Fourier series representation of signals. The coefficients D_n are related to trigonometric Fourier series coefficients a_n and b_n as

$$\begin{aligned} D_0 &= a_0 \\ D_n &= \frac{1}{2}(a_n - jb_n) \\ D_n^* &= \text{conjugate of } D_n \\ &= \frac{1}{2}(a_n + jb_n) \end{aligned} \tag{2.11}$$

2.3.3 Polar or Harmonic Form Fourier Series

The results derived in Sects. 2.31 and 2.32 are applicable whether $x(t)$ is real or complex. When $x(t)$ is real, the coefficients of trigonometric Fourier series a_n and b_n are real. In such cases, Eq. (2.4) can be expressed in a compact form as

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t - \theta_n) \tag{2.12}$$

where C_n and θ_n are related to a_n and b_n as

$$\begin{aligned} C_0 &= a_0 \\ C_n &= \sqrt{a_n^2 + b_n^2} \\ \theta_n &= \tan^{-1}\left(\frac{b_n}{a_n}\right) \end{aligned} \tag{2.13}$$

Equation (2.12) is also called as compact form Fourier series or cosine form Fourier series.

The coefficients of compact form Fourier series and exponential form Fourier series are related as

Table 2.1 Different form of FS representation their coefficients and their equivalence

FS form	Coefficients	Equivalence
1. Trigonometric	$a_0 = \frac{1}{T_0} \int_{T_0} x(t)dt$	$a_0 = C_0 = D_0$
$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$ $+ b_n \sin n\omega_0 t$	$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos n\omega_0 t \, dt$	$a_n - jb_n = C_n e^{j\theta n} = 2D_n$
	$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin n\omega_0 t \, dt$	$a_n + jb_n = C_n e^{-j\theta n} = 2D_n^*$
2. Exponential		
$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$	$D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$	$C_n = 2\|D_n\| \quad n \geq 1$
3. Polar or compact cosine	$C_0 = a_0$	$\theta_n = \angle D_n$
$x(t) =$ $C_0 + \sum_{n=1}^{\infty} C_n \cos(n^{\omega_0 t} - \theta_n)$	$C_n = \sqrt{a_n^2 + b_n^2}$	
	$\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$	

$$\begin{aligned} D_0 &= C_0 \\ |D_n| &= \left|D_n^*\right| = \frac{1}{2} C_n \\ \angle D_n &= \theta_n; \quad \angle D_n^* = -\theta_n \end{aligned} \tag{2.14}$$

For detailed derivations of Eqs. (2.13) and (2.14) one may refer to standard textbooks. Table 2.1 gives the different form of Fourier series representation, their coefficients and their equivalence.

The following examples illustrate the method of determining the Fourier series (FS) in the above three forms.

Example 2.1 Find the trigonometric Fourier series for the periodic signal shown in Fig. 2.1.

Solution 1. From Fig. 2.1, it is evident that the wave form is symmetrical with respect to the time axis t. Hence $a_0 = 0$.

2. By folding $x(t)$ across the vertical axis, it is observed that $x(t) = x(-t)$ which shows that the function of the signal is even. Hence $b_n = 0$.
3. From Fig. 2.1, it is easily obtained that the fundamental period $T_0 = 4$ seconds and the fundamental radian frequency $\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$ radians per second. From Eq. (2.4) the trigonometric Fourier series is written as

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]$$

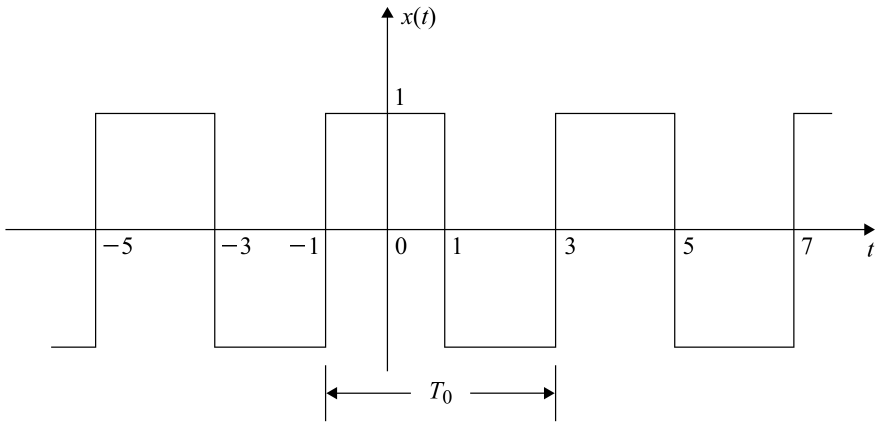


Fig. 2.1 A rectangular wave of Example 2.1

But

$$\begin{aligned} x(t) &= 1 \qquad \text{for} \quad -1 \le t \le 1 \\ &= -1 \qquad \text{for} \quad 1 \le t \le 3 \end{aligned}$$

Substituting $a_0 = 0$ and $b_n = 0$, and $\omega_0 = \frac{\pi}{2}$

$$x(t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{2} t$$

$$\begin{aligned} a_n &= \frac{2}{T_0} \int_{-1}^{3} x(t) \cos\left(\frac{n\pi}{2} t\right) dt \\ &= \frac{1}{2} \left[\int_{-1}^{1} \cos \frac{n\pi}{2} t + \int_{1}^{3} (-1) \cos \frac{n\pi}{2} t \, dt \right] \\ &= \frac{1}{2} \left[\left\{ \frac{2}{n\pi} \sin \frac{n\pi}{2} t \right\}_{-1}^{1} - \left\{ \frac{2}{n\pi} \sin \frac{n\pi}{2} t \right\}_{1}^{3} \right] \\ &= \frac{1}{n\pi} \left[\sin \frac{n\pi}{2} + \sin \frac{n\pi}{2} + \sin \frac{n\pi}{2} + \sin \frac{n\pi}{2} \right] \\ &= \frac{4}{n\pi} \sin \frac{n\pi}{2} \\ &= 0 \qquad \text{for } n = \text{even} \\ &= \frac{4}{n\pi} \qquad \text{for } n = 1, 5, 9, 13, \ldots \end{aligned}$$

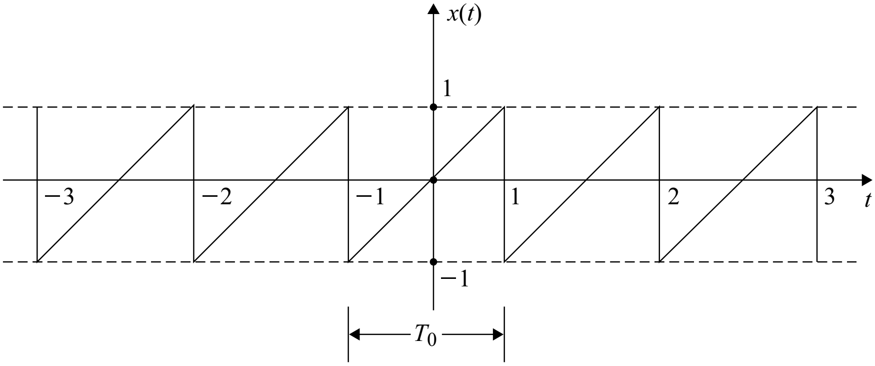


Fig. 2.2 Saw tooth wave form

$$= -\frac{4}{n\pi} \qquad \text{for } n = 3, 7, 11, 15, \ldots$$

$$x(t) = \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{2} t$$

$$x(t) = \frac{4}{\pi}\left[\cos \frac{\pi}{2} t - \frac{1}{3} \cos \frac{3\pi}{2} t + \frac{1}{5} \cos \frac{5\pi}{2} t - \frac{1}{7} \cos \frac{7\pi}{2} t\right]$$

Example 2.2 For the periodic signal shown in Fig. 2.2, determine the trigonometric Fourier series.

Solution 1. From Fig. 2.2, $T_0 = 2$ seconds and $\omega_0 = \frac{2\pi}{T_0} = \pi$. The signal is symmetrical with respect to time axis and hence $a_0 = 0$. Also, from Fig. 2.2, it is evident that $x(t) = -x(-t)$ and therefore the signal is an odd signal and $a_n = 0$. The Fourier series for such a signal is therefore

$$x(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

2. The coefficient b_n is determined as follows:

$$x(t) = t \qquad -1 \le t \le 1$$

$$b_n = \frac{2}{T_0} \int_{-1}^{1} t \sin n\omega_0 t \, dt$$

$$= \int_{-1}^{1} t \sin n\pi t \, dt$$

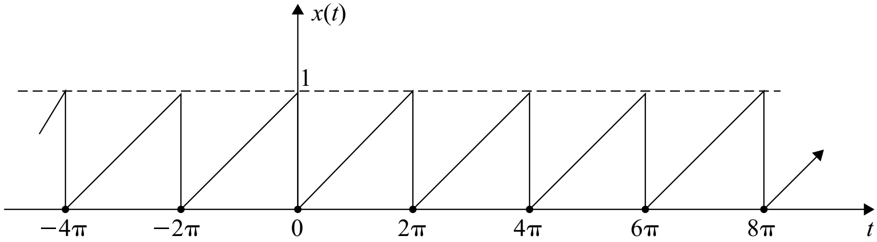


Fig. 2.3 Saw tooth signal of Example 2.3

The above integral is solved using the infinite integral

$$\int u dv = uv - \int v du$$

Let $u = t, du = dt$

$$dv = \int \sin n\pi t \, dt; \qquad v = -\frac{1}{n\pi} \cos n\pi t$$

$$b_n = \left[-\frac{t}{n\pi} \cos n\pi t \right]_{-1}^{1} + \frac{1}{n^2\pi^2} \left[\sin n\pi t \right]_{-1}^{1}$$

$$= -\frac{2}{n\pi} \cos n\pi + \frac{1}{n^2\pi^2} [\sin n\pi + \sin n\pi]$$

since $\sin n\pi = 0$,

$$b_n = -\frac{2}{n\pi} \cos n\pi$$

$$x(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t$$

$$x(t) = \frac{2}{\pi} \left[\sin \pi t - \frac{1}{2} \sin 2\pi t + \frac{1}{3} \sin 3\pi t + \cdots \right]$$

Example 2.3 Find the trigonometric Fourier series for the signal shown in Fig. 2.3.

(*Anna University, December 2006*)

Solution 1. From Fig. 2.3, $T_0 = 2\pi$ and $\omega_0 = \frac{2\pi}{T_0} = 1$. The signal is neither odd nor even. Further, it is not symmetrical with respect to the time axis. So the coefficients a_0, a_n, and b_n are to be evaluated.

2.

$$x(t) = \frac{t}{2\pi} \qquad 0 \le t \le 2\pi \ \left(\text{for a ramp signal the slope is } \frac{1}{2\pi}\right)$$

$$a_0 = \frac{1}{T_0}\int_0^{2\pi} \frac{t}{2\pi} dt = \frac{1}{4\pi^2}\left[\frac{t^2}{2}\right]_0^{2\pi}$$

$$a_0 = \frac{1}{2}$$

$$\begin{aligned} a_n &= \frac{2}{T_0}\int_0^{2\pi} \frac{t}{2\pi}\cos nt\, dt \\ &= \frac{1}{2\pi^2}\int_0^{2\pi} t\cos nt\, dt \end{aligned}$$

Let $u = t; du = dt$

$$\begin{aligned} dv &= \int \cos nt\, dt; \quad v = \frac{\sin nt}{n} \\ a_n &= uv - \int v du \\ &= \frac{1}{2\pi^2}\left[\frac{t\sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^{2\pi} \\ &= \frac{1}{2\pi^2}[0 + 0 + 1 - 1] \end{aligned}$$

$$a_n = 0$$

(This is due to half wave symmetry).

$$\begin{aligned} b_n &= \frac{2}{T_0}\int_0^{2\pi} \frac{t}{2\pi}\sin nt\, dt \\ &= \frac{1}{2\pi^2}\left[-\frac{t\cos nt}{n} + \frac{\sin nt}{n^2}\right]_0^{2\pi} \quad \text{[using } u\text{-}v \text{ method]} \\ &= \frac{1}{2\pi^2}\left[-\frac{2\pi}{n}\cos 2\pi n\right] \end{aligned}$$

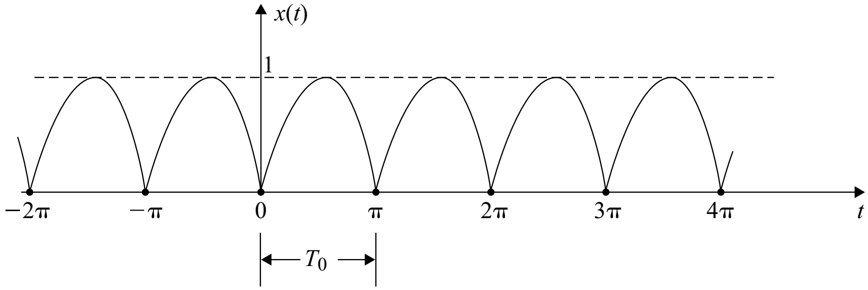


Fig. 2.4 A full wave rectifier

$$b_n = -\frac{1}{n\pi}$$

$$x(t) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin nt$$

Example 2.4 Determine the trigonometric Fourier series representation of a full wave rectified signal.

(*Anna University, April 2005*)

Solution 1. The full wave rectified signal is shown in Fig. 2.4. Here $T_0 = \pi$ and $\omega_0 = \frac{2\pi}{T_0} = 2$.

2. The signal is not symmetrical with respect to time axis. Therefore, a_0 is calculated as follows:

$$a_0 = \frac{1}{T_0} \int_0^{\pi} x(t)dt$$

where

$$x(t) = \sin t \qquad 0 \le t \le \pi$$

$$a_0 = \frac{1}{\pi} \int_0^{\pi} \sin t \, dt$$
$$= \frac{1}{\pi} \Big[-\cos t \Big]_0^{\pi} = \frac{2}{\pi}$$

$$a_0 = \frac{2}{\pi}$$

3.

$$x(t) = x(-t)$$

The given signal represents an even function and therefore

$$b_n = 0$$

4.

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} \sin t \cos n\omega_0 t \, dt \\ &= \frac{2}{\pi} \int_0^{\pi} \sin t \cos 2nt \, dt \end{aligned}$$

Using the property,

$$\sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$$

the above integral is written as

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{\pi} \sin(2n+1)t \, dt + \frac{1}{\pi} \int_0^{\pi} \sin(1-2n)t \, dt \\ &= \frac{1}{\pi} \left[-\frac{\cos(2n+1)t}{(2n+1)} \right]_0^{\pi} + \frac{1}{\pi} \left[-\frac{\cos(1-2n)t}{(1-2n)} \right]_0^{\pi} \\ &= \frac{1}{\pi} \left[-\frac{\cos(2n+1)\pi + 1}{(2n+1)} \right] + \frac{1}{\pi} \left[-\frac{\cos(1-2n)\pi + 1}{(1-2n)} \right] \\ &= \frac{1}{\pi} \left[\frac{1-(-1)^{2n+1}}{(2n+1)} + \frac{1-(-1)^{1-2n}}{(1-2n)} \right] \\ &= \frac{1}{\pi} \left[\frac{2}{(2n+1)} + \frac{2}{(1-2n)} \right] \\ &= \frac{2}{\pi} \left[\frac{1-2n+2n+1}{(1-4n^2)} \right] \end{aligned}$$

$$a_n = \frac{4}{\pi(1-4n^2)}$$

$$x(t) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(1-4n^2)} \cos 2nt$$

Example 2.5 Obtain the Fourier series expression of a half wave sine wave.

(*Anna University, December 2007*)

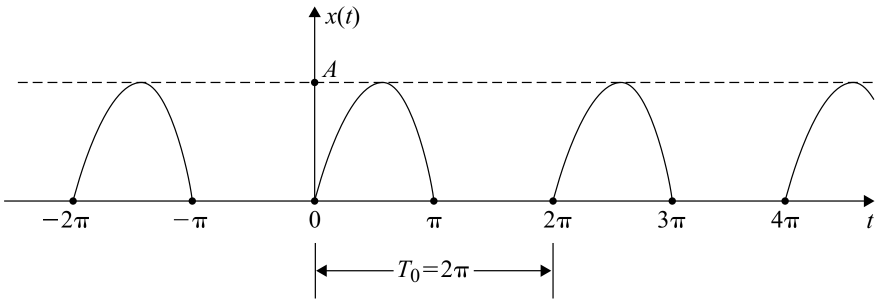


Fig. 2.5 A half wave rectified sine wave

Solution 1. $T_0 = 2\pi$ and $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1$ (Fig. 2.5)

$$\begin{aligned} x(t) &= A \sin t \qquad 0 \le t \le \pi \\ &= 0 \qquad\qquad \pi \le t \le 2\pi \\ a_0 &= \frac{1}{2\pi} \int_0^{\pi} A \sin t \, dt \\ &= \frac{A}{2\pi} \Big[-\cos t \Big]_0^{\pi} = \frac{A}{\pi} \end{aligned}$$

$$a_0 = \frac{A}{\pi}$$

2.

$$\begin{aligned} a_n &= \frac{2}{2\pi} \int_0^{\pi} A \sin t \cos nt \, dt \\ &= \frac{A}{2\pi} \left[\int_0^{\pi} \sin(1+n)t \, dt + \int_0^{\pi} \sin(1-n)t \, dt \right] \\ &= \frac{A}{2\pi} \left[-\frac{\cos(1+n)t}{(1+n)} - \frac{\cos(1-n)t}{(1-n)} \right]_0^{\pi} \\ &= \frac{A}{2\pi} \left[\frac{1-\cos(1+n)\pi}{(1+n)} + \frac{1-\cos(1-n)\pi}{(1-n)} \right] \\ &= \frac{A}{2\pi} \left[\frac{2}{(1+n)} + \frac{2}{(1-n)} \right] = \frac{2A}{\pi(1-n^2)} \end{aligned}$$

$$a_n = \frac{2A}{\pi(1-n^2)} \qquad n \ne 1$$

Since for $n = 1$, $a_n = \infty$, a_1 is calculated as follows.
For $n = 1$,

$$\begin{aligned} a_1 &= \frac{1}{2\pi}\int_0^{\pi} A \sin t \cos t \, dt \\ &= \frac{A}{2\pi}\int_0^{\pi} \sin 2t \, dt \\ &= \frac{A}{4\pi}\left[-\cos 2t\right]_0^{\pi} = 0 \end{aligned}$$

$$a_1 = 0$$

3.

$$\begin{aligned} b_n &= \frac{2}{2\pi}\int_0^{\pi} A \sin t \sin nt \, dt \\ &= \frac{A}{2\pi}\left[\int_0^{\pi} \{\cos(1-n)t - \cos(1+n)t\} \, dt\right] \\ &= \frac{A}{2\pi}\left[\frac{\sin(1-n)t}{(1-n)} - \frac{\sin(1+n)t}{(1+n)}\right]_0^{\pi} \\ &= \frac{A}{2\pi}\left[\frac{\sin(1-n)\pi - \sin 0}{(1-n)} - \frac{\sin(1+n)\pi + \sin 0}{(1+n)}\right] \end{aligned}$$

$$b_n = 0 \quad n \neq 1$$

For $n = 1$, $b_1 = \infty$ and therefore b_1 is calculated as follows:

$$\begin{aligned} b_1 &= \frac{2}{2\pi}\int_0^{\pi} A \sin t \sin t \, dt \\ &= \frac{A}{\pi}\int_0^{\pi} A \sin^2 t \, dt \\ &= \frac{A}{2\pi}\left[\int_0^{\pi} (1 - \cos 2t) dt\right] = \frac{A}{2\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{\pi} \end{aligned}$$

$$b_1 = \frac{A}{2}$$

$$x(t) = \frac{A}{\pi} + \frac{A}{2}\sin t + \sum_{n=2}^{\infty} \frac{2A}{\pi(1-n^2)} \cos nt$$

Example 2.6 Determine the Fourier series representation of the signal $x(t) = t^2$ for all values of "t" which exists in the interval $(-1, 1)$.

(*Anna University, May 2007*)

Solution 1. For the given signal $T_0 = 2$ and $\omega_0 = \frac{2\pi}{T_0} = \pi$.

$$a_0 = \frac{1}{2}\int_{-1}^{1} t^2 dt = \frac{1}{2}\left[\frac{t^3}{3}\right]_{-1}^{1} = \frac{1}{3}$$

$$a_0 = \frac{1}{3}$$

2.

$$\begin{aligned} a_n &= \frac{2}{2}\int_{-1}^{1} t^2 \cos n\pi t \, dt \\ &= \int_{-1}^{1} t^2 \cos n\pi t \, dt \end{aligned}$$

Applying $\int u dv = uv - \int v du$ twice for the above equation, we get

$$\begin{aligned} a_n &= \left[t^2 \frac{\sin n\pi t}{n\pi} + \frac{2t}{n^2\pi^2}\cos n\pi t - \frac{2}{n^3\pi^3}\sin n\pi t\right]_{-1}^{1} \\ &= \left[\frac{\sin n\pi}{n\pi} + \frac{2}{n^2\pi^2}\cos n\pi - \frac{2}{n^3\pi^3}\sin n\pi + \frac{\sin n\pi}{n\pi} \right. \\ &\quad \left. + \frac{2}{n^2\pi^2}\cos n\pi - \frac{2}{n^3\pi^3}\sin n\pi\right] \end{aligned}$$

$$\sin n\pi = 0 \quad \text{for all } n$$

$$a_n = \frac{4}{n^2\pi^2}\cos n\pi$$

$$a_n = \frac{4}{n^2\pi^2}(-1)^n$$

3. From Fig. 2.6, it is evident that $x(t)$ is an even function and therefore $b_n = 0$.

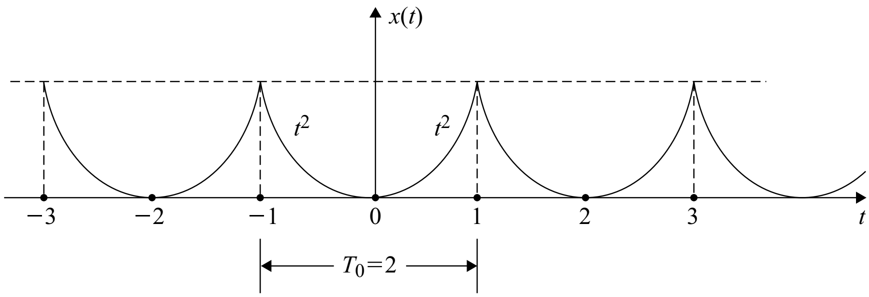


Fig. 2.6 Representation of $x(t) = t^2$

4.

$$x(t) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi t$$

$$x(t) = \frac{1}{3} + \frac{4}{\pi^2}\left[-\cos \pi t + \frac{1}{4}\cos 2\pi t - \frac{1}{9}\cos 3\pi t + \ldots\right]$$

2.4 Properties of Fourier Series

2.4.1 Linearity

Let $x_1(t)$ and $x_2(t)$ be two periodic signals with the same period T_0. Let D_{n1} and D_{n2} be the Fourier series coefficients in complex exponential form. Let $x(t)$ be the composite signals of $x_1(t)$ and $x_2(t)$ which are related as

$$x(t) = Ax_1(t) + Bx_2(t) \tag{2.15}$$

where A and B are constants.

From Eq. (2.10)

$$D_{n1} = \frac{1}{T_0}\int_{T_0} x_1(t)e^{-jn\omega_0 t}dt \tag{2.16}$$

$$D_{n2} = \frac{1}{T_0}\int_{T_0} x_2(t)e^{-jn\omega_0 t}dt \tag{2.17}$$

Let D_n be the Fourier series coefficient of $x(t)$

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt \tag{2.18}$$

$$= \frac{1}{T_0}\int_{T_0} [Ax_1(t) + Bx_2(t)]e^{-jn\omega_0 t}dt \tag{2.19}$$

$$= \frac{1}{T_0}\int_{T_0} Ax_1(t)e^{-jn\omega_0 t}dt + \frac{1}{T_0}\int_{T_0} Bx_2(t)e^{-jn\omega_0 t}dt \tag{2.20}$$

$$D_n = AD_{n1} + BD_{n2} \tag{2.21}$$

The Fourier series coefficient of the composite signal $x(t)$ is the linear combination of individual signal.

2.4.2 Time Shifting Property

According to time shifting property, if the periodic signal $x(t)$ with fundamental period T_0 is time shifted, the periodicity remains the same and the FS coefficient is multiplied by the factor $e^{-jnw_0t_0}$.

Proof Let $x(t)$ be time shifted by t_0. Now the time shifted signal is $x(t - t_0)$. The Fourier series coefficient of $x(t)$ is

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt \tag{2.22}$$

Let D_{n0} be the FS coefficient for the time shifted signal.

$$D_{n0} = \frac{1}{T_0}\int_{T_0} x(t - t_0)e^{-jn\omega_0 t}dt \tag{2.23}$$

Substitute $\tau = (t - t_0)$ in the above equation

$$D_{n0} = \frac{1}{T_0}\int_{T_0} x(\tau)e^{-jn\omega_0(\tau+t_0)}d\tau$$

$$= e^{-jn\omega_0 t_0}\frac{1}{T_0}\int_{T_0} x(\tau)e^{-jn\omega_0\tau}d\tau \tag{2.24}$$

$$D_{n0} = e^{-jn\omega_0 t_0} D_n \quad (2.25)$$

2.4.3 Time Reversal Property

According to time reversal property, if the signal $x(t)$ is time reversed, the periodicity remains the same with the time reversal in the FS coefficient.

Proof Let $x(t)$ be the signal with period T_0 and the FS coefficient D_n. If $x(t)$ is time reversed, the signal becomes $x(-t)$. Let D_{-n} be the FS coefficient of $x(-t)$.

$$D_n = \frac{1}{T_0} \int_{T_0} x(-t) e^{-jn\omega_0 t} dt \quad (2.26)$$

Let us substitute $\tau = -t$

$$D_n = \frac{1}{T_0} \int_{T_0} x(\tau) e^{-j(-n)\omega_0 \tau} (-d\tau) \quad (2.27)$$

$$= -\frac{1}{T_0} \int_{T_0} x(\tau) e^{-j(-n)\omega_0 \tau} d\tau \quad (2.28)$$

$$D_n = -D_{-n}$$

2.4.4 Time Scaling Property

According to time scaling property if $x(t)$ is periodic with fundamental period T_0, then $x(at)$ where a is any positive real number, and is also periodic but with a fundamental period of $\frac{T_0}{a}$.

Proof Let D_s be the FS coefficient of $x(at)$.

$$D_s = \frac{1}{T_0} \int_{T_0} x(at) e^{-jn\omega_0 t} dt \quad (2.29)$$

Let $at = \tau$

$$D_s = \frac{1}{aT_0} \int_{T_0} x(\tau) e^{-jn\omega_0 \frac{\tau}{a}} d\tau$$

$$D_s = \frac{1}{a} D_{n/a} \tag{2.30}$$

2.4.5 Multiplication Property

According to multiplication property, if $x_1(t)$ and $x_2(t)$ are the two signals having the periodicity T_0, then the Fourier coefficient of the product of these two signals is given by

$$\boldsymbol{D_n = \sum_{l=-\infty}^{\infty} A_l B_{n-l}}$$

where A_l and B_l are the FS coefficients of $x_1(t)$ and $x_2(t)$ respectively.

Proof Let

$$x(t) = x_1(t) \times x_2(t)$$
$$D_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

$$D_n = \frac{1}{T_0} \int_{To} [x_1(t) \times x_2(t)]\, e^{-jn\omega_0 t} dt$$
$$= \frac{1}{T_0} \int_{T_0} \left[\sum_{l=-\infty}^{\infty} A_l e^{jl\omega_0 t} \right] x_2(t) e^{-jn\omega_0 t} dt$$
$$= \sum_{l=-\infty}^{\infty} A_l \frac{1}{T_0} \int_{T_0} x_2(t) e^{-j(n-l)\omega_0 t} dt$$

$$D_n = \sum_{l=-\infty}^{\infty} A_l B_{n-l} \tag{2.31}$$

2.4.6 Conjugation Property

According to this property, the FS coefficients have conjugate symmetric property

$$D_{-n} = D_n^*$$

Proof

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$x^*(t) = \left[\sum_{n=-\infty}^{\infty} Dn^{ejn\omega_0 t}\right]^*$$

$$= \sum_{n=-\infty}^{\infty} D_n^* e^{-jn\omega_0 t}$$

Let $l = -n$,

$$x^*(t) = \sum_{l=-\infty}^{\infty} D_{-l}^* e^{jl\omega_0 t} \tag{2.32}$$

Thus during conjugation, FS coefficient becomes conjugate and time reversed.

2.4.7 Differentiation Property

If a periodical signal $x(t)$ is differentiated, the FS coefficient is multiplied by the factor $jn\omega_0$.

Proof

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt}$$

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} j\omega_0 n D_n e^{j\omega_0 nt}$$

$$= \sum_{n=-\infty}^{\infty} D_n^1 e^{j\omega_0 nt} \tag{2.33}$$

where $D_n^1 = j\omega_0 n D_n$. Thus, when the signal $x(t)$ is differentiated, its FS coefficient is multiplied by the factor $j\omega_0 n$.

2.4.8 Integration Property

According to the integration property, the FS coefficient of $x(t)$ when $x(t)$ is integrated becomes

$$\frac{1}{j\omega_0 n} D_n$$

Proof

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

Integrating both sides we get

$$\begin{aligned}
\int_{-\infty}^{t} x(t) &= \int_{-\infty}^{t} \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} dt \\
&= \sum_{n=-\infty}^{\infty} \frac{D_n e^{jn\omega_0 t}}{j\omega_0 n} \\
&= \sum_{n=-\infty}^{\infty} D_n^1 e^{jn\omega_0 t} \qquad (2.34)
\end{aligned}$$

where $D_n^1 = \frac{1}{j\omega_0 n} D_n$. Thus, when the signal $x(t)$ is integrated, its FS coefficient is divided by the factor $j\omega_0 n$.

2.4.9 Parseval's Theorem

According to Parseval's theorem, the total average power in a periodic signal is the sum of the average powers in all its components which is the sum of the squared value of FS coefficients.

Proof The average power in a periodic signal is given by

$$P = \frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt$$

$$P = \frac{1}{T_0}\int_{T_0} x(t)\,[x(t)]^*\,dt$$
$$= \frac{1}{T_0}\int_{T_0} x(t)\left[\sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt}\right]^* dt$$
$$= \sum_{n=-\infty}^{\infty} D_n^* \frac{1}{T_0}\int_{T_0} x(t)e^{-j\omega_0 nt}dt = \sum_{n=-\infty}^{\infty} D_n^* D_n$$

$$P = \sum_{n=-\infty}^{\infty} |D_n|^2 \tag{2.35a}$$

For a real $x(t)$, $|D_{-n}| = |D_n|$

$$P = D_0^2 + 2\sum_{n=1}^{\infty} |D_n|^2 \tag{2.35b}$$

For a trigonometric Fourier series,

$$P = C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2 \tag{2.35c}$$

Example 2.7 Find the Fourier series representation for the signal

$$x(t) = 3\cos\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$$

and hence find the power.

(*Anna University, April 2008*)

Solution

$$x(t) = 3\cos\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$$
$$= \frac{3}{2}\left[e^{j(\pi/2t+\pi/4)} + e^{-j(\frac{\pi}{2}t+\pi/4)}\right]$$
$$= \frac{3}{2}e^{j\pi/4}e^{j(\pi/2)t} + \frac{3}{2}e^{-j\pi/4}e^{-j\pi/2t}$$

Compare this with complex exponential Fourier series

$$\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \text{where } \omega_0 = \frac{\pi}{2} \\
&= \sum_{n=-\infty}^{\infty} D_n e^{jn\frac{\pi}{2}t} \\
&= D_{-1}e^{-j\frac{\pi}{2}t} + D_1 e^{j\frac{\pi}{2}t} \\
D_1 &= \frac{3}{2}e^{j\frac{\pi}{4}} = \frac{3}{2}\left[\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right]
\end{aligned}$$

$$\begin{aligned}
D_1 &= \frac{3}{2\sqrt{2}}(1+j); \quad |D_1| = \frac{3}{2} \\
D_{-1} &= \frac{3}{2\sqrt{2}}(1-j); \quad |D_{-1}| = \frac{3}{2}
\end{aligned}$$

$$P = \sum_{n=-\infty}^{\infty} |D_n|^2 = D_{-1}^2 + D_1^2 = \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = \frac{9}{2}$$

Example 2.8 Find the Fourier series of the following signals. Also find the power using Fourier series coefficients.

(a) $x(t) = 2\cos 3t + 3\sin 2t$

(b) $x(t) = \cos^2 t$

Solution (a) $\boldsymbol{x(t) = 2\cos 3t + 3\sin 2t}$

1.

$$\begin{aligned}
\omega_{01} &= 3; \qquad T_{01} = \frac{2\pi}{\omega_{01}} = \frac{2\pi}{3} \\
\omega_{02} &= 2; \qquad T_{02} = \frac{2\pi}{\omega_{02}} = \frac{2\pi}{2} = \pi \\
\frac{T_{01}}{T_{02}} &= \frac{2\pi}{3\pi} = \frac{2}{3} \\
T_0 &= 3T_{01} = 2T_{02} = 2\pi \\
\omega_0 &= \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1
\end{aligned}$$

2. Using Euler's Formula, $x(t)$ can be expressed as

$$x(t) = (e^{j3t} + e^{-j3t}) + \frac{3}{j2}(e^{j2t} - e^{-j2t})$$
$$= e^{-j3t} + j\frac{3}{2}e^{-j2t} + e^{j3t} - j\frac{3}{2}e^{j2t}$$

$x(t)$ can also be expressed in complex exponential form as

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$
$$= \sum_{n=-\infty}^{\infty} D_n e^{jnt}$$

Equating the two equations for $x(t)$, we get

$$e^{-j3t} + j\frac{3}{2}e^{-j2t} + e^{j3t} - j\frac{3}{2}e^{j2t} = \sum_{n=-\infty}^{\infty} D_n e^{jnt}$$

Putting $n = \pm 3$

$$D_3 = 1 \quad \text{and} \quad D_{-3} = 1$$

Putting $n = \pm 2$

$$D_2 = -j\frac{3}{2} \quad \text{and} \quad D_{-2} = j\frac{3}{2}$$

All other $D_n = 0$.

$$\text{Power } P = |D_{-3}|^2 + |D_{-2}|^2 + |D_3|^2 + |D_2|^2$$
$$= 1^2 + \left(\frac{3}{2}\right)^2 + 1^2 + \left(\frac{3}{2}\right)^2 = \frac{13}{2}$$

(b) $\boldsymbol{x(t) = \cos^2 t}$

$$x(t) = \cos^2 t$$
$$= \frac{1}{2}[1 + \cos 2t]$$
$$\omega_0 = 2$$
$$x(t) = \frac{1}{2} + \frac{1}{2}\frac{\left[e^{j2t} + e^{-j2t}\right]}{2} = \sum_{n=-\infty}^{\infty} D_n e^{j2nt}$$

For $n = 0$,

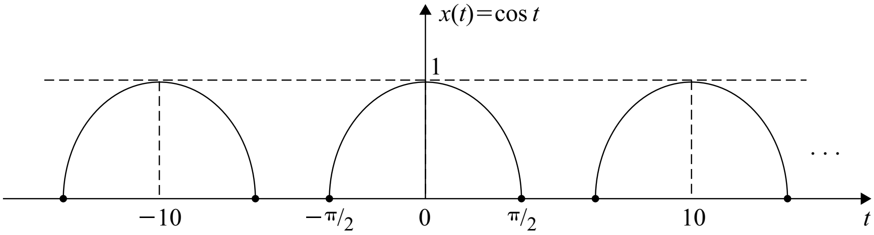


Fig. 2.7 Signal of Example 2.9

$$D_0 = \frac{1}{2}$$

For $n = \pm 1$,

$$D_1 = \frac{1}{4} \quad \text{and} \quad D_{-1} = \frac{1}{4}$$

$$\text{Power } P = D_0^2 + D_{-1}^2 + D_1^2 = \frac{1}{4} + \frac{1}{16} + \frac{1}{16} = \frac{3}{8}$$

Example 2.9 Find the exponential Fourier series for the signal shown in Fig. 2.7.

(*Anna University, December 2007*)

Solution

$$\begin{aligned}
x(t) &= \cos t \\
T_0 &= 10 \\
\omega_0 &= \frac{2\pi}{T_0} = 0.2\pi \\
D_n &= \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 nt} dt \\
&= \frac{1}{10} \int_{-\pi/2}^{\pi/2} \cos t \, e^{-j\omega_0 nt} dt \\
&= \frac{1}{20} \int_{-\pi/2}^{\pi/2} (e^{jt} + e^{-jt}) e^{-j0.2n\pi t} dt
\end{aligned}$$

$$= \frac{1}{20}\left[\int_{-\pi/2}^{\pi/2} e^{j(1-.2n\pi)t}dt + \int_{-\pi/2}^{\pi/2} e^{-j(1+.2n\pi)t}dt\right]$$

$$= \frac{1}{20}\left\{\frac{1}{j(1-.2n\pi)}\left[e^{j(1-.2n\pi)t}\right]_{-\pi/2}^{\pi/2} - \frac{1}{j(1+.2n\pi)}\left[e^{-j(1+.2n\pi)t}\right]_{-\pi/2}^{\pi/2}\right\}$$
$$= \frac{1}{20}\left\{\frac{1}{j(1-.2n\pi)}\left[e^{j\frac{\pi}{2}(1-.2n\pi)} - e^{-j\pi/2(1-.2n\pi)}\right]\right.$$
$$\left. - \frac{1}{j(1+.2n\pi)}\left[e^{-j\pi/2(1+.2n\pi)} - e^{-j\pi/2(1+.2n\pi)}\right]\right\}$$
$$= \frac{1}{10}\left[\frac{1}{(1-.2n\pi)}\sin\frac{\pi}{2}(1-.2n\pi) + \frac{1}{(1+.2n\pi)}\sin\frac{\pi}{2}(1+.2n\pi)\right]$$
$$= \frac{1}{10(1-.04n^2\pi^2)}\left[(1+.2n\pi)\cos 0.1n\pi^2 + (1-.2n\pi)\cos(0.1n\pi^2)\right]$$

$$D_n = \frac{0.2\cos 0.1n\pi^2}{(1-0.04n^2\pi^2)}$$
$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j0.2\pi nt}$$

Example 2.10 Consider the wave form shown in Fig. 2.8. Determine the complex exponential Fourier series.

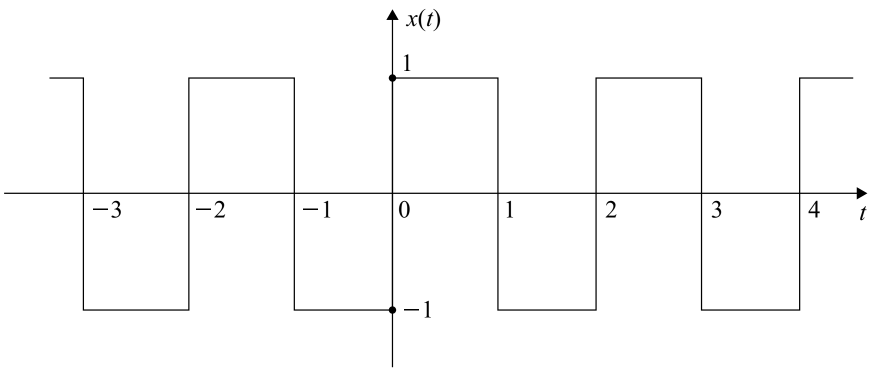


Fig. 2.8 Signal of Example 2.10

Solution 1. From Fig. 2.8, $T_0 = 2$ and $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2} = \pi$.
2.

$$\begin{aligned}
D_n &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jn\omega_0 t}dt \\
&= \frac{1}{2}\int_0^1 e^{-jn\pi t}dt - \frac{1}{2}\int_1^2 e^{-jn\pi t}dt \\
&= \frac{1}{2}\frac{1}{(-jn\pi)}\left[e^{-jn\pi t}\right]_0^1 - \frac{1}{2(-jn\pi)}\left[e^{-jn\pi t}\right]_1^2 \\
&= \frac{1}{2(-jn\pi)}\left[e^{-jn\pi} - 1 - e^{-jn\pi 2} + e^{-jn\pi}\right] \\
&= \frac{1}{-2n\pi j}\left[2e^{-jn\pi} - 2\right] \qquad \left[\because\ e^{-jn\pi 2} = 1\right] \\
&= \frac{1}{jn\pi}\left[1 - e^{-jn\pi}\right] \\
&= \frac{1}{jn\pi}\left[1 - \cos n\pi\right]
\end{aligned}$$

$$D_n = \frac{2}{jn\pi}$$

where n is an odd number.
3.

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\pi t}$$

$$x(t) = \frac{2}{j\pi}\sum_{m=-\infty}^{\infty}\frac{1}{2m+1}e^{j(2m+1)\pi t}$$

where m is any integer which will be equivalent to n being odd integer.

Example 2.11 Let

$$x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \end{cases}$$

be a periodic signal with fundamental period $T_0 = 2$ and Fourier coefficients a_k.

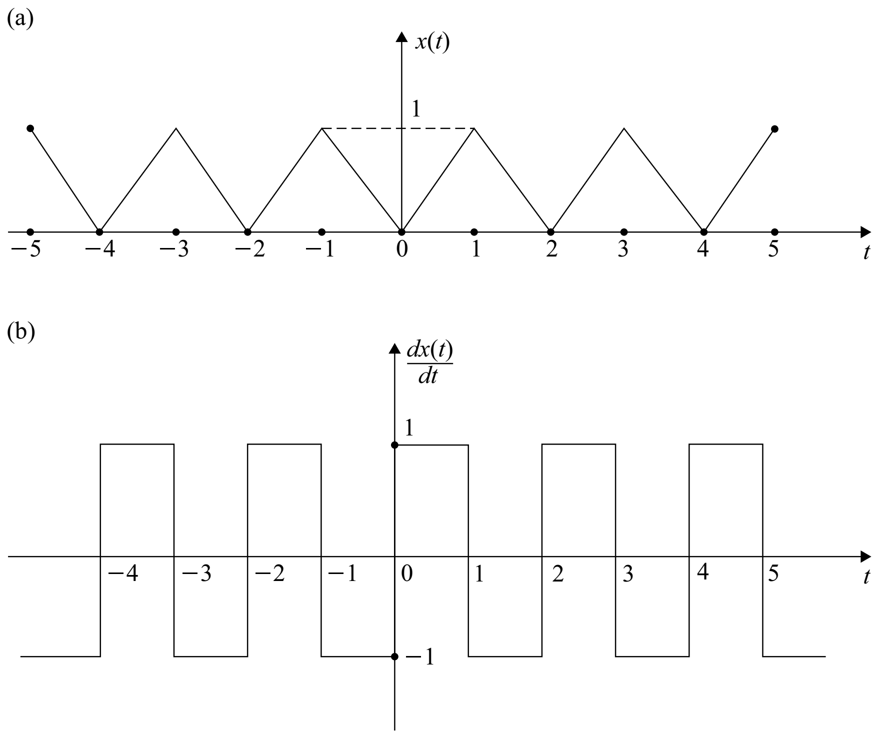


Fig. 2.9 **a** A triangular wave and **b** Derivative of triangular wave

(a) Determine the value of a_0.
(b) Determine the Fourier series representation of $\frac{dx(t)}{dt}$.
(c) Use the result of part (b) and the differentiation property of FS to help determine the Fourier series coefficients of $x(t)$.

(*Anna University, May 2008*)

Solution (a)

$$x(t) = \begin{cases} t & 0 \le t \le 1 \\ 2 - t & 1 \le t \le 2 \end{cases}$$

The above equation represents a triangle in the given time interval and the periodical signal with period $T_0 = 2$ is shown in Fig. 2.9.

$$\omega_0 = \frac{2\pi}{T_0} = \pi$$

The Fourier series coefficient a_0 is determined as follows:

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$
$$= \frac{1}{2}\int_0^1 t\,dt + \int_1^2 (2-t)dt$$
$$= \frac{1}{2}\left[\frac{t^2}{2}\right]_0^1 + \left[2t - \frac{t^2}{2}\right]_1^2$$
$$= \frac{1}{4} + \frac{1}{2}\left[4 - 2 - 2 + \frac{1}{2}\right]$$

$$a_0 = \frac{1}{2}$$

(b) Differentiating the given $x(t)$ we get

$$\frac{dx(t)}{dt} = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 \le t \le 2 \end{cases}$$

This is the square wave and is shown in Fig. 2.9b. Figures 2.8 and 2.9b are the rectangular waves with the amplitude and periodicity. The exponential FS coefficient of Fig. 2.8 has been determined as

$$D_n = \frac{2}{jn\pi} \quad \text{where } n \text{ is an odd integer}$$
$$= \frac{2}{j(2m+1)\pi} \quad \text{where } m \text{ is any integer}$$

$$\frac{dx(t)}{dt} = \dot{x}(t) = \frac{2}{j\pi}\sum_{m=-\infty}^{\infty}\frac{1}{(2m+1)}e^{j(2m+1)\pi t}$$

(c) $x(t)$ in the Fourier exponential form can be written as follows:

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$
$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} (jn\omega_0) D_n e^{jn\omega_0 t}$$

From the result derived in part (b),

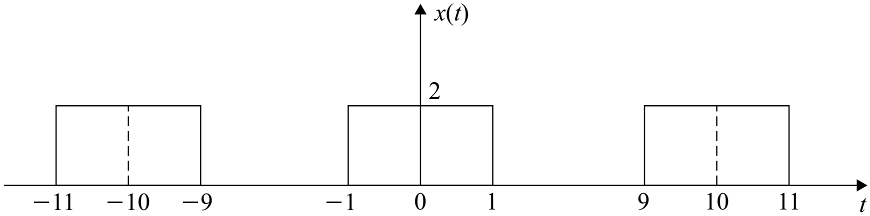


Fig. 2.10 Signal of Example 2.12

$$jn\omega_0 D_n = \frac{2}{jn\pi}$$

$$D_n = \frac{-2}{n^2\pi^2}$$

where n is an odd integer.

$$x(t) = D_0 + \sum_{n=-\infty}^{\infty} D_n e^{jn\pi t}$$
$$D_0 = a_0 = \frac{1}{2}$$
$$n = 2m + 1 \qquad \text{where } m \text{ is any integer}$$

$$x(t) = \frac{1}{2} - \frac{2}{\pi^2} \sum_{m=-\infty}^{\infty} \frac{1}{(2m+1)^2} e^{j(2m+1)\pi t}$$

Example 2.12 For the signal shown in Fig. 2.10. Determine the exponential Fourier series.

Solution

$$T_0 = 10$$
$$\omega_0 = \frac{2\pi}{10} = \frac{\pi}{5}$$
$$D_n = \frac{1}{T_0} \int_{-1}^{1} 2e^{-j\omega_0 nt} dt = \frac{2}{10} \int_{-1}^{1} e^{-j\frac{\pi}{5}nt} dt$$

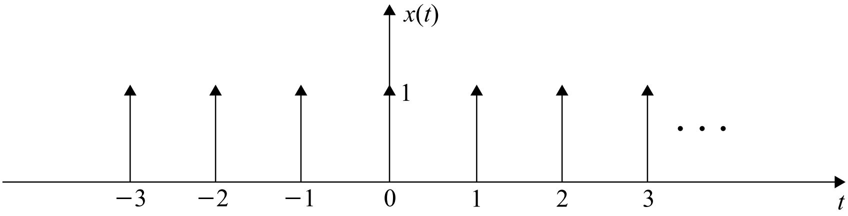


Fig. 2.11 Periodic train of impulses

$$= -\frac{1}{5}\frac{5}{\pi jn}\left[e^{-j\frac{\pi n}{5}t}\right]_{-1}^{1}$$
$$= -\frac{1}{j\pi n}\left[e^{-j\frac{\pi n}{5}} - e^{+j\frac{n\pi}{5}}\right]$$

$$D_n = \frac{2}{\pi n}\sin\frac{\pi n}{5} \quad \text{for all } n \text{ but } n \neq 0$$

For $n = 0$,

$$D_0 = \underset{n\to 0}{Lt}\,\frac{2}{5}\frac{\sin\frac{\pi n}{5}}{\frac{\pi n}{5}}$$
$$D_0 = \frac{2}{5} = 0.4$$

$$x(t) = 0.4 + \frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{n}\sin\frac{\pi}{5}ne^{\frac{-j\pi nt}{5}}$$

Example 2.13 Determine the exponential and trigonometric Fourier series of a train of impulse with periodicity $T_0 = 1$. Verify the exponential and trigonometric coefficients relationship (Fig. 2.11).

Solution

$$T_0 = 1 \qquad \text{and} \qquad \omega_0 = 2\pi$$

To determine the exponential FS coefficients

$$D_n = \frac{1}{T_0}\int_0^{T_0}\delta(t)\,e^{-jn\omega_0 t}dt = \frac{1}{T_0}\int_{-1/2}^{1/2}\delta(t)\,e^{-j2\pi nt}dt$$

Over this interval, $D_n = \frac{1}{T_0}$

$$D_n = \frac{1}{T_0} = 1$$
$$D_0 = 1$$

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$x(t) = \sum_{n=-\infty}^{\infty} e^{j2\pi nt}$$

To determine the trigonometric Fourier series

$$a_0 = \frac{1}{T}\int_0^{T_0} \delta(t)\, dt$$
$$a_0 = \frac{1}{T_0} = 1$$

Since the train of impulses is an even signal $b_n = 0$.

$$a_n = \frac{2}{T_0}\int_0^{T_0} \delta(t) \cos n\omega_0 t\, dt$$
$$= \frac{2}{T_0} = 2$$
$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$x(t) = 1 + \sum_{n=1}^{\infty} 2\cos 2\pi nt$$

$$a_0 = D_0 = 1$$
$$D_n = \frac{a_n}{2} = \frac{2}{2} = 1$$

Thus, the relationships between trigonometric and exponential Fourier series coefficients are verified.

Example 2.14 For the periodic signal $x(t) = e^{-t}$ with a period $T_0 = 1$ second, find the Fourier series in

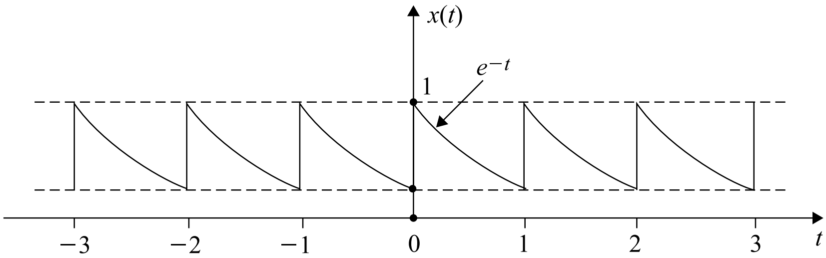


Fig. 2.12 Exponentially decaying periodic signal

(a) Exponential form,
(b) Trigonometric form,
(c) Polar form, and
(d) Verify the relationships of FS coefficients.

Solution (a) **Exponential Fourier series**

$$
\begin{aligned}
T_0 &= 1 \\
\omega_0 &= \frac{2\pi}{T_0} = 2\pi \\
D_n &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-jn\omega_0 t}dt \\
&= \int_0^1 e^{-t}e^{-jn2\pi t}dt \\
&= \int_0^1 e^{-(1+j2\pi n)t}dt \\
&= -\frac{1}{(1+j2\pi n)}\left[e^{-(1+j2\pi n)t}\right]_0^1 \\
&= \frac{1}{(1+j2\pi n)}\left[1-e^{-(1+j2\pi n)}\right] \\
&= \frac{1}{(1+j2\pi n)}\left[1-e^{-1}\right] \qquad \left[\because\ e^{-j2\pi n}=1\right]
\end{aligned}
$$

$$D_n = \frac{0.632}{(1+j2\pi n)}; \quad |D_n| = \frac{0.632}{\sqrt{1+4\pi^2 n^2}}$$

$$D_0 = 0.632$$

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$x(t) = 0.632 \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{(1 + 4\pi^2 n^2)}} e^{j2\pi nt}$$

(b) **Trigonometric Fourier series**

$$\begin{aligned} a_0 &= \frac{1}{T_0} \int_0^{T_0} x(t) dt \\ &= \int_0^1 e^{-t} dt \\ &= -\left[e^{-t}\right]_0^1 \\ &= (1 - e^{-1}) \end{aligned}$$

$$a_0 = 0.632$$

$$\begin{aligned} a_n &= \frac{2}{T_0} \int_0^{T_0} x(t) \cos \omega_0 nt \, dt \\ &= 2 \int_0^1 e^{-t} \cos 2\pi nt \, dt \end{aligned}$$

Using the property

$$\int_a^b e^{at} \cos bt \, dt = \left[\frac{e^{at}(a \cos bt + b \sin bt)}{(a^2 + b^2)}\right]_a^b$$

$$\begin{aligned} a_n &= \frac{2}{(1 + 4\pi^2 n^2)} \left[- \cos 2\pi nt (e^{-t}) + e^{-t} 2\pi n \sin 2\pi n\right]_0^1 \\ &= \frac{2}{(1 + 4\pi^2 n^2)} \left[e^{-1} \{- \cos 2\pi n + 2\pi n \sin 2\pi n\} + 1\right] \\ &= \frac{2}{(1 + 4\pi^2 n^2)} [1 - e^{-1}] \end{aligned}$$

$$a_n = \frac{1.264}{(1 + 4\pi^2 n^2)}$$

$$b_n = \frac{2}{T_0} \int_0^{To} x(t) \sin \omega_0 nt \, dt$$
$$= 2 \int_0^1 e^{-t} \sin 2\pi nt \, dt$$

Using the property

$$\int_a^b e^{at} \sin bt \, dt = \frac{1}{(a^2 + b^2)} \left\{ e^{at} [a \sin bt - b \cos bt] \right\}_a^b$$

we get,

$$b_n = \frac{2}{\left(1 + 4\pi^2 n^2\right)} \left\{ e^{-t} \left[-\sin 2\pi nt - 2\pi n \cos 2\pi nt \right] \right\}_0^1$$
$$= \frac{2}{\left(1 + 4\pi^2 n^2\right)} \left[-e^{-1} (\sin 2\pi n + 2\pi n \cos 2\pi n) + 2\pi n \right]$$
$$= \frac{4\pi n}{\left(1 + 4\pi^2 n^2\right)} (1 - e^{-1})$$

$$b_n = \frac{2.53\pi n}{1 + 4\pi^2 n^2}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \omega_0 nt + \sum_{n=1}^{\infty} b_n \sin \omega_0 nt$$

$$x(t) = 0.632 + 1.264 \sum_{n=1}^{\infty} \frac{n}{\left(1 + 4\pi^2 n^2\right)} \cos 2\pi nt$$
$$+ 2.53\pi \sum_{n=1}^{\infty} \frac{n}{\left(1 + 4\pi^2 n^2\right)} \sin 2\pi nt$$

(c) **Polar form Fourier series (cosine form FS)**

$$\begin{aligned} x(t) &= C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_0 nt - \theta_n) \\ C_0 &= a_0 \\ C_n &= \sqrt{a_n^2 + b_n^2} \\ \theta_n &= \tan^{-1}\left(\frac{b_n}{a_n}\right) \end{aligned}$$

$$C_0 = 0.632$$

$$C_n = \frac{\sqrt{(1.6 + 6.4\pi^2 n^2)}}{(1 + 4\pi^2 n^2)};$$

$$C_n = \frac{1.265}{\sqrt{1 + 4\pi^2 n^2}}$$

$$\begin{aligned} \theta_n &= \tan^{-1} \frac{2.53\pi n}{1.264} = \tan^{-1} 2\pi n = 0 \\ x(t) &= 0.632 + \sum_{n=1}^{\infty} \frac{\sqrt{1.6 + 6.4\pi^2 n^2}}{(1 + 4\pi^2 n^2)} \cos[2\pi nt] \end{aligned}$$

$$x(t) = 0.632 + \sum_{n=1}^{\infty} \frac{1.265}{\sqrt{(1 + 4\pi^2 n^2)}} \cos 2\pi nt$$

(d) 1. $a_0 = C_0 = D_0 = 0.632$
   2.

$$|D_n| = \frac{C_n}{2} = \frac{1.265}{2\sqrt{1 + 4\pi^2 n^2}} = \frac{0.632}{\sqrt{1 + 4\pi^2 n^2}}$$

3.

$$\begin{aligned} C_n &= \sqrt{a_n^2 + b_n^2} \\ &= \frac{\sqrt{(1.264)^2 + 2.53^2\pi^2 n^2}}{(1 + 4\pi^2 n^2)} \\ &= \frac{\sqrt{1.6(1 + 4\pi^2 n^2)}}{(1 + 4\pi^2 n^2)} \\ &= \frac{1.265}{\sqrt{(1 + 4\pi^2 n^2}} \end{aligned}$$

2.5 Existence of Fourier Series—The Dirichlet Conditions

The continuous Fourier series of the signal $x(t)$ is represented in the following form:

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2\pi nt} \tag{2.36}$$

where

$$D_n = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi nt} dt \tag{2.37}$$

and n represents the harmonic member.

If the integral in Eq. (2.37) diverges, CTFS cannot be found for $x(t)$. If certain constraints are put on $x(t)$, Eq. (2.37) converges and the conditions are called Dirichlet conditions. The Dirichlet conditions are

1. The signal $x(t)$ must be absolutely integrable over the time interval $t_0 < t < t_0 + T_0$. The above condition implies that

$$\int_{t_0}^{t_0+T_0} |x\,(t)|\, dt < \infty \tag{2.38}$$

2. The signal $x(t)$ must have a finite number of maxima and minima in the time interval $t_0 < t < t_0 + T_0$.
3. The signal $x(t)$ must have finite number of discontinuities in the time interval $t_0 < t < t_0 + T_0$.

2.6 Convergence of Continuous-Time Fourier Series

The arbitrary signal $x(t)$ can be expressed by FS in Eq. (2.4) if it is periodic. It does not mean that every periodic signal can be expressed by FS. When the series uses a fixed number of terms, then it guarantees convergence. If the energy difference between the signal $x(t)$ and the corresponding finite term series approaches zero as the number of terms approaches infinity, such a series is said to be convergent in the mean. The Fourier series of $x(t)$ converges in the mean if it has a finite energy over one period. This can be expressed as

$$E = \int_{T_0} |x(t)|^2 dt < \infty \tag{2.39}$$

When condition (2.39) is satisfied, the Fourier series converges in the mean and also guarantees that the Fourier coefficients are finite.

2.7 Fourier Series Spectrum

The plot of Fourier series coefficients with respect to ω is called Fourier series spectrum. In exponential Fourier series and in polar Fourier series, the Fourier series, the FS coefficients D_n and C_n are complex. Thus, these coefficients have magnitude and angle. Thus, the plots of D_n *versus* ω and $\angle D_n$ *verses* ω are called exponential Fourier spectra. Similarly the plots of $|C_n|$ *versus* ω and $\angle C_n$ *versus* ω are called trigonometric Fourier spectra. The following examples illustrate the above methods.

Example 2.15 For Example 2.14, plot the exponential Fourier spectra for the periodic signal $x(t)$ shown in Fig. 2.12.

Solution The exponential Fourier series coefficient of Fig. 2.12 has been derived as

$$D_n = \frac{0.632}{1 + j2\pi n} = \frac{0.632}{\sqrt{1 + 4\pi^2 n^2}} \angle - \tan^{-1} 2\pi n$$

For $n = 0$,

$$D_0 = 0.632\angle 0^\circ$$

For $n = \pm 1$,

$$D_1 = D_{-1} = 0.1\angle \mp 81^\circ$$
$$D_2 = D_{-2} = 0.05\angle \mp 85.5^\circ$$
$$D_3 = D_{-3} = 3.35 \times 10^{-2}\angle \mp 87^\circ$$

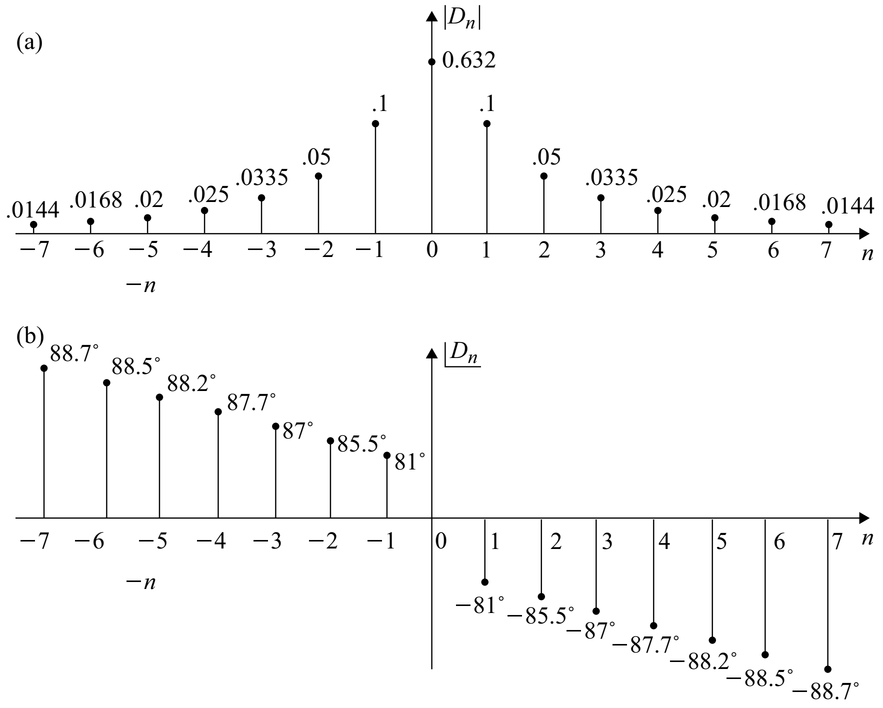


Fig. 2.13 Frequency spectra of Example 2.15. **a** Magnitude spectrum and **b** Phase angle spectrum

$$D_4 = D_{-4} = 2.5 \times 10^{-2} \angle \mp 87.7^\circ$$
$$D_5 = D_{-5} = 2 \times 10^{-2} \angle \mp 88.2^\circ$$
$$D_6 = D_{-6} = 1.68 \times 10^{-2} \angle \mp 88.5^\circ$$
$$D_7 = D_{-7} = 1.44 \times 10^{-2} \angle \mp 88.7^\circ$$

The magnitude spectrum of D_n is shown in Fig. 2.13a and the phase spectrum is Fig. 2.13b. **Note:** $\omega = n\omega_0 = 2\pi n$ or $n = \frac{\omega}{2\pi}$ which is a function of frequency.

Summary

1. Any arbitrary periodic signal $x(t)$ can be represented in the form of a linear combination of complex sinusoids. Such a representation is called Fourier series. The higher frequency sines and cosines have frequencies that are integer multiples of the fundamental frequency.
2. The Fourier series can be represented in any one of the following forms:

   (a) Trigonometry form.

   (b) Complex exponential form.
   (c) Polar or harmonic or cosine form.

   The coefficients of the above forms have definite relationships between them.
3. The Fourier series possesses the following properties:

   (a) Linearity,
   (b) Time shifting,
   (c) Time reversal,
   (d) Time scaling,
   (e) Multiplication,
   (f) Conjugation,
   (g) Differentiation, and
   (h) Integration.

4. Parseval's theorem on Fourier series states that the total average power in a periodic signal is the sum of the average powers in all its components which is the sum of the squared value of Fourier series coefficients.
5. Dirichlet showed that if $x(t)$ satisfies certain conditions, the Fourier series of $x(t)$ is guaranteed. These conditions are called Dirichlet conditions.
6. The magnitude and phase angle of Fourier series coefficients plotted versus frequency ω are called Fourier spectra of the signal $x(t)$.
7. The exponential form of Fourier series representation is better preferred compared to other forms because it is more compact and the system response is also simpler.

Exercises

I. Short Answer Type Questions

1. **What is a Fourier series?**
   Any arbitrary periodic signal $x(t)$ can be expressed as a sum of sinusoids and all its harmonics. Such an infinite series is known as Fourier series.
2. **What are the different forms of representing Fourier series?**
   The different forms of representing Fourier series are

   (a) Trigonometric Fourier series.
   (b) Polar (compact or cosine form) Fourier series.
   (c) Exponential form Fourier series.

3. **Give mathematical expression for trigonometric Fourier series?**

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$
$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t dt$$
$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t dt$$

a_0, a_n, and b_n are called the coefficients of trigonometric Fourier series.

4. **What is the effect of symmetry in trigonometric Fourier series?**
If $x(t)$ has an odd symmetry, $a_n = 0$. If $x(t)$ has even symmetry $b_n = 0$. If $x(t)$ is symmetrical with respect to the time axis, $a_0 = 0$.
5. **What is half wave symmetry?**
If the periodic signal $x(t)$ when shifted by half the period remains unchanged except for a sign, the signal is said to be half wave symmetry. Mathematically, it is expressed as

$$x\left(t - \frac{T_0}{2}\right) = -x(t)$$

For the signal with half wave symmetry, all the even numbered harmonics vanish.
6. **Give the mathematical expression for the cosine Fourier series.**

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos n(n\omega_0 t - \theta_n)$$

where

$$C_0 = a_0$$
$$C_n = \sqrt{a_n^2 + b_n^2}$$
$$\theta_n = \tan^{-1}\frac{b_n}{a_n}$$

7. **Give mathematical expression for the exponential Fourier series?**

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt}$$

where

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\omega_0 nt}dt$$

8. **How the coefficients of exponential Fourier series are related to the coefficients of trigonometric and cosine Fourier series?**

$$D_0 = a_0 = C_0$$
$$D_n = \frac{1}{2}\left[a_n - jb_n\right]$$
$$|D_n| = \frac{1}{2}C_n$$

9. **Why exponential Fourier series is preferred to represent the Fourier series?**
The exponential Fourier series is more compact and the system response to exponential signal is simpler.
10. **What do you understand by Fourier spectrum?**
The Fourier series expresses a periodic signal $x(t)$ as a sum of sinusoids of fundamental frequency ω_0 and their higher harmonics $2\omega_0,\ 3\omega_0, \ldots, n\omega_0$. Corresponding to these frequencies, the amplitudes and phases are determined. The plot of these amplitudes versus n which is proportional to $n\omega_0$ is termed as amplitude spectrum. The plot of phase angle θ_n versus n is called phase spectrum.
11. **What do you understand by existence of Fourier series?**
For the existence of Fourier series, its coefficients should exist. The existence of these coefficients is guaranteed iff $x(t)$ is absolutely integrable. In other words

$$\int_{T_0} |x(t)|\, dt < \infty$$

12. **What do you understand by convergence of Fourier series in the mean?**
The periodic signal $x(t)$ which has finite energy over one period guarantees the convergence in the mean of its Fourier series. Mathematically, it is expressed as

$$\int_{T_0} |x(t)|^2\, dt < \infty$$

13. **What are Dirichlet conditions?**
Fourier at the time of presenting his papers could not successfully defend the existence Fourier series which is infinite. He could not also give convincing reply when there is discontinuities in $x(t)$. The answers to these questions came from the great mathematician Dirichlet in the form of **certain constraints**. These constraints are called Dirichlet conditions and they are

(a) The function $x(t)$ must be absolutely integrable.
(b) The function $x(t)$ should have finite number discontinuities in one period.
(c) The function $x(t)$ should contain only a finite number of maxima and minima in one period.

14. **What do you understand by Parseval's theorem as applied to Fourier series?**
According to Parseval's theorem, the power of the periodic signal is equal to the sum of the powers of its Fourier coefficients

$$P = C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2 \qquad \text{(For cosine FS)}$$

$$P = \sum_{n=-\infty}^{\infty} |D_n|^2 \qquad \text{(Exponential FS)}$$

$$P = D_0^2 + 2\sum_{n=1}^{\infty} |D_n|^2 \qquad (x(t) = \text{real})$$

15. **What are differentiating and integrating properties of Fourier series?**
If a periodical signal $x(t)$ is differentiated the Fourier series coefficient gets multiplied by the factor $jn\omega_0$. Suppose D_n is the Fourier series coefficient of $x(t)$. Then the Fourier series coefficient of $\frac{dx(t)}{dt}$ is $j\omega_0 n D_n$. This is the differentiation property of Fourier series. If the periodic signal $x(t)$ is integrated, then the Fourier series coefficient gets divided by $j\omega_0 n$. If D_n is the coefficient of exponential Fourier series of $x(t)$, then the Fourier series coefficient of $\int_{T_0} x(t)dt$ is $\frac{1}{j\omega_0 n} D_n$.
This is called the integration property of Fourier series.

II. Long Answer Type Questions

1. **Determine the trigonometric and exponential Fourier series representation of the signal $x(t)$ shown in Fig. 2.14?**

$$T_0 = T; \qquad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{T}$$

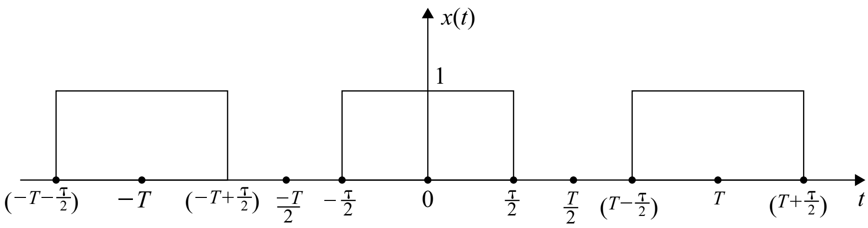


Fig. 2.14 Signal $x(t)$ for Problem 1

(a) **Trigonometric or quadratic Fourier series.**

$$
\begin{aligned}
a_0 &= \frac{\tau}{T} \\
b_n &= 0 \quad \text{since } x(t) \text{ is even} \\
a_n &= \frac{2}{n\pi}\sin\left(\frac{n\pi\tau}{T}\right) \\
x(t) &= \frac{\tau}{T} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi\tau}{T}\right)\cos n\frac{2\pi t}{T}
\end{aligned}
$$

(b) **Exponential Fourier series.**

$$
\begin{aligned}
D_n &= \frac{\tau}{2}\sin\mathrm{c}\left(\frac{n\pi\tau}{T}\right) \\
x(t) &= \sum_{n=1}^{\infty}\frac{\tau}{2}\sin\mathrm{c}\left(\frac{n\pi\tau}{T}\right)e^{jn\frac{2\pi t}{T}}
\end{aligned}
$$

2. **Consider the following signal:**

$$
\boldsymbol{x(t) = \cos\left(\frac{1}{3}t + 30^\circ\right) + \sin\left(\frac{2}{5}t + 60^\circ\right)}
$$

Determine (a) whether the signal is periodic, (b) find the fundamental period and frequency, (c) what harmonics are present in $x(t)$, (d) Determine the coefficients of exponential Fourier series, and (e) Determine the power of the signal using Parseval's theorem.

(a) The signal is periodic.
(b) The fundamental period $T_0 = 30\pi$ and the fundamental radian frequency $\omega_0 = \frac{1}{15}$.
(c) Fifth and sixth harmonics are present.
(d)

$$
\begin{aligned}
D_5 &= \frac{1}{4}[\sqrt{3} + j]; \quad D_{-5} = \frac{1}{4}[\sqrt{3} - j] \\
D_6 &= \frac{1}{4}[\sqrt{3} - j]; \quad D_{-6} = \frac{1}{4}[\sqrt{3} + j]
\end{aligned}
$$

(e)

$$
P = |D_5|^2 + |D_{-5}|^2 + |D_6|^2 + |D_{-6}|^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1
$$

3. **For the signal shown in Fig. 2.15, determine the coefficients of exponential Fourier series.**

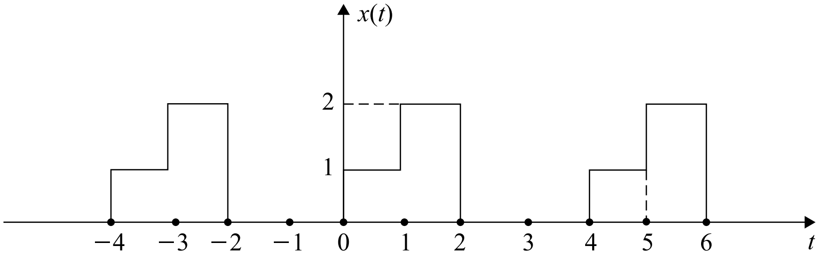


Fig. 2.15 Signal $x(t)$ for Problem 3

$$T_0 = 4; \quad \omega_0 = \frac{\pi}{2}; \quad D_0 = \frac{3}{4}$$
$$D_n = \frac{1}{jn\pi}\left[\frac{1}{2} - (-1)^n - \frac{1}{2}e^{-j\frac{n\pi}{2}}\right]$$

4. **Find the exponential Fourier series coefficients for the signal shown in Fig. 2.16a and plot its amplitude and phase spectrum.**

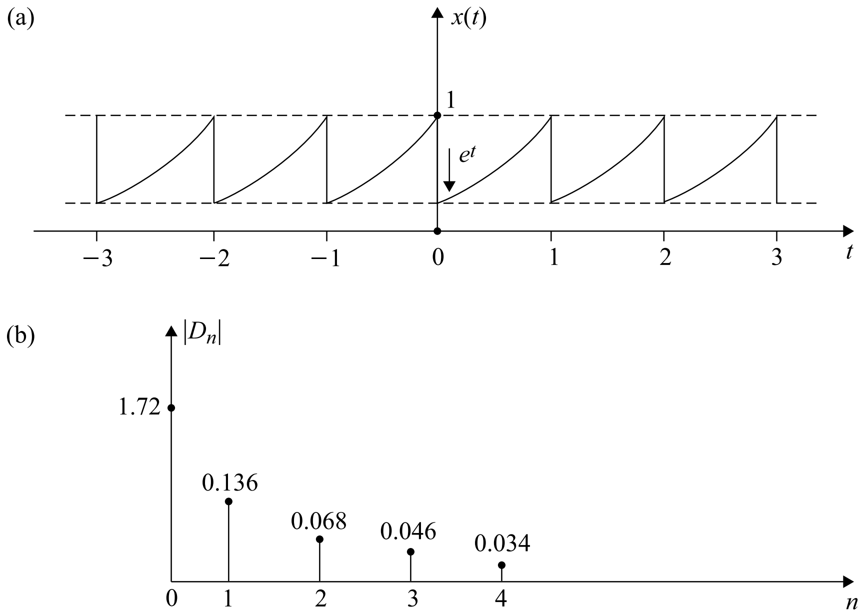


Fig. 2.16 **a** $x(t)$ signal and **b** Amplitude spectrum of D_n

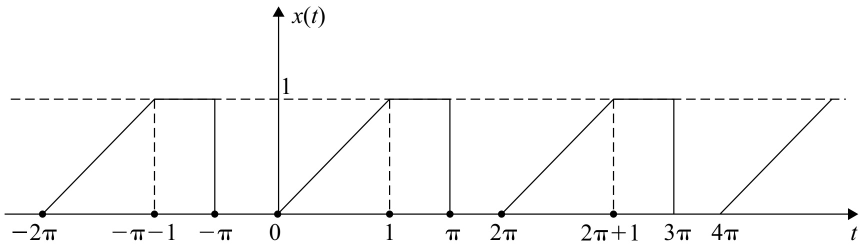


Fig. 2.17 Signal of Problem 5

$$T_0 = 1; \quad \omega_0 = 2\pi$$
$$D_n = \frac{1.72}{\sqrt{1 + 4\pi^2 n^2}}$$
$$\theta_n = 0$$

The amplitude spectrum is shown in Fig. 2.16b.

5. **Consider the signal shown in Fig. 2.17. Determine the exponential Fourier series coefficients.**

$$D_0 = \frac{(2\pi - 1)}{4\pi}$$
$$D_n = \frac{1}{2\pi n^2}\left[e^{-jn} - 1\right].$$

Chapter 3
Fourier Transform Analysis of Continuous Time Signals

Chapter Objectives

- To define the Fourier transform for continuous time signal which is aperiodic.
- To derive the properties of Fourier transform and demonstrate with examples.
- To find the magnitude and phase angle spectrum of Fourier transform.
- To solve the differential equation by partial fraction method using Fourier transform (FT).

3.1 Introduction

In Chap. 2, periodic signals were represented as a sum of **everlasting sinusoids or exponentials**. The Fourier series method of analysis of such periodic signals is indeed a very powerful tool. However, FS fails when applied to aperiodic signals. To overcome this major limitation, an aperiodic signal $x(t)$ is expressed as a continuous sum (integral) of **everlasting exponentials**. Such a representation is called Fourier integral which is basically a Fourier series with fundamental frequency tending to zero. By such representation the aperiodic signal $x(t)$ in the time domain is transformed to $X(j\omega)$ in the frequency domain. The transformations from $x(t)$ to $X(j\omega)$ and from $X(j\omega)$ to $x(t)$ are called Fourier transform and inverse Fourier transform respectively. They are also called Fourier transform pairs.

S. Palani, *Basic System Analysis*,
https://doi.org/10.1007/978-3-031-28280-5_3

3.2 Representation of Aperiodic Signal by Fourier Integral–The Fourier Transform

If an aperiodic signal is viewed as a periodical signal with an infinite period, then it can be represented by the Fourier series. In such a situation, as the period increases, the fundamental frequency decreases, and the frequency components become closer. Now the Fourier series sum becomes integral.

Consider the periodic signal $x(t)$ defined as follows:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < \frac{T}{2} \end{cases}$$

The above signal is represented as a periodic square wave in Fig. 3.1. The exponential Fourier series coefficients D_n can be determined as

$$D_n = \frac{2\sin(n\omega_0 T_1)}{(n\omega_0 T)} \tag{3.1}$$

where $\omega_0 = \frac{2\pi}{T}$. The Fourier series coefficient TD_n is obtained as

$$TD_n = \frac{2\sin(n\omega_0 T_1)}{(n\omega_0)} \tag{3.2}$$

For a fixed value of T_1, the plot of TD_n represents a sinc function. Equation (3.2) is plotted for $2\omega_0$, $4\omega_0$ and $8\omega_0$ and they are represented in Fig. 3.2a–c respectively.

From Fig. 3.2, it is evident that as T increases (the fundamental frequency $\omega_0 = \frac{2\pi}{T}$ decreases) the samples of TD_n become closer and closer. As T becomes very large, the original periodic square wave becomes a rectangular pulse. As $T \to \infty$, TD_n becomes continuous.

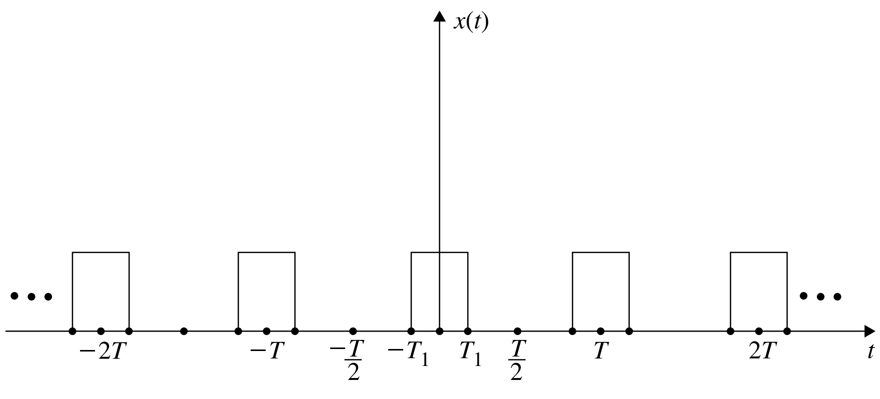


Fig. 3.1 A continuous time periodic square wave

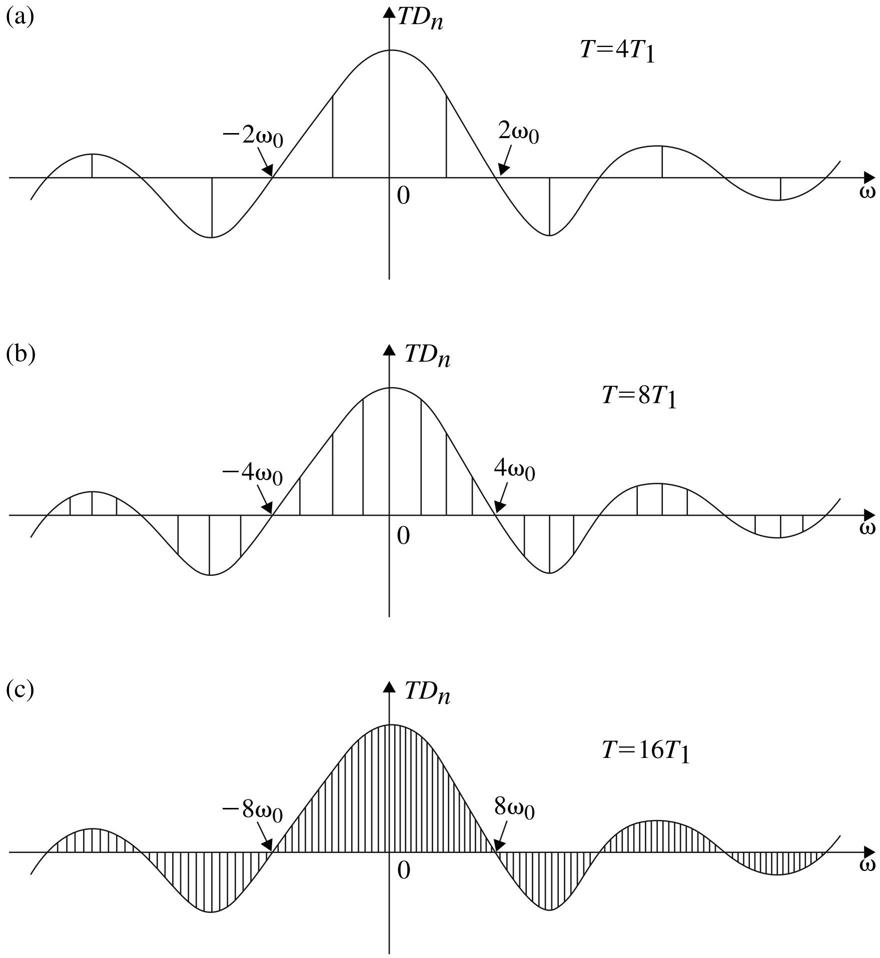


Fig. 3.2 Fourier series coefficients for different values of T

Let $\bar{x}(t)$ be non-periodic square wave as represented in Fig. 3.3.

$$\bar{x}(t) = 0 \qquad |t| > T_1$$

The periodic signal $x(t)$ formed by repeating $\bar{x}(t)$ with fundamental period T is shown in Fig. 3.1. If $T \to \infty$

$$\underset{T\to\infty}{Lt}\, x(t) = \bar{x}(t).$$

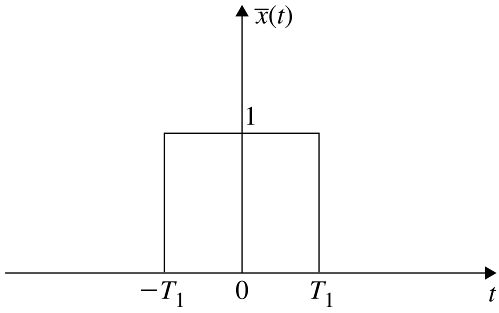


Fig. 3.3 A continuous time aperiodic square wave

The Fourier series coefficients of periodical signal are written as (Fig. 3.1)

$$D_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_0 t}\,dt \tag{3.3}$$

The periodical signal $x(t)$ can be expressed in the Fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} D_n\, e^{jn\omega_0 t} \tag{3.4}$$

$$Tx(t) = \sum_{n=-\infty}^{\infty} TD_n\, e^{jn\omega_0 t} \tag{3.5}$$

Let

$$\begin{aligned} X(n\omega_0) &= TD_n \\ &= \int_{-T/2}^{T/2} x(t)e^{-jn\omega_0 t}\,dt \end{aligned}$$

$$\begin{aligned} x(t) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} TD_n\, e^{jn\omega_0 t} \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(n\omega_0)e^{jn\omega_0 t}\omega_0 \end{aligned} \tag{3.6}$$

As $T \to \infty$, $\omega_0 = \frac{2\pi}{T} \to 0$ and $n\omega_0 = \omega$ which is continuous. Further, the summation in Equation (3.6) becomes integration. Thus, Equation (3.6) is written as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \quad \text{for all } \omega \tag{3.7}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{for all } t \tag{3.8}$$

Equations (3.7) and (3.8) are called Fourier transform pair. Equation (3.7) transforms the time function $x(t)$ to frequency function $X(j\omega)$ and so it is called Fourier transform. Equation (3.8) converts the frequency function to time function, and hence, it is called inverse Fourier transform. These transformations are also denoted as given below:

$$\begin{aligned} X(j\omega) &= F[x(t)] \\ x(t) &\stackrel{\text{FT}}{\longleftrightarrow} X(j\omega) \\ x(t) &= F^{-1}[X(j\omega)] \\ X(j\omega) &\stackrel{\text{IFT}}{\longleftrightarrow} x(t) \end{aligned} \tag{3.9}$$

Note: The time function $x(t)$ is always denoted by lower case letter and the frequency function $X(j\omega)$ by capital letter. Further, when $x(t)$ is Fourier transformed, it becomes complex and so it is denoted as $X(j\omega)$. In some literature, $X(j\omega)$ is also represented simply as $X(\omega)$.

3.3 Convergence of Fourier Transforms–The Dirichlet Conditions

As in the case of continuous time periodic signals, the following conditions (Dirichlet Conditions) are sufficient for the convergence of $X(j\omega)$.

1. $x(t)$ is absolutely integrable or square integrable. That is

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$
$$\int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty$$

2. $x(t)$ should have finite number of maxima and minima within any finite interval.
3. $x(t)$ has a finite number of discontinuities within any finite interval.

However, signals which do not satisfy these conditions can have Fourier transforms if impulse functions are included in the transform.

3.4 Fourier Spectra

The Fourier transform of $X(j\omega)$ of $x(t)$ is in general, complex and can be expressed as

$$X(j\omega) = |X(j\omega)| \; \underline{\big|X(j\omega)}$$

The plot of $|X(j\omega)|$ versus ω is called magnitude spectrum of $X(j\omega)$. The plot of $\underline{\big|X(j\omega)}$ versus ω is called phase spectrum. The amplitude (magnitude) and phase spectra are together called Fourier spectrum which is nothing but the frequency response of $X(j\omega)$ for the frequency range $-\infty < \omega < \infty$.

3.5 Connection Between the Fourier Transform and Laplace Transform

By definitions,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \tag{3.10}$$

and the Laplace transform is given by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt. \tag{3.11}$$

From Equations (3.10) and (3.11), it is observed that the Fourier transform is a special case of the Laplace transform in which $s = j\omega$. Substituting $s = \sigma + j\omega$ in Equation (3.11) we get

$$\begin{aligned} X(\sigma + j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt \\ &= \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt \\ &= F[x(t)e^{-\sigma t}] \end{aligned}$$

Thus, the bilateral Laplace transform of $x(t)$ is nothing but the Fourier transform of $x(t)e^{-\sigma t}$.

Note: The statement that Fourier transform can be obtained from Laplace transform by replacing s by $j\omega$ is true only if $x(t)$ is absolutely integrable. If $x(t)$ is not absolutely integrable the above statement is erroneous.

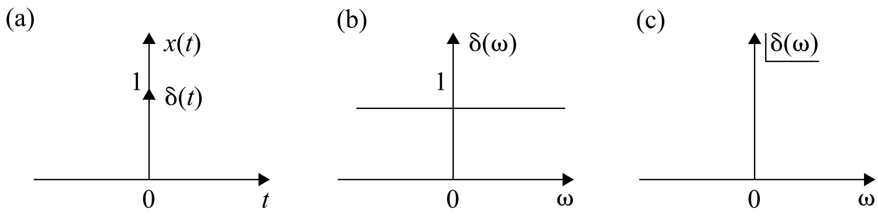


Fig. 3.4 Representation of $\delta(t)$ and its spectra

The following examples illustrate the method of finding the Fourier transform of non-periodic signals:

Example 3.1 Find the Fourier transform of the following time functions and sketch their Fourier spectra (amplitude and phase).

(a) $x(t) = \delta(t)$
(b) $x(t) = \text{sgn}\,(t)$
(c) $x(t) = 1 \quad \text{for all } t$
(d) $x(t) = u(t)$ and $x(t) = u(-t)$
(e) $x(t) = e^{-at}u(t); \quad a > 0$
(f) $x(t) = e^{-|a|t}; \quad a > 0$
(g) $x(t) = e^{at}u(t); \quad a > 0$
$x(t) = e^{at}u(-t)$

Solution

(a) $\boldsymbol{x(t) = \delta(t)}$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt$$

$$= 1 \qquad [\delta(t) = 0 \quad \text{for } t \neq 0$$
$$= 1 \quad \text{for } t = 0]$$

$$\delta(t) \overset{\text{FT}}{\longleftrightarrow} 1$$

Fourier Spectra of $\boldsymbol{\delta(t)}$

$\delta(j\omega) = 1$ which is independent of frequency. Hence, the amplitude spectrum is constant at all ω and the phase spectrum is zero at all ω. $\delta(t)$ and its Fourier spectra are shown in Fig. 3.4a–c respectively.

(b) $\boldsymbol{x(t) = \operatorname{sgn}(t)}$

$$\operatorname{sgn}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

$$\begin{aligned} F[\operatorname{sgn}(t)] &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\ &= -\int_{-\infty}^{0} e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-j\omega t}\,dt \end{aligned}$$

The first integral on the right side of the above equation is not integrable. $x(t)$ is multiplied by $e^{-a|t|}$ and the limiting value of $a \to 0$ is considered.

$$F[e^{-a|t|}\operatorname{sgn}(t)] = \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt$$

$$\begin{aligned} F[e^{-a|t|}\operatorname{sgn}(t)] &= \int_{-\infty}^{0} -e^{(a-j\omega)t}\,dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt \\ &= \underset{a\to 0}{Lt}\left[\frac{-1}{a-j\omega}\left\{e^{(a-j\omega)t}\right\}_{-\infty}^{0} - \frac{1}{(a+j\omega)}\left\{e^{-(a+j\omega)t}\right\}_{0}^{\infty}\right] \\ &= \underset{a\to 0}{Lt}\left[\frac{-1}{(a-j\omega)} + \frac{1}{a+j\omega}\right] = \frac{1}{j\omega} + \frac{1}{j\omega} = \frac{2}{j\omega} \end{aligned}$$

$$\operatorname{sgn}(t) \overset{\text{FT}}{\longleftrightarrow} \frac{2}{j\omega}$$

Fourier Spectra of $\boldsymbol{\operatorname{sgn}(t)}$

$$X(j\omega) = \frac{2}{j\omega}\begin{cases} \dfrac{2}{\omega}\angle -90^\circ & \omega \geq 0 \\ \dfrac{2}{\omega}\angle 90^\circ & \omega < 0 \end{cases}$$

$x(t) = \operatorname{sgn}(t)$, $|X(j\omega)| = \frac{2}{\omega}$ and $\underline{|X(j\omega)}$ are represented in Fig. 3.5a–c respectively.

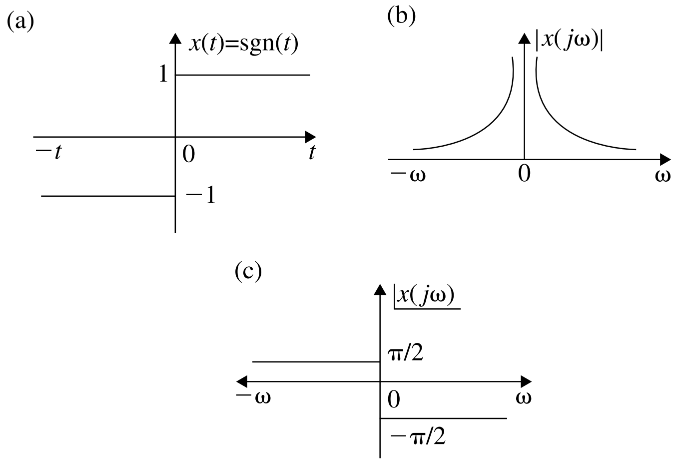


Fig. 3.5 Representation of sgn(t) and its spectra

(c) $\boldsymbol{x(t) = 1;}$ **for all** $\boldsymbol{t}$

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t}\, d\omega$$
$$= \frac{1}{2\pi} \qquad \delta(\omega) = \begin{cases} 1 & \omega = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{2\pi} \stackrel{\text{FT}}{\longleftrightarrow} = \delta(\omega)$$
$$1 \stackrel{\text{FT}}{\longleftrightarrow} = 2\pi\delta(\omega)$$

The above result shows that a constant signal $x(t) = 1$ for all t, when Fourier transformed becomes an impulse $2\pi\ \delta(\omega)$. $x(t)$ and $X(j\omega)$ are represented in Fig. 3.6a, b respectively.

(d) $\boldsymbol{x(t) = u(t)}$ **and** $\boldsymbol{x(t) = u(-t)}$

$$x(t) = \begin{cases} u(t) & \\ 1 & t \geq 0 \end{cases}$$

To find the FT of unit step $u(t)$ by direct integration yields an indeterminate value as is evident from the following equation because it has a jump discontinuity at $t = 0$.

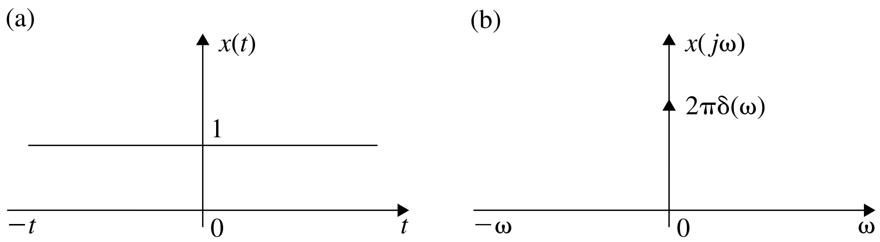


Fig. 3.6 Representation of $x(t) = 1$ and its FT

$$X(j\omega) = \int_0^{\infty} e^{-j\omega t}\, dt = -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_0^{\infty}$$

So, the problem is approached by considering $u(t)$ as

$$u(t) = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t)$$

Figure 3.7 represents $\frac{1}{2}\mathrm{sgn}(t)$ and $u(t)$

$$F[u(t)] = F\left[\frac{1}{2}\right] + \frac{1}{2}F\,\mathrm{sgn}(t)$$

$$F\left[\frac{1}{2}\right] = \pi\,\delta(\omega)$$

$$F\left[\frac{1}{2}\mathrm{sgn}(t)\right] = \frac{1}{j\omega}$$

$$F[u(t)] = \pi\,\delta(\omega) + \frac{1}{j\omega}$$

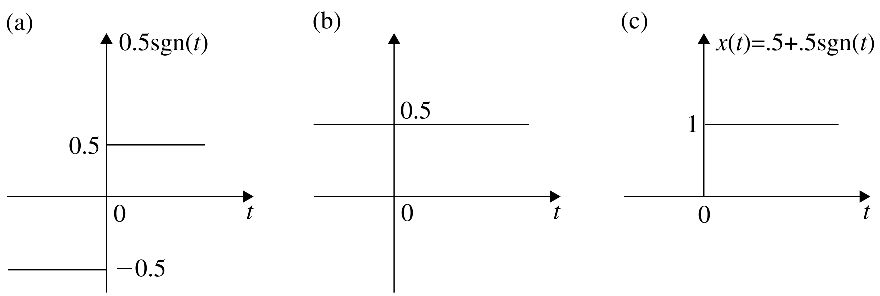


Fig. 3.7 Representation of $u(t)$ in terms of signum function

$$F[u(-t)] = X(-j\omega)$$

$$F[u(-t)] = \pi\,\delta(\omega) - \frac{1}{j\omega}$$

(e) $\boldsymbol{x(t) = e^{-at}u(t);\ a > 0}$

$$\begin{aligned} X(j\omega) &= \int_0^\infty e^{-at}e^{-j\omega t}\,dt = \int_0^\infty e^{-(a+j\omega)t}\,dt \\ &= -\frac{1}{(a+j\omega)}\left[e^{-(a+j\omega)t}\right]_0^\infty \end{aligned}$$

$$X(j\omega) = \frac{1}{(a+j\omega)}$$

$$\begin{aligned} |X(j\omega)| &= \frac{1}{\sqrt{a^2+\omega^2}} \\ \underline{|X(j\omega)} &= -\tan^{-1}\frac{\omega}{a} \end{aligned}$$

The signal $x(t)$, the amplitude spectrum $|X(j\omega)|$ and phase spectrum $\underline{|X(j\omega)}$ are shown in Fig. 3.8a–c respectively.

(f) $\boldsymbol{x(t) = e^{-a|t|};\ a > 0}$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^\infty x(t)e^{-j\omega t}\,dt \\ &= \int_{-\infty}^0 e^{at}e^{-j\omega t}\,dt + \int_0^\infty e^{-at}e^{-j\omega t}\,dt \\ &= \int_{-\infty}^0 e^{(a-j\omega)t}\,dt + \int_0^\infty e^{-(a+j\omega)t}\,dt \\ X(j\omega) &= \frac{1}{(a-j\omega)}\left[e^{(a-j\omega)t}\right]_{-\infty}^0 - \frac{1}{(a+j\omega)}\left[e^{-(a+j\omega)t}\right]_0^\infty \\ &= \frac{1}{(a-j\omega)} + \frac{1}{(a+j\omega)} \end{aligned}$$

$$X(j\omega) = \frac{2a}{a^2+\omega^2}$$

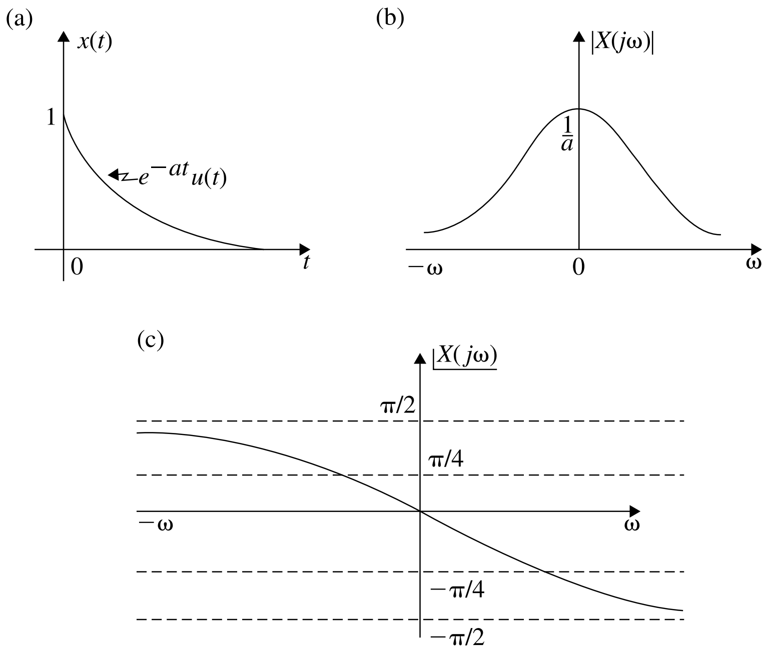


Fig. 3.8 Representation of $x(t) = e^{-at}u(t)$ and its FT spectra

$$\left[e^{-a|t|}\right] \stackrel{\text{FT}}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$$

Fourier Spectra

$$|X(j\omega)| = \frac{2a}{a^2+\omega^2}$$
$$\underline{/X(j\omega)} = 0$$

The Fourier phase spectrum is zero at all frequencies. The representation of $x(t)$ and its Fourier amplitude spectrum are shown in Fig. 3.9a, b respectively.

(g) $\boldsymbol{x(t) = e^{at}u(t);\ a > 0}$

$$\begin{aligned} X(j\omega) &= \int_0^\infty e^{at}e^{-j\omega t}\,dt \\ &= \int_0^\infty e^{(a-j\omega)t}\,dt \\ &= \frac{1}{(a-j\omega)}\left[e^{(a-j\omega)t}\right]_0^\infty \end{aligned}$$

If the upper limit is applied to the above integral, the Fourier integral does not converge. **Hence, FT does not exist for $x(t) = e^{at}u(t)$.**

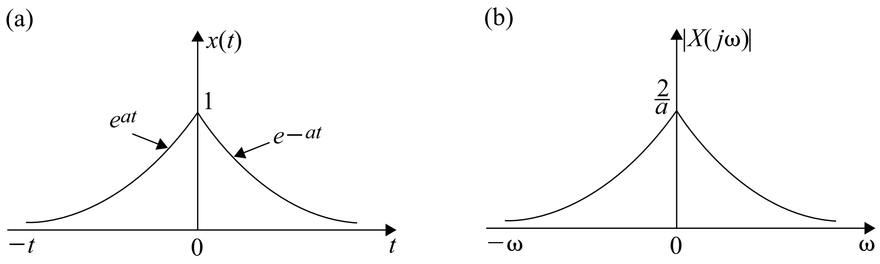


Fig. 3.9 Representation of $e^{-a|t|}$ and its amplitude spectrum

$$x(t) = e^{at}u(-t) \qquad a > 0$$
$$x(-t) = e^{-at}u(t)$$

From Example 3.1(e), it is derived

$$F[e^{-at}u(t)] = \frac{1}{(a + j\omega)}$$
$$F[x(-t)] = X(-j\omega)$$

$$F[e^{at}u(-t)] = \frac{1}{a - j\omega}$$

The above result can be derived from the first principle as explained below:

$$\begin{aligned} F[e^{at}u(-t)] &= \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt \\ &= \frac{1}{(a - j\omega)}\left[e^{(a-j\omega)t}\right]_{-\infty}^{0} \end{aligned}$$

$$F[e^{at}u(-t)] = \frac{1}{(a - j\omega)}$$

Example 3.2 Consider the rectangular pulse shown in Fig. 3.10 which is the gate function. Find the FT and sketch the Fourier spectra.

(*Anna University, April, 2004*)

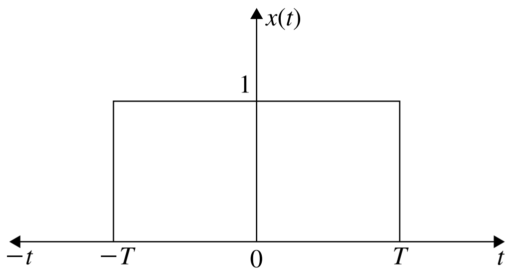


Fig. 3.10 Representation of gate function

Solution

$$x(t) = 1 \qquad |t| \leq T$$

$$X(j\omega) = \int_{-T}^{T} 1e^{-j\omega t}\, dt = \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{-T}^{T}$$

$$= \frac{\left[e^{j\omega T} - e^{-j\omega T}\right]}{j\omega}$$

$$= \frac{2T \sin \omega T}{\omega T} = 2T \text{sinc}\, \omega T$$

$$X(j\omega) = 2T \text{sinc}\, \omega T$$

Frequency Spectra of Gate Function

Amplitude Spectrum

At $\omega = 0$,

$$|X(j\omega)| = \frac{2 \sin \omega T}{\omega T} = \frac{2 \sin 0}{0} = 2$$

At $\omega = \pm\frac{n\pi}{T}$,

$$|X(j\omega)| = 0, \qquad \text{where } n = 1, 2, 3, \ldots$$

Phase Spectrum

$$\text{For sinc}\, \omega > 0, \qquad \angle X(j\omega) = 0$$

$$\text{For sinc}\, \omega < 0, \qquad \angle X(j\omega) = \pi$$

The amplitude and phase spectra are shown in Fig. 3.11a, b respectively.

Note: Since $\pi = -\pi$, in Fig. 3.11b, $\angle X(j\omega)$ is marked as π.

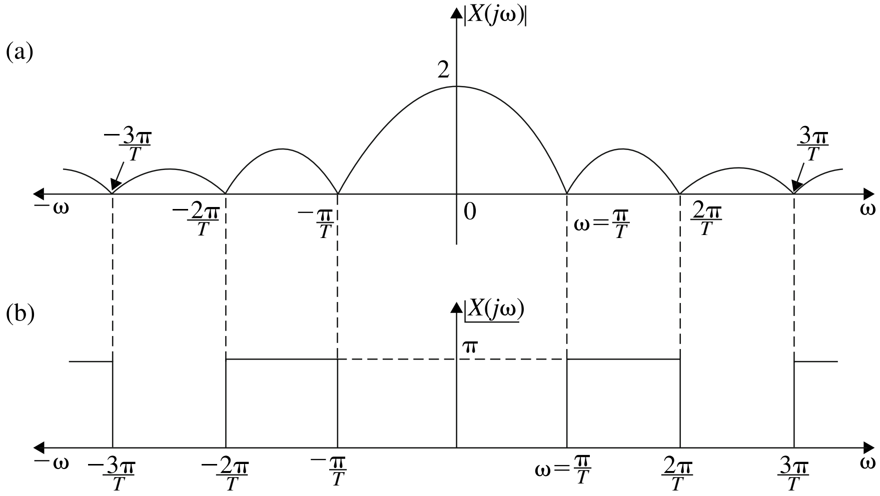


Fig. 3.11 Fourier spectra of gate function

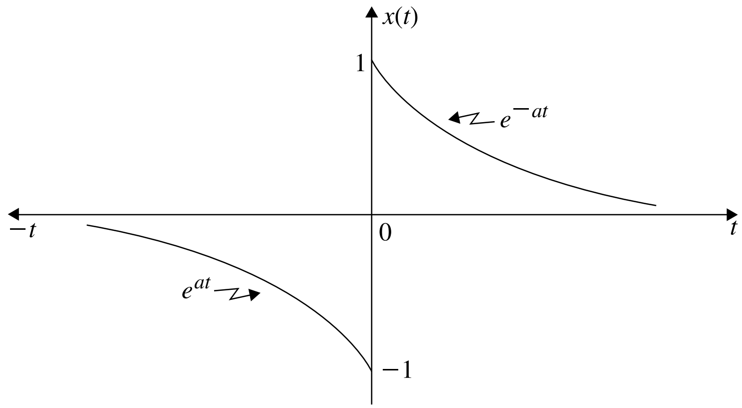


Fig. 3.12 Antisymmetry exponential decay pulse

Example 3.3 For the following signal $x(t)$, find the FT and FT spectra

$$x(t) = \begin{cases} e^{-at} & t > 0 \\ |1| & t = 0 \\ -e^{+at} & t < 0 \end{cases}$$

Solution The signal $x(t)$ is sketched as shown in Fig. 3.12.

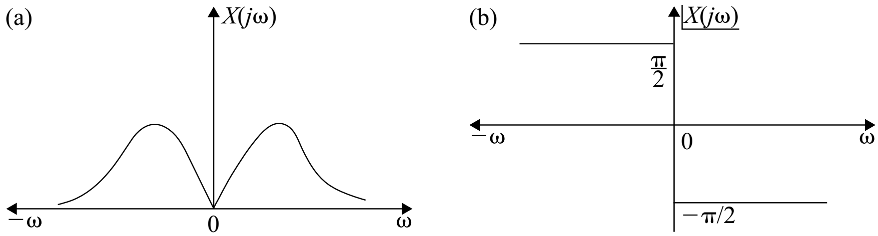


Fig. 3.13 **a** Amplitude spectra and **b** Phase spectra

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\ &= \int_{-\infty}^{0} -e^{at}e^{-j\omega t}\,dt + \int_{0^-}^{0^+} 1e^{-j\omega t}\,dt + \int_{0^+}^{\infty} e^{-at}e^{-j\omega t}\,dt \\ &= -\int_{-\infty}^{0} e^{(a-j\omega)t}\,dt + \int_{0^-}^{0^+} e^{-j\omega t}\,dt + \int_{0^+}^{\infty} e^{-(a+j\omega)t}\,dt \end{aligned}$$

$$\begin{aligned} X(j\omega) &= \frac{-1}{(a-j\omega)}\left[e^{(a-j\omega)t}\right]_{-\infty}^{0} + 0 - \frac{1}{(a+j\omega)}\left[e^{-(a+j\omega)t}\right]_{0^+}^{\infty} \\ &= \frac{-1}{(a-j\omega)} + \frac{1}{(a+j\omega)} \end{aligned}$$

$$X(j\omega) = \frac{-2j\omega}{(a^2+\omega^2)}$$

Fourier Transform Spectra

$$|X(j\omega)| = \frac{2\omega}{(a^2+\omega^2)}$$

$$\underline{|X(j\omega)} = \begin{cases} -\dfrac{\pi}{2} & \omega > 0 \\ \dfrac{\pi}{2} & \omega < 0 \end{cases}$$

The frequency spectra for $-\infty < \omega < \infty$ are shown in Fig. 3.13a, b.

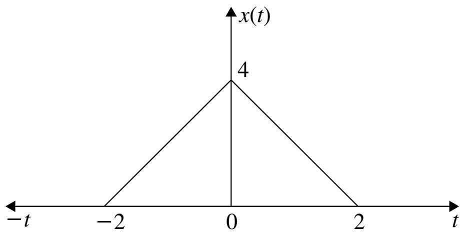


Fig. 3.14 Representation of triangular pulse

Example 3.4 Consider the triangular pulse shown in Fig. 3.14. Find the FT and its amplitude spectrum.

Solution

$$x(t) = \begin{cases} (2t+4) & -2 \le t \le 0 \\ (4-2t) & 0 \le t \le 2 \end{cases}$$

$$X(j\omega) = \int_{-2}^{0} (2t+4)e^{-j\omega t}\,dt + \int_{0}^{2} (4-2t)e^{-j\omega t}\,dt = X_1(j\omega) + X_2(j\omega)$$

$$X_1(j\omega) = \int_{-2}^{0} (2t+4)e^{-j\omega t}\,dt$$

Let $u = 2t + 4$; $du = 2\,dt$; $dv = e^{-j\omega t}\,dt$; and $v = -\frac{1}{j\omega}e^{-j\omega t}$

$$\begin{aligned} X_1(j\omega) &= uv - \int v\,du \\ &= \left[(2t+4)\left(\frac{-1}{j\omega}\right)e^{-j\omega t}\right]_{-2}^{0} + \frac{2}{j\omega}\int_{-2}^{0} e^{-j\omega t}\,dt \\ X_1(j\omega) &= \frac{-4}{j\omega} + \frac{2}{\omega^2} - \frac{2}{\omega^2}e^{j2\omega} \\ X_2(j\omega) &= \int_{0}^{2} (4-2t)e^{-j\omega t}\,dt \end{aligned}$$

Let $u = (4 - 2t)$; $du = -2\,dt$; $dv = e^{-j\omega t}\,dt$; and $v = -\frac{1}{j\omega}e^{-j\omega t}$

$$\begin{aligned} X_2(j\omega) &= uv - \int v\,du \\ &= \left[(4-2t)\left(\frac{-1}{j\omega}\right)e^{-j\omega t}\right]_{0}^{2} - \frac{2}{j\omega}\int_{0}^{2} e^{-j\omega t}\,dt \end{aligned}$$

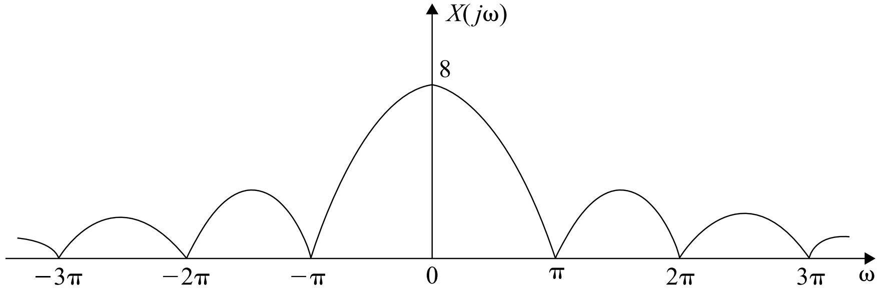


Fig. 3.15 Magnitude spectrum of a triangular wave

$$
\begin{aligned}
X_2(j\omega) &= \frac{4}{j\omega} - \frac{2}{\omega^2}\left[e^{-j\omega t}\right]_0^2 \\
&= \frac{4}{j\omega} - \frac{2}{\omega^2}\left[e^{-j2\omega} - 1\right] \\
X(j\omega) &= X_1(j\omega) + X_2(j\omega) \\
&= -\frac{4}{j\omega} + \frac{2}{\omega^2} - \frac{2}{\omega^2}e^{j2\omega} + \frac{4}{j\omega} - \frac{2}{\omega^2}e^{-j2\omega} + \frac{2}{\omega^2} \\
&= \frac{4}{\omega^2} - \frac{4}{\omega^2}\cos 2\omega \\
&= \frac{4}{\omega^2}[-\cos 2\omega + 1] \\
&= \frac{8}{\omega^2}\sin^2\omega \\
&= 8\left[\frac{\sin\omega}{\omega}\right]^2
\end{aligned}
$$

$$X(j\omega) = 8\text{sinc}^2\,\omega$$

Fourier Spectra

$$
\begin{aligned}
|X(j\omega)| &= 8\text{sinc}^2\,\omega \\
\underline{|X(j\omega)} &= 0^\circ \quad \text{for all } \omega
\end{aligned}
$$

The magnitude spectra is represented in Fig. 3.15.

Note: The FT of rectangular, triangular, and other signals can be easily determined by following the properties of FT which are discussed below.

3.6 Properties of Fourier Transform

The Fourier transform possesses the following properties and using them same results are easily obtained. These properties are:

1. Linearity
2. Time shifting
3. Conjugation and conjugation symmetry
4. Differentiation
5. Integration
6. Time scaling and time reversal
7. Frequency shifting
8. Duality
9. Time convolution
10. Parseval's Theorem.

3.6.1 Linearity

If

$$x_1(t) \overset{\text{FT}}{\longleftrightarrow} X_1(j\omega)$$
$$x_2(t) \overset{\text{FT}}{\longleftrightarrow} X_2(j\omega)$$

then

$$[A\, x_1(t) + B\, x_2(t)] \overset{\text{FT}}{\longleftrightarrow} [A\, X_1(j\omega) + B\, X_2(j\omega)]$$

Proof Let $x(t) = A\, x_1(t) + B\, x_2(t)$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} [A\, x_1(t) + B\, x_2(t)]e^{-j\omega t}\, dt \\ &= A \int_{-\infty}^{\infty} x_1(t)e^{-j\omega t} + B \int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}\, dt \end{aligned}$$

$$X(j\omega) = A\, X_1(j\omega) + B\, X_2(j\omega) \tag{3.12}$$

3.6.2 Time Shifting

If

$$x(t) \overset{\text{FT}}{\longleftrightarrow} X(j\omega)$$

then

$$x(t - t_0) \overset{\text{FT}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

Proof

$$F[x(t - t_0)] = \int_{-\infty}^{\infty} x(t - t_0)e^{-j\omega t}\, dt$$

Let $(t - t_0) = p$ and $dt = dp$

$$\begin{aligned} F[x(t - t_0)] &= \int_{-\infty}^{\infty} x(p)e^{-j\omega(p+t_0)}\, dp \\ &= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(p)e^{-j\omega p}\, dp \end{aligned}$$

$$F[x(t - t_0)] = e^{-j\omega t_0}\, X(j\omega) \tag{3.13}$$

3.6.3 Conjugation and Conjugation Symmetry

If

$$x(t) \overset{\text{FT}}{\longleftrightarrow} X(j\omega)$$

then

$$x^*(t) \overset{\text{FT}}{\longleftrightarrow} X^*(-j\omega)$$

Proof

$$\begin{aligned} F[x^*(t)] = X^*(j\omega) &= \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt\right]^* \\ &= \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}\, dt \end{aligned}$$

Replacing ω by $(-\omega)$,

$$X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}\, dt$$

$$X^*(-j\omega) = X(j\omega) \qquad \text{if } x(t) \text{ is real } x^*(t) = x(t)$$

Also

$$X(-j\omega) = X^*(j\omega) \tag{3.14}$$

3.6.4 Differentiation in Time

If

$$x(t) \overset{\text{FT}}{\longleftrightarrow} X(j\omega)$$

then

$$\frac{dx(t)}{dt} \overset{\text{FT}}{\longleftrightarrow} j\omega X(j\omega)$$

Proof

$$F[x(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega$$
$$F\left[\frac{dx(t)}{dt}\right] = \frac{j\omega}{2\pi}\int_{0}^{\infty} X(j\omega)e^{j\omega t}\, d\omega$$
$$= j\omega\, X(j\omega)$$

$$\frac{dx(t)}{dt} \overset{\text{FT}}{\longleftrightarrow} j\omega\, X(j\omega) \tag{3.15}$$

In general,

$$F\left[\frac{d^n\, x(t)}{dt^n}\right] = (j\omega)^n\, X(j\omega)$$

3.6.5 Differentiation in Frequency

If

$$F[x(t)] = X(j\omega)$$

then

$$F[tx(t)] = j\frac{d}{d\omega}X(j\omega)$$

Proof

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\ \frac{d}{d\omega}[X(j\omega)] &= \int_{-\infty}^{\infty} -jtx(t)e^{-j\omega t}\,dt \\ &= -jF[tx(t)] \end{aligned}$$

$$[tx(t)] \xleftrightarrow{\text{FT}} j\frac{dX(j\omega)}{d\omega} \tag{3.16}$$

3.6.6 Time Integration

If

$$F[x(t)] = X(j\omega)$$

then

$$F\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{1}{j\omega}X(j\omega) + \pi X(0)\,\delta(\omega)$$

Proof Let

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

Differentiating the above equation, we get

$$x(t) = \frac{dy(t)}{dt}$$

Using differentiation property, we get

$$X(j\omega) = j\omega Y(j\omega)$$

The differentiation in the time domain corresponds to multiplication by $j\omega$ in frequency domain.

$$Y(j\omega) = \left(\frac{1}{j\omega}\right) X(j\omega)$$

if the initial condition $X(0) = 0$.

If $X(j\omega) \neq 0$ at $\omega = 0$, then $y(t)$ is not integrable and FT does not exist. However, this problem is overcome by including impulses in the transform. The value at $\omega = 0$ is modified by adding $\pi X(0)$ and the FT is written as

$$F\left[\int_{-\infty}^{\infty} x(\tau)\, d\tau\right] \stackrel{\text{FT}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0)\, \delta(\omega) \tag{3.17}$$

3.6.7 Time Scaling

If

$$F[x(t)] = X(j\omega)$$

then

$$F[x(at)] = \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

Proof

$$F[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-j\omega t}\, dt$$

Let $at = p$; and $dt = \frac{1}{a} dp$, $a > 0$

$$F[x(p)] = \frac{1}{a} \int_{-\infty}^{\infty} x(p) e^{-\frac{j\omega p}{a}}\, dp$$

$$F[x(at)] = \frac{1}{a} X\left(j\frac{\omega}{a}\right)$$

For $a < 0$,

$$F[x(at)] = \frac{-1}{a} X\left(j\frac{\omega}{a}\right)$$

Hence,

$$F[x(at)] = \frac{1}{|a|} X\left(j\frac{\omega}{a}\right) \tag{3.18}$$

For time reversal,

$$F[x(-t)] = X(-j\omega) \tag{3.19}$$

3.6.8 Frequency Shifting

If

$$F[x(t)] = X(j\omega)$$

then

$$F[x(t)e^{j\omega_0 t}] = X[j(\omega - \omega_0)]$$

Proof

$$\begin{aligned} F[x(t)e^{j\omega_0 t}] &= \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t} e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}\, dt \end{aligned}$$

$$F[x(t)e^{j\omega_0 t}] = X[j(\omega - \omega_0)] \tag{3.20}$$

3.6.9 Duality

If

$$F[x(t)] = X(j\omega)$$

then

$$F[X(t)] = 2\pi x(j\omega)$$

Proof

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega \\ x(-t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{-j\omega t}\,d\omega \\ 2\pi x(-t) &= \int_{-\infty}^{\infty} X(j\omega)e^{-j\omega t}\,d\omega \\ &= F[X(j\omega)] \end{aligned}$$

Changing t to $j\omega$, we get

$$2\pi x(j\omega) = F[X(t)] \tag{3.21}$$

3.6.10 The Convolution

Let

$$\begin{aligned} y(t) &= x(t) * h(t) \\ F[y(t)] &= Y(j\omega) = X(j\omega)H(j\omega) \end{aligned}$$

Proof

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\ F[y(t)] = Y(j\omega) &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau\right]e^{-j\omega t}\,dt \end{aligned}$$

Interchanging the order of integration, we get

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega t}\, dt \right] d\tau$$

By time shifting property, the term inside the bracket becomes $e^{-j\omega\tau} H(j\omega)$.

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} H(j\omega)\, d\tau = H(j\omega) \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau}\, d\tau$$

$$Y(j\omega) = H(j\omega) X(j\omega) \tag{3.22}$$

3.6.11 Parseval's Theorem

According to Parseval's theorem, that the total energy in a signal is obtained by integrating the energy per unit frequency $\frac{|X(j\omega)|^2}{2\pi}$.

Proof

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} x(t) x^*(t)\, dt \\ &= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) e^{-j\omega t}\, d\omega \right] dt \end{aligned}$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \right] d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) X(j\omega)\, d\omega$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$

The Fourier transform properties are summarized and given in Table 3.1. The basic Fourier transform pairs are given in Table 3.2.

3.7 Fourier Transform of Periodic Signal

Example 3.5 Find the Fourier transform of the following periodic signals:

(a) $x(t) = e^{j\omega_0 t}$

(b) $x(t) = e^{-j\omega_0 t}$

(c) $x(t) = \cos \omega_0 t$

(d) $x(t) = \sin \omega_0 t$

Table 3.1 Fourier transform properties

Property	Time signal $x(t)$	Fourier transform $X(j\omega)$
1. Linearity	$x(t) = A\,x_1(t) + B\,x_2(t)$	$X(j\omega) = A\,X_1(j\omega) + B\,X_2(j\omega)$
2. Time shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
3. Conjugation	$x^*(t)$	$X^*(-j\omega)$
4. Differentiation in time	$\frac{d^n\,x(t)}{dt^n}$	$(j\omega)^n X(j\omega)$
5. Differentiation in frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
6. Time integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
7. Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(j\frac{\omega}{a}\right)$
8. Time reversal	$x(-t)$	$X(-j\omega)$
9. Frequency shifting	$x(t)e^{j\omega_0 t}$	$X[j(\omega - \omega_0)]$
10. Duality	$X(t)$	$2\pi x(j\omega)$
11. Time convolution	$x(t) * h(t)$	$X(j\omega)H(j\omega)$
12. Parseval's theorem	$E = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt$	$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\,d\omega$

Table 3.2 Basic Fourier transform pairs

Signal	Fourier transform
1. $\delta(t)$	1
2. $u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
3. $\delta(t - t_0)$	$e^{-j\omega t_0}$
4. $te^{-at}u(t)$	$\frac{1}{(a + j\omega)^2}$
5. $u(-t)$	$\pi\delta(\omega) - \frac{1}{j\omega}$
6. $e^{at}u(-t)$	$\frac{1}{(a - j\omega)}$
7. $e^{-a\lvert t\rvert}$	$\frac{2a}{a^2 + \omega^2}$
8. $\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
9. $\sin\omega_0 t$	$-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
10. $\frac{1}{(a^2 + t^2)}$	$e^{-a\lvert\omega\rvert}$
11. $\mathrm{sgn}(t)$	$\frac{2}{j\omega}$
12. 1; for all t	$2\pi\,\delta(\omega)$

Solution

(a) $\boldsymbol{x(t) = e^{j\omega_0 t}}$

Let $y(t) = 1$

$$Y(j\omega) = 2\pi\delta(\omega)$$

By using the frequency shifting property, we get

$$X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

(b) $\boldsymbol{x(t) = e^{-j\omega_0 t}}$

$$\begin{aligned} x(t) &= e^{-j\omega_0 t} \\ &= e^{-j\omega_0 t}1 \end{aligned}$$

By using the frequency shifting property, we get

$$X(j\omega) = 2\pi\delta(\omega + \omega_0)$$

(c) $\boldsymbol{x(t) = \cos(\omega_0 t)}$

$$\begin{aligned} x(t) &= \cos(\omega_0 t) \\ &= \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] \end{aligned}$$

$$X(j\omega) = \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

The frequency spectrum is shown in Fig. 3.16.

(d) $\boldsymbol{x(t) = \sin\omega_0 t}$

$$\begin{aligned} x(t) &= \sin\omega_0 t \\ &= \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] \end{aligned}$$

$$X(j\omega) = -j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$

The Fourier spectra of $\sin\omega_0 t$ are shown in Fig. 3.17.

Example 3.6 Consider the signal $x(t)$ shown in Fig. 3.18a. The rectangular pulse $\bar{x}(t)$ is shown in Fig. 3.18b. From $\bar{X}(j\omega)$, determine $X(j\omega)$ using shift property.

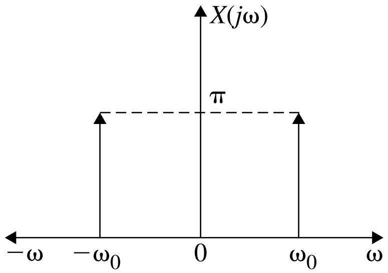


Fig. 3.16 FT of $\cos(\omega_0 t)$

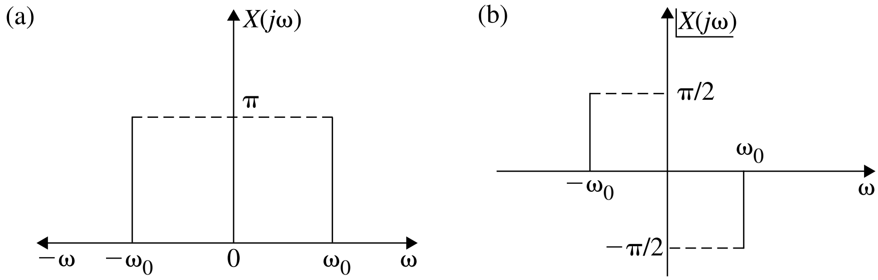


Fig. 3.17 Fourier spectra of $\sin \omega_0 t$

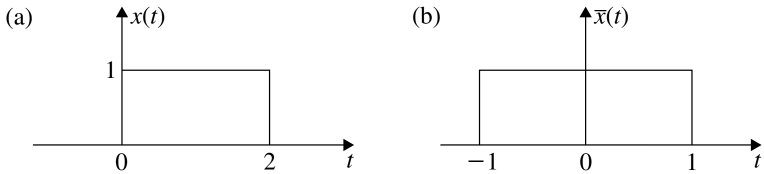


Fig. 3.18 **a** Rectangular time-shifted pulse and **b** Rectangular or gate pulse

Solution In Example 3.2, the FT of $\bar{x}(t)$ has been derived as

$$\bar{X}(j\omega) = 2\text{sinc}\,\omega$$

Using shift property, the FT of $x(t)$ is obtained as

$$X(j\omega) = 2e^{-j\omega}\text{sinc}\,\omega$$

Example 3.7 Find the Fourier transform of the signal shown in Fig. 3.19 and plot its magnitude.

(*Anna University, April, 2005*)

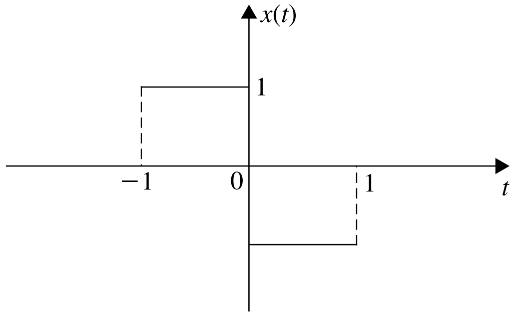


Fig. 3.19 $x(t)$ signal of Example 3.7

Solution
Method 1

$$x(t) = \begin{cases} 1 & -1 \le t \le 0 \\ -1 & 0 \le t \le 1 \end{cases}$$

$$\begin{aligned} X(j\omega) &= \int_{-1}^{0} e^{-j\omega t}\, dt - \int_{0}^{1} e^{-j\omega t}\, dt \\ &= \frac{-1}{j\omega}\left\{\left[e^{-j\omega t}\right]_{-1}^{0} - \left[e^{-j\omega t}\right]_{0}^{1}\right\} \\ &= \frac{-1}{j\omega}\left[1 - e^{j\omega} - e^{-j\omega} + 1\right] \end{aligned}$$

$$X(j\omega) = \frac{2}{j\omega}[\cos\omega - 1]$$

Method 2
Differentiating the signal in Fig. 3.19, $\frac{dx(t)}{dt}$ is obtained and is represented in Fig. 3.20.

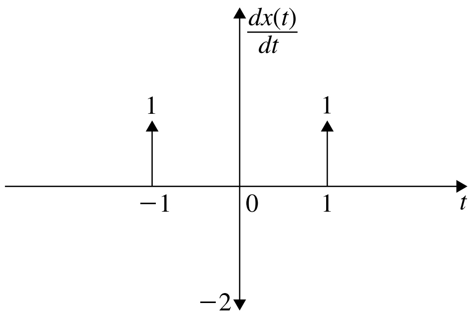


Fig. 3.20 Differentiated signal of Fig. 3.19

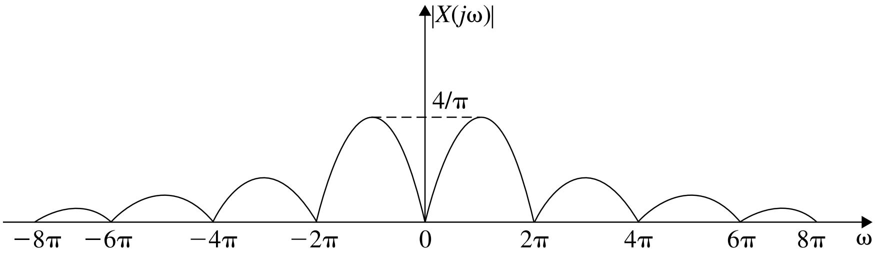


Fig. 3.21 Amplitude spectrum of $\omega\,\mathrm{sinc}^2(\omega/2)$

Using time shifting property, FT of Fig. 3.20 is written as follows:

$$F\left[\frac{dx(t)}{dt}\right] = \left[e^{j\omega} - 2 + e^{-j\omega}\right] = 2[\cos\omega - 1]$$

Using the time integration property ω get

$$F[x(t)] = X(j\omega) = \frac{2}{j\omega}[\cos\omega - 1]$$

$$X(j\omega) = \frac{2}{j\omega}[\cos\omega - 1]$$

To Plot the Magnitude Spectrum

$$\begin{aligned}|X(j\omega)| &= \frac{2}{\omega}[\cos\omega - 1] = \frac{2}{\omega}\left[\cos^2\frac{\omega}{2} - \sin^2\frac{\omega}{2} - 1\right] \\ &= \frac{-4}{\omega}\sin^2\omega/2 = -\omega\left[\frac{\sin\omega/2}{\frac{\omega}{2}}\right]^2\end{aligned}$$

$$|X(j\omega)| = \left|\omega\,\mathrm{sinc}^2\frac{\omega}{2}\right|$$

The amplitude spectrum of $X(j\omega)$ is shown in Fig. 3.21.

Example 3.8 Using Fourier transform properties, find the Fourier transform of the signal shown in Fig. 3.22a.

(*Anna University, December, 2007*)

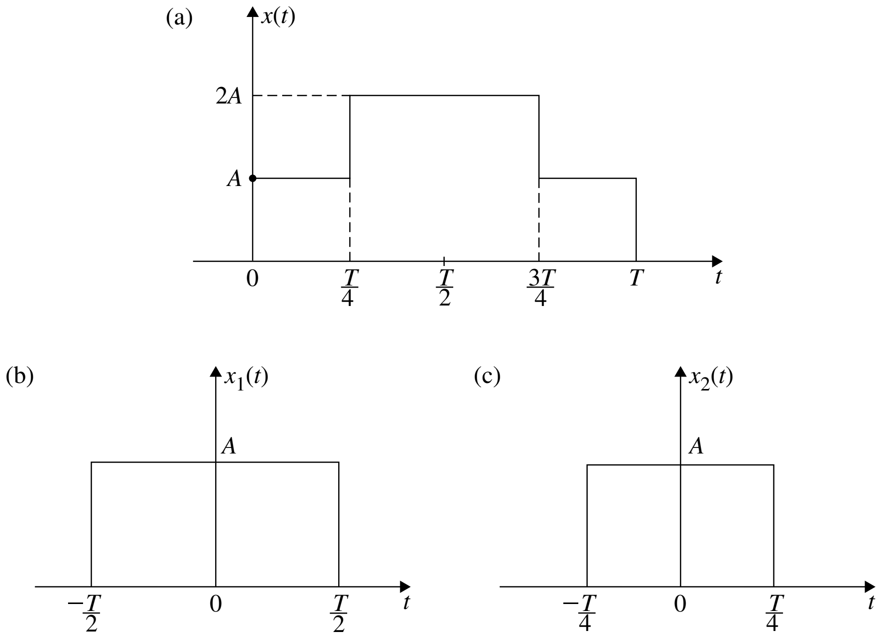


Fig. 3.22 Decomposition of signal

Solution The given signal $x(t)$ represented in Fig. 3.22a can be decomposed as $x_1(t)$ and $x_2(t)$ and represented in Fig. 3.22b, c respectively. $x(t)$ can be represented as

$$x(t) = A\left[x_1\left(t - \frac{T}{2}\right) + x_2\left(t - \frac{T}{2}\right)\right]$$

Thus, the FT of $x(t)$ can be obtained using linearity and time shifting. From Example 3.2,

$$X_1(j\omega) = AT\,\text{sinc}\frac{\omega T}{2}$$
$$X_2(j\omega) = AT\,\text{sinc}\frac{\omega T}{4}$$
$$X(j\omega) = [X_1(j\omega) + X_2(j\omega)]e^{-j\frac{\omega T}{2}}$$

$$X(j\omega) = AT\left[\text{sinc}\frac{\omega T}{2} + \text{sinc}\frac{\omega T}{4}\right]e^{-j\frac{\omega T}{2}}$$

Example 3.9 Find the Fourier transform $X(j\omega)$ of the signal $x(t)$ represented in Fig. 3.23a using differentiation property of FT. Verify the same using Fourier integral.

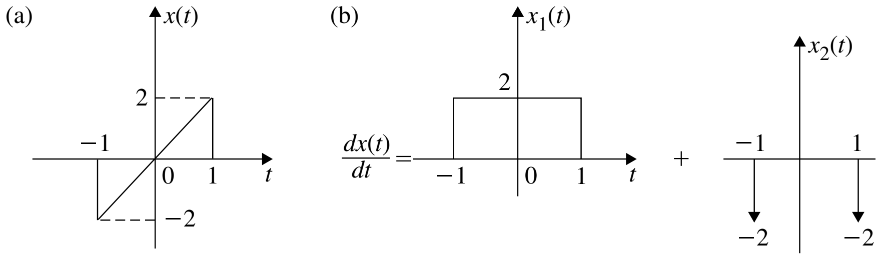


Fig. 3.23 Representation of $x(t)$ and $\frac{dx(t)}{dt}$

Solution
(a) FT Using Differentiation Property

$$\begin{aligned}
x(t) &= 2t \qquad\qquad\qquad\qquad -1 \le t \le 1 \\
\frac{dx(t)}{dt} &= 2 - 2\delta(t-1) - 2\delta(t+1) \qquad -1 \le t \le 1
\end{aligned}$$

$x(t)$ is represented in Fig. 3.23a and $\frac{dx(t)}{dt}$ is shown in Fig. 3.23b. In Fig. 3.23b, $x_1(t)$ represents the gate function and $x_2(t)$ represents impulse functions. From Example 3.2,

$$\begin{aligned}
X_1(j\omega) &= 4\,\text{sinc}\,\omega \\
X_2(j\omega) &= -2\left(e^{j\omega} + e^{-j\omega}\right) \\
&= -4\cos\omega \\
F\left[\frac{dx(t)}{dt}\right] &= X_1(j\omega) + X_2(j\omega) \\
&= 4[\text{sinc}\,\omega - \cos\omega]
\end{aligned}$$

Using integration property, FT of $x(t)$ is obtained by dividing by $j\omega$. Thus,

$$X(j\omega) = \frac{4}{j\omega}[\text{sinc}\,\omega - \cos\omega]$$

The above result can be obtained using the Fourier integral as explained below.
(b) FT Using Fourier Integral

$$\begin{aligned}
x(t) &= 2t \\
X(j\omega) &= \int_{-1}^{1} 2te^{-j\omega t}\,dt
\end{aligned}$$

Let $u = 2t$; $du = 2dt$ and $dv = \int e^{-j\omega t}\,dt$; $v = \frac{-1}{j\omega}e^{-j\omega t}$

$$\begin{aligned}
X(j\omega) &= uv - \int v\,du = \left[\frac{-2t}{j\omega}e^{-j\omega t}\right]_{-1}^{1} + j\frac{2}{\omega}\int e^{-j\omega t}\,dt \\
&= \left[\frac{-2t}{j\omega}e^{-j\omega t} + \frac{2}{\omega^2}e^{-j\omega t}\right]_{-1}^{1} \\
&= 2\left[\frac{-e^{-j\omega}}{j\omega} + \frac{1}{\omega^2}e^{-j\omega} - \frac{1}{j\omega}e^{j\omega} - \frac{1}{\omega^2}e^{j\omega}\right] \\
&= 2\left[-\frac{1}{j\omega}\left(e^{j\omega} + e^{-j\omega}\right) - \frac{1}{\omega^2}\left(e^{j\omega} - e^{-j\omega}\right)\right] \\
&= 4\left[-\frac{1}{j\omega}\cos\omega + \frac{1}{j\omega}\frac{\sin\omega}{\omega}\right]
\end{aligned}$$

$$X(j\omega) = \frac{4}{j\omega}[\operatorname{sinc}\omega - \cos\omega]$$

Example 3.10 Find the Fourier transform of impulse train shown in Fig. 3.24.

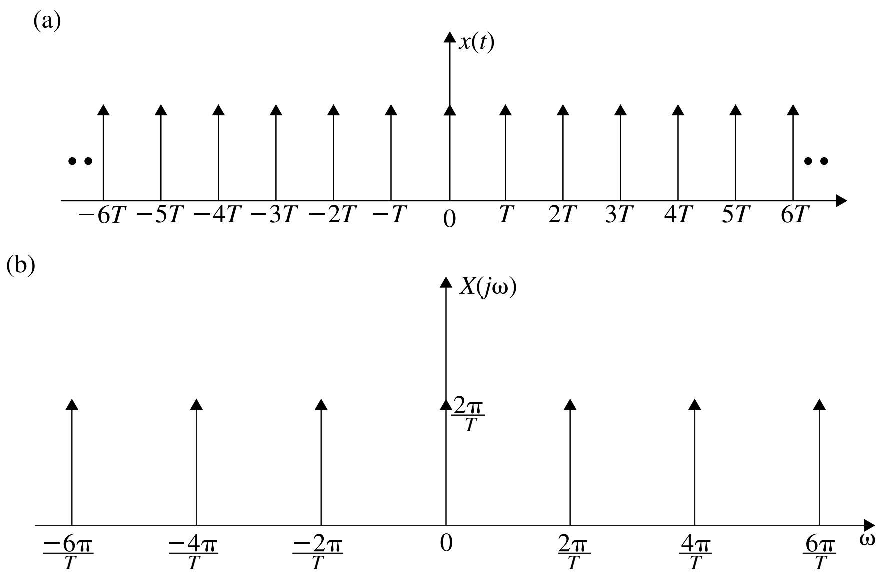


Fig. 3.24 **a** Impulse train and **b** FT of Impulse train

Solution For Fig. 3.24a,

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

where T is periodic. The Fourier series coefficients are determined as

$$\begin{aligned} D_n &= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_0 t}\, dt \\ &= \frac{1}{T} \end{aligned}$$

For a periodical signal,

$$X(j\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T}\right)$$

The above expression is represented in Fig. 3.24b.

Example 3.11 For the triangular wave shown in Fig. 3.25, find the Fourier transform using double-differentiation property.

Solution The triangular signal $x(t)$ is represented in Fig. 3.25a. It is mathematically expressed as

$$x(t) = \begin{cases} 2t + 4 & -2 \le t < 0 \\ 4 - 2t & 0 \le t \le 2 \end{cases}$$

$$\frac{dx(t)}{dt} = \begin{cases} 2 & -2 \le t < 0 \\ -2 & 0 \le t \le 2 \end{cases}$$

$\frac{dx(t)}{dt}\Big|_{t=0}$ varies from $+2$ to -2. $\frac{dx(t)}{dt}$ is represented in Fig. 3.25b.

$$\frac{d^2x(t)}{dt^2} = \begin{cases} 2\delta(t + 2) & t = -2 \\ -4 & t = 0 \\ 2\,\delta(t - 2) & t = 2 \end{cases}$$

$\frac{d^2x(t)}{dt^2}$ is shown in Fig. 3.25c. From Fig. 3.25c, using linearity and time shifting properties of FT, we get

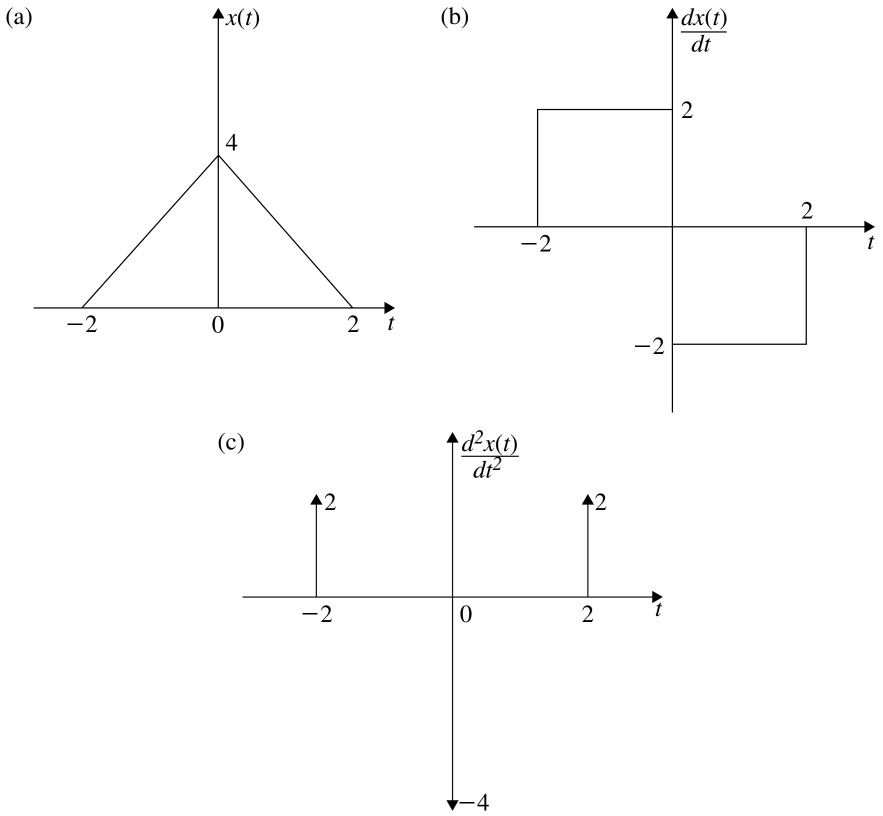


Fig. 3.25 **a** Triangular wave; **b** First derivative and **c** Second derivative

$$F\left[\frac{d^2x(t)}{dt^2}\right] = 2e^{j2\omega} - 4 + 2e^{-j2\omega}$$
$$= 4[\cos 2\omega - 1]$$
$$= -8\sin^2\omega$$

$F[x(t)]$ is obtained by dividing $F\left[\frac{d^2x(t)}{dt^2}\right]$ by $(j\omega)^2$. Thus

$$X(j\omega) = \frac{-8}{(j\omega)^2}\ \sin^2\omega$$
$$= 8\left[\sin\frac{\omega}{\omega}\right]^2$$

$$X(j\omega) = 8\text{sinc}^2\,\omega$$

The same result is obtained in Example 3.4 which is obtained directly using Fourier integral.

Example 3.12 Find the Fourier transform of

$$x(t) = \frac{2a}{a^2 + t^2}$$

using the duality property of FT.

Solution
Method 1
From Example 3.1(f), the FT of $x(t) = e^{-a|t|}$ is obtained as

$$x(t) = e^{-a|t|} \stackrel{\text{FT}}{\longleftrightarrow} \frac{2a}{a^2 + \omega^2}$$

By the application of the inverse Fourier transform, we get

$$e^{-a|t|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2 + \omega^2} e^{j\omega t}\, d\omega$$
$$2\pi e^{-a|t|} = \int_{-\infty}^{\infty} \frac{2a}{a^2 + \omega^2} e^{j\omega t}\, d\omega$$

Replacing t by $-t$ in the above equation, we get

$$2\pi e^{-a|t|} = \int_{-\infty}^{\infty} \frac{2a}{a^2 + \omega^2} e^{-j\omega t}\, d\omega$$

Interchanging t and ω in the above equation, we get

$$2\pi e^{-a|\omega|} = \int_{-\infty}^{\infty} \frac{2a}{(a^2 + t^2)} e^{-j\omega t}\, dt$$

The right-hand side of the above equation is nothing but the FT of $\frac{2a}{a^2+t^2}$.

$$2\pi e^{-a|\omega|} = F\left[\frac{2a}{(a^2 + t^2)}\right]$$

$$\left[\frac{2a}{(a^2 + t^2)}\right] \stackrel{\text{FT}}{\longleftrightarrow} 2\pi e^{-a|\omega|}$$

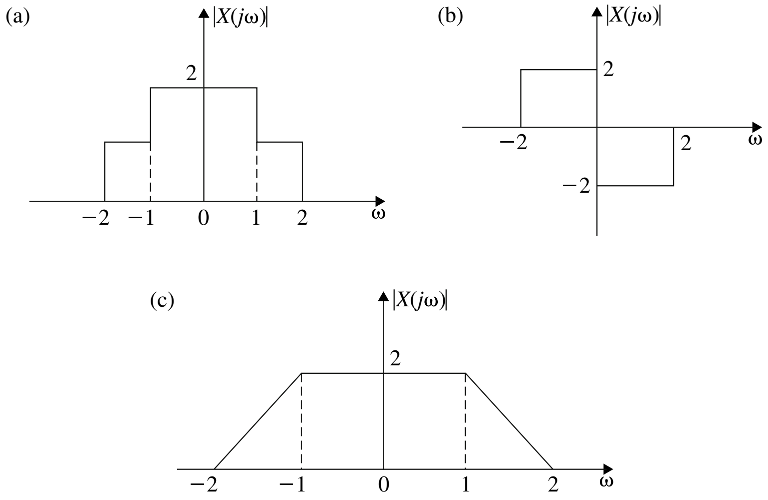


Fig. 3.26 Fourier transformed signal

Method 2

The duality property of $X(t) = 2\pi x(-\omega)$. From Example 3.1(f), the FT of $e^{-|t|}$ is obtained as

$$e^{-a|t|} \overset{\text{FT}}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$$

$$X(t) = \frac{2a}{a^2+t^2}$$

$$x(-\omega) = e^{-a|\omega|}$$

$$X(t) \overset{\text{FT}}{\longleftrightarrow} 2\pi x(-\omega)$$

$$\frac{2a}{a^2+t^2} \overset{\text{FT}}{\longleftrightarrow} 2\pi e^{-a|\omega|}$$

Example 3.13 For the Fourier transforms shown in Fig. 3.26a–c. Find the energy of the signals using Parseval's theorem

Solution

(a)

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$

$$E = \frac{1}{2\pi}\left\{\int_{-2}^{1} 1^2\,d\omega + \int_{-1}^{1} 2^2\,d\omega + \int_{1}^{2} 1^2\,d\omega\right\}$$
$$= \frac{1}{2\pi}\left\{\Big[\omega\Big]_{-2}^{-1} + 4\Big[\omega\Big]_{-1}^{1} + \Big[\omega\Big]_{1}^{2}\right\}$$
$$= \frac{1}{2\pi}\{-1+2+4+4+2-1\}$$

$$E = \frac{5}{\pi}$$

(b)

$$E = \frac{1}{2\pi}\left\{\int_{-2}^{0} 2^2\,d\omega + \int_{0}^{2} (-2)^2\,d\omega\right\}$$
$$= \frac{1}{2\pi}\left\{4\Big[\omega\Big]_{-2}^{0} + 4\Big[\omega\Big]_{0}^{2}\right\}$$

$$E = \frac{8}{\pi}$$

(c)

$$|X(j\omega)| = \begin{cases} 2\omega + 4 & -2 \le \omega \le -1 \\ 2 & -1 \le \omega \le 1 \\ (4-2\omega) & 1 \le \omega < 2 \end{cases}$$

$$E = \frac{1}{2\pi}\left\{\int_{-2}^{-1} (2\omega+4)^2\,d\omega + \int_{-1}^{1} (2)^2\,d\omega + \int_{1}^{2} (4-2\omega)^2\,d\omega\right\}$$
$$= \frac{1}{2\pi}\left\{\int_{-2}^{-1} (4\omega^2+16\omega+16)\,d\omega + 4\int_{-1}^{1} d\omega + \int_{1}^{2} (4\omega^2-16\omega+16)\,d\omega\right\}$$
$$= \frac{1}{2\pi}\left\{\left[\frac{4}{3}\omega^3 + 8\omega^2 + 16\omega\right]_{-2}^{-1} + 4\Big[\omega\Big]_{-1}^{1} + \left[\frac{4}{3}\omega^3 - 8\omega^2 + 16\omega\right]_{1}^{2}\right\}$$
$$= \frac{1}{2\pi}\left\{\left[-\frac{4}{3} + 8 - 16 + \frac{32}{3} - 32 + 32\right] + [4+4]\right.$$
$$\left. + \left[\frac{32}{3} - 32 + 32 - \frac{4}{3} + 8 - 16\right]\right\}$$

$$E = \frac{16}{3\pi}$$

Example 3.14 Find the Fourier transform of the following continuous time functions by applying Fourier transform properties or otherwise.

1. $x(t) = \delta(t-2)$
2. $x(t) = \delta(t-1) - \delta(t+1)$
3. $x(t) = \delta(t+2) + \delta(t-2)$
4. $x(t) = u(t+2) - u(t-2)$
5. $x(t) = \frac{d}{dt}[u(-t-3) + u(t-3)]$
6. $x(t) = e^{-3t}u(t-1)$
7. $x(t) = te^{-at}u(t)$
8. $x(t) = e^{-a(t-2)}u(t-2)$
9. $x(t) = e^{-a|t-2|}$
10. $x(t) = \sin\left(2\pi t + \frac{\pi}{4}\right)$
11. $x(t) = \cos\left(3\pi t + \frac{\pi}{8}\right) + 1$
12. $x(t) = \cos\left(6\pi t - \frac{\pi}{8}\right)$
13. $x(t) = x(4t-8)$
14. $x(t) = \frac{d^2}{dt^2}x(t-2)$
15. $x(t) = x(2-t) + x(-2-t)$
16. $x(t) = \text{rect}\left(\frac{t+2}{4}\right)$
17. $x(t) = \text{tri}\left(\frac{t-4}{10}\right)$
18. $x(t) = \frac{d}{dt}\left[5\ \text{rect}\frac{t}{8}\right]$
19. $x(t) = \delta(t+2) + 5\delta(t+1) + \delta(t-1) + 5\delta(t-2)$
20. $x(t) = \begin{cases} e^{j6|t|} & |t| \le \pi \\ 0 & \text{elsewhere} \end{cases}$
21. $x(t) = \cos(\omega_0 t + \phi)$
22. $x(t) = \sin(\omega_0 t + \phi)$
23. $x(t) = \begin{cases} 0 & |t| > 1 \\ \frac{(t+1)}{2} & -1 \le t \le 1 \end{cases}$

24. $$x(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

25. $$x(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

26. $$x(t) = \begin{cases} 1 & |t| < 1 \\ 2 - |t| & 1 < |t| < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Solution

1. $\boldsymbol{x(t) = \delta(t-2)}$
   The impulse is time shifted by $t_0 = 2$.

$$F[\delta(t-2)] = e^{-j\omega t_0}\, F[\delta(t)]$$
$$= e^{-j2\omega}$$

$$F[\delta(t-2)] = e^{-j2\omega}$$

2. $\boldsymbol{x(t) = \delta(t-1) - \delta(t+1)}$

$$F[\delta(t-1)] = e^{-j\omega}$$
$$F[\delta(t+1)] = e^{j\omega}$$
$$F[\delta(t-1) + \delta(t+1)] = e^{-j\omega} - e^{j\omega}$$
$$= -2j\sin\omega$$

$$F[\delta(t-1) - \delta(t+1)] = -2j\sin\omega$$

3. $\boldsymbol{x(t) = \delta(t+2) + \delta(t-2)}$

$$F[\delta(t+2)] = e^{j2\omega}$$
$$F[\delta(t-2)] = e^{-j2\omega}$$
$$F[\delta(t+2) + \delta(t-2)] = e^{j2\omega} + e^{-j2\omega}$$
$$= 2\cos 2\omega$$

$$X(j\omega) = 2\cos 2\omega$$

4. $\boldsymbol{x(t) = u(t+2) - u(t-2)}$

$$F[u(t+2)] = \frac{1}{j\omega}e^{j2\omega}$$
$$F[u(t-2)] = \frac{1}{j\omega}e^{-j2\omega}$$
$$F[u(t+2) - u(t-2)] = \frac{1}{j\omega}\left[e^{j2\omega} - e^{-j2\omega}\right]$$
$$= \frac{2}{\omega}\sin 2\omega$$
$$X(j\omega) = 4\text{sinc}\, 2\omega$$

5. $\boldsymbol{x(t) = \frac{d}{dt}[u(-t-3) + u(t-3)]}$

$x(t)$ and $\frac{dx(t)}{dt}$ are shown in Fig. 3.27a–b respectively.

From Fig. 3.27b,

$$F\left[\frac{dx(t)}{dt}\right] = e^{-j3\omega} - e^{+j3\omega}$$
$$= -2j\frac{\left[e^{j3\omega} - e^{-j3\omega}\right]}{2j}$$

$$F\left[\frac{dx(t)}{dt}\right] = -2j\sin 3\,\omega$$

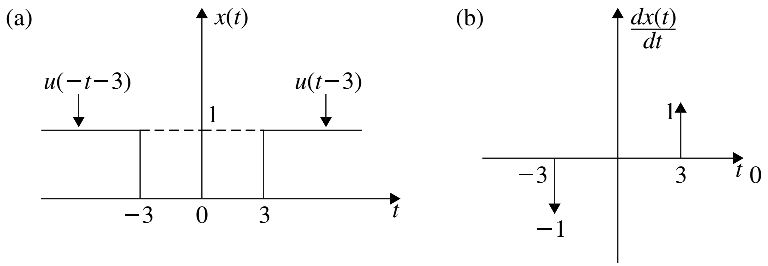


Fig. 3.27 **a** Representation of $x(t)$ and **b** Representation of $\frac{dx(t)}{dt}$

6. $\boldsymbol{x(t) = e^{-3t}u(t-1)}$

Method 1

$$F\left[e^{-3t}u(t)\right] = \frac{1}{(3+j\omega)}$$

Using time shifting property, we get

$$F\left[e^{-3(t-1)}u(t-1)\right] = \frac{e^{-j\omega}}{(3+j\omega)}$$
$$e^{3}F\left[e^{-3t}u(t-1)\right] = \frac{e^{-j\omega}}{(3+j\omega)}$$

$$F\left[e^{-3t}u(t-1)\right] = \frac{e^{-(j\omega+3)}}{(3+j\omega)}$$

Method 2

Using FT definition, from Fig. 3.28, we get
Rect $\left(\frac{t}{4}\right)$ and rect $\left(\frac{t}{4}+0.5\right)$ are represented in Figs. 3.29a, b respectively

$$F[x(t)] = \int_{1}^{\infty} e^{-3t}e^{-j\omega t}\,dt$$
$$= \int_{1}^{\infty} e^{-(3+j\omega)t}\,dt$$
$$= \frac{-1}{(3+j\omega)}\left[e^{-(3+j\omega)t}\right]_{1}^{\infty}$$

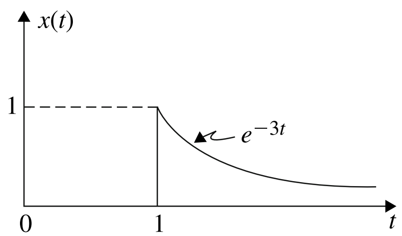


Fig. 3.28 Representation of $x(t) = e^{-3t}u(t-1)$

$$F[x(t)] = \frac{e^{-(3+j\omega)}}{j\omega + 3}$$

7. $\boldsymbol{x(t) = te^{-at}u(t)}$

$$F[e^{-at}u(t)] = \frac{1}{(a + j\omega)}$$

Using the FT property of differentiation in frequency, we get

$$\begin{aligned} F\left[te^{-at}u(t)\right] &= j\frac{d}{d\omega}\left[\frac{1}{(a + j\omega)}\right] \\ &= \frac{j(-j)}{(a + j\omega)^2} \end{aligned}$$

$$X(j\omega) = \frac{1}{(a + j\omega)^2}$$

8. $\boldsymbol{x(t) = e^{-a(t-2)}u(t-2)}$
Method 1

$$\begin{aligned} x(t) &\stackrel{\text{FT}}{\longleftrightarrow} X(j\omega) \\ x(t - t_0) &\stackrel{\text{FT}}{\longleftrightarrow} X(j\omega)e^{-j\omega t_0} \end{aligned}$$

$$F\left[e^{-a(t-2)}u(t-2)\right] = \frac{1}{(a + j\omega)}e^{-j2\omega}$$

Method 2
Using the definition of FT, we get

$$\begin{aligned} X(j\omega) &= \int_2^{\infty} e^{-a(t-2)}e^{-j\omega t}\,dt \\ &= e^{2a}\int_2^{\infty} e^{-(a+j\omega)t}\,dt \\ &= \frac{-e^{2a}}{(a + j\omega)}\left[e^{-(a+j\omega)t}\right]_2^{\infty} \\ &= \frac{+e^{2a}}{(a + j\omega)}e^{-(a+j\omega)2} \end{aligned}$$

$$X(j\omega) = \frac{e^{-j2\omega}}{(a + j\omega)}$$

9. $\boldsymbol{x(t) = e^{-a|t-2|}}$

$$x(t) = \begin{cases} e^{-a(t-2)} & 0 \le t \le \infty \\ e^{a(t+2)} & -\infty \le t < 0 \end{cases}$$

Let $|t - 2| = \tau$

$$x(\tau) = e^{-a|\tau|}$$

From Example 3.1(e),

$$F\left[e^{-a|\tau|}\right] = \frac{2a}{a^2 + \omega^2}$$

Using time shifting property,

$$F\left[e^{-a|t-2|}\right] = \frac{2a}{a^2 + \omega^2} e^{-j2\omega}$$

10. $\boldsymbol{x(t) = \sin\left(2\pi t + \frac{\pi}{4}\right)}$. (*Anna University, December, 2006*)

Let $\omega_0 = 2\pi$ and $\phi = \frac{\pi}{4}$

$$F[x(t)] = -j\pi\left[e^{j\phi}\delta(\omega - \omega_0) - e^{-j\phi}\delta(\omega + \omega_0)\right]$$

(For proof, *see* Example 3.21 below).

$$X(j\omega) = -j\pi\left[e^{\frac{j\pi}{4}}\delta(\omega - 2\pi) - e^{-\frac{j\pi}{4}}\delta(\omega + 2\pi)\right]$$

11. $\boldsymbol{x(t) = \cos\left(3\pi t + \frac{\pi}{8}\right) + 1}$

$$F[\cos(\omega t + \phi)] = \pi\left[e^{j\phi}\delta(\omega - \omega_0) + e^{-j\phi}\delta(\omega + \omega_0)\right]$$

Let $\omega_0 = 3\pi$ and $\phi = \frac{\pi}{8}$

$$F\left[\cos 3\pi t + \frac{\pi}{8}\right] = \pi\left[e^{j\frac{\pi}{8}}\delta(\omega - 3\pi) + e^{-j\frac{\pi}{8}}\delta(\omega + 3\pi)\right]$$
$$F[1] = 2\pi\delta(\omega)$$

$$\left[\cos\left(3\pi t + \frac{\pi}{8}\right) + 1\right] \overset{\text{FT}}{\longleftrightarrow} \pi\left[e^{j\frac{\pi}{8}}\delta(\omega - 3\pi) + e^{-j\frac{\pi}{8}}\delta(\omega + 3\pi) + 2\delta(\omega)\right]$$

12. $\boldsymbol{x(t) = \cos\left(6\pi t - \frac{\pi}{8}\right)}$

Let $\omega_0 = 6\pi$ and $\phi = \frac{-\pi}{8}$

$$F[\cos\omega_0 t + \phi] = \pi\left[e^{j\phi}\delta(\omega - \omega_0) + e^{-j\phi}\delta(\omega + \omega_0)\right]$$

$$F\left[\cos\left(6\pi t - \frac{\pi}{8}\right)\right] = \pi\left[e^{-j\frac{\pi}{8}}\delta(\omega - 6\pi) + e^{j\frac{\pi}{8}}\delta(\omega + 6\pi)\right]$$

13. $\boldsymbol{x(t) = x(4t - 8)}$
By time scaling,

$$F[x(4t)] = \frac{1}{4}X\left(\frac{j\omega}{4}\right)$$

$x(4t)$ is time shifted by $-\frac{8}{4} = -2$. Hence,

$$F[x(4t - 8)] = \frac{1}{4}X\left(\frac{j\omega}{4}\right)e^{-j2\omega}$$

14. $\boldsymbol{x(t) = \frac{d^2}{dt^2}x(t - 2)}$

$$F\left[\frac{d^2x(t)}{dt^2}\right] = -\omega^2\, X(j\omega)$$

For the time delay t_0,

$$F[x(t - t_0)] = e^{-j\omega t_0}\, X(j\omega)$$

Here, $t_0 = 2$.

$$F\left[\frac{d^2}{dt^2}x(t - 2)\right] = -\omega^2 e^{-j2\omega}\, X(j\omega)$$

15. $\boldsymbol{x(t) = x(2 - t) + x(-2 - t)}$

$$x(t) = x_1(t) + x_2(t)$$

where

$$x_1(t) = x(2-t)$$
$$x_2(t) = x(-2-t)$$
$$F[x(-t)] = X(-j\omega)$$

Using time shifting property of FT, we get

$$X_1(j\omega) = F[x(2-t)] = e^{-j2\omega}X(-j\omega)$$
$$X_2(j\omega) = F[x(-2-t)] = e^{j2\omega}X(-j\omega)$$
$$X(j\omega) = X_1(j\omega) + X_2(j\omega)$$
$$= X(-j\omega)\left[e^{-j2\omega} + e^{j2\omega}\right]$$

$$X(j\omega) = 2X(-j\omega)\cos 2\,\omega$$

16. $\boldsymbol{x(t) = \mathrm{rect}\left(\frac{t+2}{4}\right)}$

$$x(t) = \mathrm{rect}\left(\frac{t}{4} + 0.5\right)$$

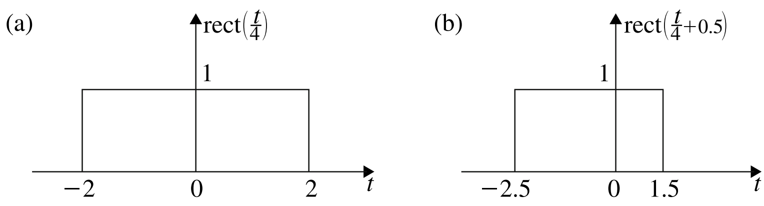


Fig. 3.29 **a** Representation of rect $\left(\frac{t}{4}\right)$ and **b** Representation of rect $\left(\frac{t}{4} + 0.5\right)$

$$\mathrm{rect}\left(\frac{t}{4}\right) \overset{\mathrm{FT}}{\longleftrightarrow} \frac{2}{\omega}\sin 2\omega$$
$$\mathrm{rect}\left(\frac{t}{4} + 0.5\right) \overset{\mathrm{FT}}{\longleftrightarrow} \frac{2}{\omega}\sin 2\omega\, e^{+0.5j\omega}$$

$$X(j\omega) = \frac{2}{\omega}\sin 2\omega\, e^{j0.5\omega}$$

17. $\boldsymbol{x(t) = \text{tri}\left(\frac{t-4}{10}\right)}$

$$\text{tri}\left(\frac{t-4}{10}\right) = \text{tri}\left(\frac{t}{10} - 0.4\right)$$

$\text{tri}(t)$ is represented in Fig. 3.30a and $\text{tri}\left(\frac{t}{10}\right)$ in Fig. 3.30b.

$$\text{tri}\left(\frac{t}{10}\right) = \begin{cases} (1 + 0.1t) & -10 \le t \le 0 \\ (1 - 0.1t) & 0 \le t \le 10 \end{cases}$$

Let $x(t) = \text{tri}0.1t$.

$\frac{dx(t)}{dt}$ and $\frac{d^2x(t)}{dt}$ are represented in Fig. 3.30c.

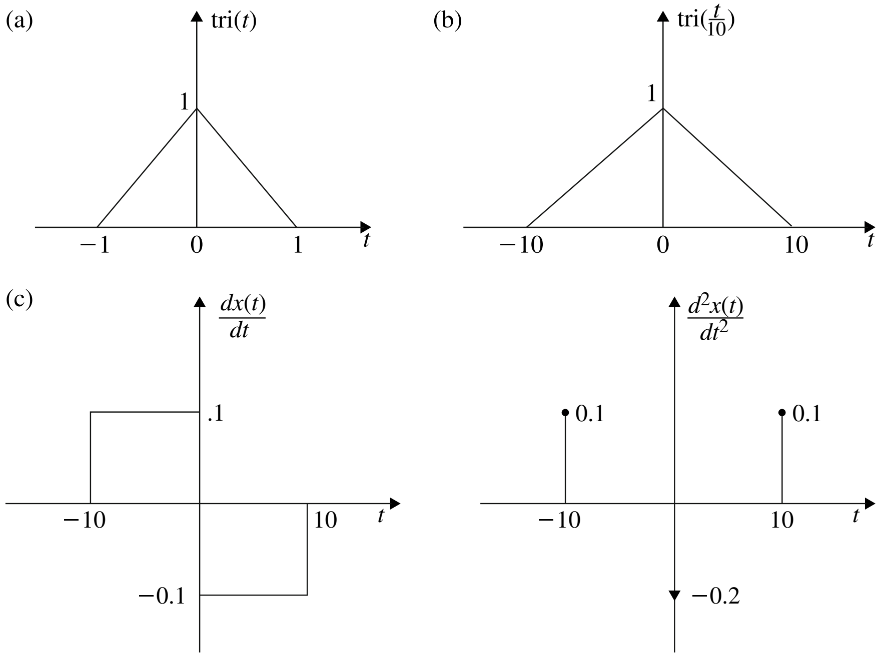


Fig. 3.30 **a** $x(t) = \text{tri}(t)$ and **b** $x(t) = \text{tri}\left(\frac{t}{10}\right)$. **c** Representation of $\frac{dx(t)}{dt}$ and $\frac{d^2x(t)}{dt^2}$

$$F\left[\frac{d^2x(t)}{dt^2}\right] = 0.1e^{j10\omega} + 0.1e^{-j10\omega} - 0.2 \quad \text{[Refer to Example 3.11]}$$
$$= 0.2[\cos 10\,\omega - 1]$$
$$= -0.4\sin^2 5\,\omega$$

Using the double integration property of FT, we get

$$F[\text{tri}0.1t] = \frac{-0.4}{-\omega^2}\sin^2 5\,\omega = 0.4\frac{\sin^2 5\,\omega}{\omega^2}$$

By time shifting, we get

$$F[\text{tri}\,(0.1t - 0.4)] = 0.4\frac{\sin^2 5\,\omega}{\omega^2}e^{-j0.4\omega}$$

18. $\boldsymbol{x(t) = \frac{d}{dt}\left[5\ \text{rect}\left(\frac{t}{8}\right)\right]}$

Figure 3.31a represents $5\,\text{rect}(t)$. The time expanded $5\,\text{rect}\left(\frac{t}{8}\right)$ is shown in Fig. 3.31b and its derivative is shown in Fig. 3.31c. From Fig. 3.31c,

$$X(j\omega) = 5e^{j8\omega} - 5e^{-j8\omega}$$

$$X(j\omega) = j10\sin 8\,\omega$$

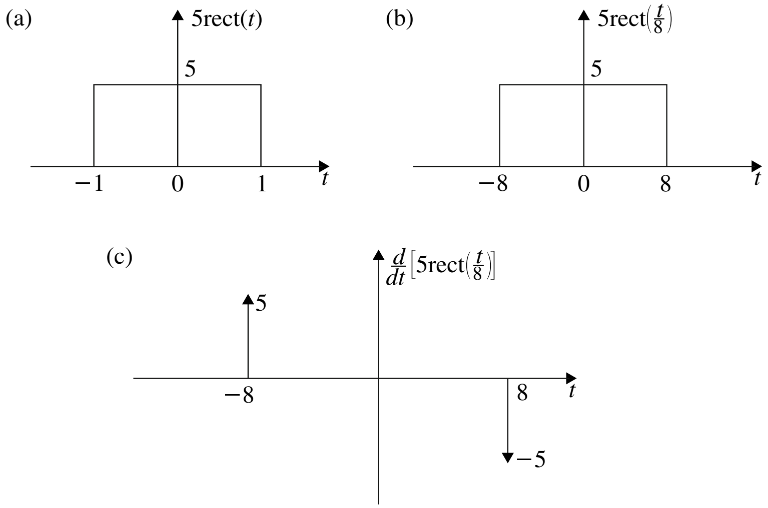


Fig. 3.31 Representation of rectangular wave and its derivatives

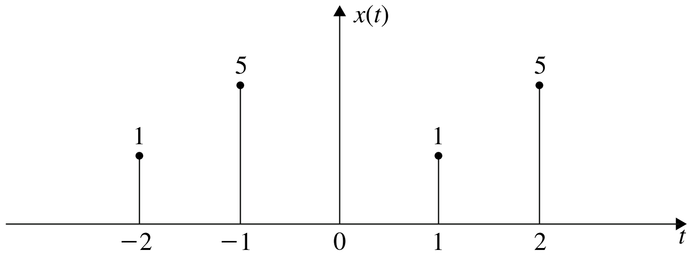


Fig. 3.32 Discrete time signal

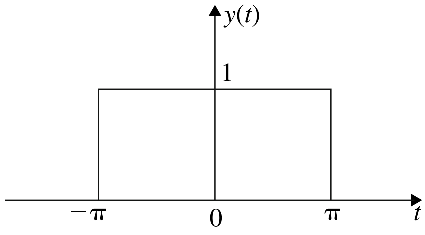


Fig. 3.33 Representation of rectangular pulse

19. $\boldsymbol{x(t) = \delta(t+2) + 5\delta(t+1) + \delta(t-1) + 5\delta(t-2)}$

The given $x(t)$ is represented in Fig. 3.32. By applying time shifting property to each impulse, we get

$$X(j\omega) = e^{j2\omega} + 5e^{j\omega} + e^{-j\omega} + 5e^{-j2\omega}$$

20. $\boldsymbol{x(t) = \begin{cases} e^{j6t} & |t| \leq \pi \\ 0 & \text{elsewhere} \end{cases}}$

The above signal is represented as a product of a rectangular pulse of width 2π and a complex sinusoid e^{j6t}.

$$x(t) = \begin{cases} 1\, e^{j6t} & |t| \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

For $-\pi \leq t \leq \pi$, the rectangular pulse $y(t)$ is shown in Fig. 3.33. The FT of the rectangular pulse shown in Fig. 3.33 can be easily derived as

$$Y(j\omega) = \frac{2}{\omega} \sin \omega\pi$$

Using frequency shifting property,

$$y(t)e^{j6t} \stackrel{\text{FT}}{\longleftrightarrow} = Y(j(\omega-6))$$
$$X(j\omega) = F\left[y(t)e^{j6t}\right]$$

$$X(j\omega) = \frac{2\sin((\omega-6)\pi)}{(\omega-6)}$$

21. $\boldsymbol{x(t) = \cos(\omega_0 t + \phi)}$

$$\cos(\omega_0 t + \phi) = \frac{1}{2}\left[e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}\right] = \frac{1}{2}\left[e^{j\phi}e^{j\omega_0 t} + e^{-j\phi}e^{-j\omega_0 t}\right]$$

By frequency shifting property,

$$F\left[e^{j\omega_0 t}\right] = 2\pi\delta(\omega-\omega_0)$$
$$F\left[e^{-j\omega_0 t}\right] = 2\pi\delta(\omega+\omega_0)$$
$$F[x(t)] = X(j\omega) = \frac{2\pi}{2}\left[e^{j\phi}\delta(\omega-\omega_0) + e^{-j\phi}\delta(\omega+\omega_0)\right]$$

$$X(j\omega) = \pi\left[e^{j\phi}\delta(\omega-\omega_0) + e^{-j\phi}\delta(\omega+\omega_0)\right]$$

22. $\boldsymbol{x(t) = \sin(\omega_0 t + \phi)}$

$$\sin(\omega_0 t + \phi) = \frac{1}{2j}\left[e^{+j(\omega_0 t+\phi)} - e^{-j(\omega_0 t+\phi)}\right]$$
$$= \frac{1}{2j}\left[e^{j\phi}e^{j\omega_0 t} - e^{-j\phi}e^{-j\omega_0 t}\right]$$
$$F[x(t)] = X(j\omega) = \frac{2\pi}{2j}\left[e^{j\phi}\delta(\omega-\omega_0) - e^{-j\phi}\delta(\omega+\omega_0)\right]$$

$$X(j\omega) = -j\pi\left[e^{j\phi}\delta(\omega-\omega_0) - e^{-j\phi}\delta(\omega+\omega_0)\right]$$

23. $\boldsymbol{x(t) = \begin{cases} 0 & |t| > 1 \\ \dfrac{(t+1)}{2} & -1 \le t \le 1 \end{cases}}$

Figure 3.34a gives $x(t)$ and Fig. 3.34b gives $\dfrac{dx(t)}{dt}$.

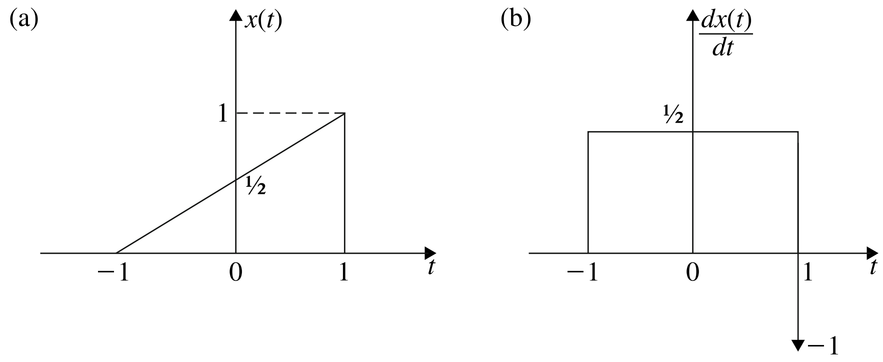


Fig. 3.34 **a** Representation of $x(t)$ and **b** Representation of $\frac{dx(t)}{dt}$

$$F[\text{rect}(t)] = \frac{\sin\omega}{\omega}$$

$$F\left[\frac{dx(t)}{dt}\right] = F\left[\frac{\text{rect}(t)}{2}\right] - e^{-j\omega}$$

Using the integration property of FT,

$$X(j\omega) = \frac{1}{j\omega}\left[\frac{\sin\omega}{\omega} - e^{-j\omega}\right]$$

24. $\boldsymbol{x(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & \textbf{otherwise} \end{cases}}$

$x(t) = t;\ 0 \le t \le 1$ is shown in Fig. 3.35a $\frac{dx(t)}{dt}$ is shown in Fig. 3.35b. The Fourier transform of the time-shifted rectangle is $2\frac{\sin(\omega/2)}{\omega}e^{-j\omega/2}$ and that of the negative impulse is $-e^{-j\omega}$.

$$F\left[\frac{dx(t)}{dt}\right] = \left[2\frac{\sin(\omega/2)}{\omega}e^{-\frac{j\omega}{2}} - e^{-j\omega}\right]$$

Using the integration property of FT,

$$F[x(t)] = \frac{1}{j\omega}F\left[\frac{dx(t)}{dt}\right]$$

$$X(j\omega) = \frac{1}{j\omega}\left[\frac{2\sin(\omega/2)}{\omega}e^{-\frac{j\omega}{2}} - e^{-j\omega}\right]$$

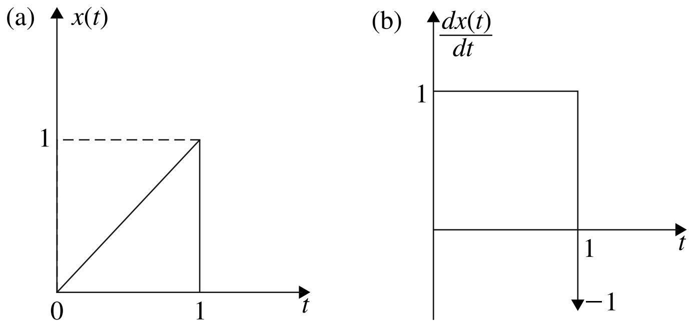


Fig. 3.35 **a** Representation of $x(t)$ and **b** Representation of $\frac{dx(t)}{dt}$

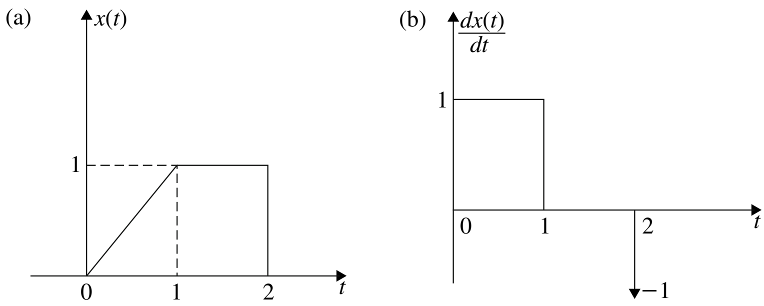


Fig. 3.36 **a** Representation of $x(t)$ and **b** Representation of $\frac{dx(t)}{dt}$

25. $\boldsymbol{x(t) = \begin{cases} t & 0 \leq t < 1 \\ 1 & 1 \leq t \leq 2 \\ 0 & \text{elsewhere} \end{cases}}$

The signal $x(t)$ shown in Fig. 3.36a when differentiated takes the shape as shown in Fig. 3.36b. For the square pulse, the FT is

$$X_1(j\omega) = \frac{2\sin\frac{\omega}{2}}{\omega} e^{-\frac{j\omega}{2}}$$

For the negative impulse, the FT is

$$X_2(j\omega) = -e^{-j2\omega}$$

$$X_1(j\omega) + X_2(j\omega) = \left[\frac{2}{\omega}\sin\frac{\omega}{2} e^{-\frac{j\omega}{2}} - e^{-j2\omega}\right]$$

The Fourier transform of the given signal is obtained using the integration property.

$$X(j\omega) = \frac{1}{j\omega}\left[X_1(j\omega) + X_2(j\omega)\right]$$

$$X(j\omega) = \frac{1}{j\omega}\left[\frac{2}{\omega}\sin\frac{\omega}{2}e^{-\frac{j\omega}{2}} - e^{-j2\omega}\right]$$

26. $\boldsymbol{x(t) = \begin{cases} 1 & |t| < 1 \\ 2 - |t| & 1 < |t| < 2 \\ 0 & \text{elsewhere} \end{cases}}$

The given signal $x(t)$ is represented in Fig. 3.37a. The first and second derivatives are shown in Figs. 3.37b, c respectively. From Fig. 3.37c the FT of the impulses are obtained using time shifting property.

$$F\left[\frac{d^2x(t)}{dt^2}\right] = \left[e^{j2\omega} - (e^{j\omega} + e^{-j\omega}) + e^{-j2\omega}\right] = 2\left[\cos 2\omega - \cos\omega\right]$$

Using the integration property of FT

$$F\left[x(t)\right] = \frac{2}{(j\omega)^2}F\left[\frac{d^2x(t)}{dt^2}\right]$$

$$x(j\omega) = \frac{2}{\omega^2}\left[\cos\omega - \cos 2\omega\right]$$

Example 3.15 Let the FT of a signal $x(t)$ be as shown in Fig. 3.38a. Determine the FT of $\frac{dx(t)}{dt}$, $tx(t)$ and $\int_0^t x(t)dt$ using property.

(*Anna University, December, 2005*)

Solution

(a)

$$F\left[\frac{dx(t)}{dt}\right] = j\omega X(j\omega) = |\omega X(j\omega)|$$

$$X(j\omega_1) = \omega_1$$

$$X(j\omega_2) = \omega_2$$

$$X(j\omega) = \omega \qquad \omega_1 \leq \omega \leq \omega_2$$

This is the straight line with slope 1. This is represented in Fig. 3.38b.

(b)

$$F[tx(t)] = \frac{d}{d\omega}X(j\omega)$$

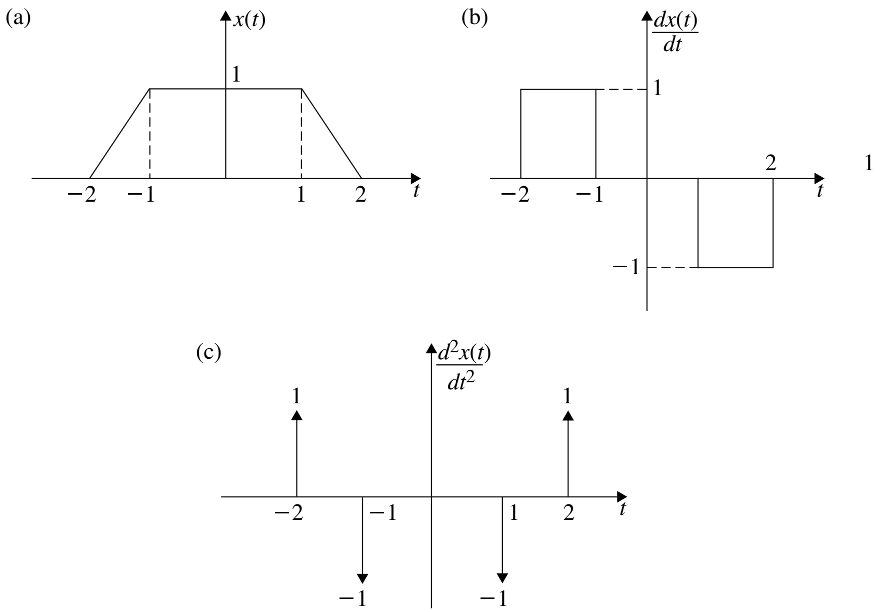


Fig. 3.37 **a** Representation of signal $x(t)$; **b** Representation of the signal $\frac{dx(t)}{dt}$ and **c** Representation of the signal $\frac{d^2x(t)}{dt^2}$

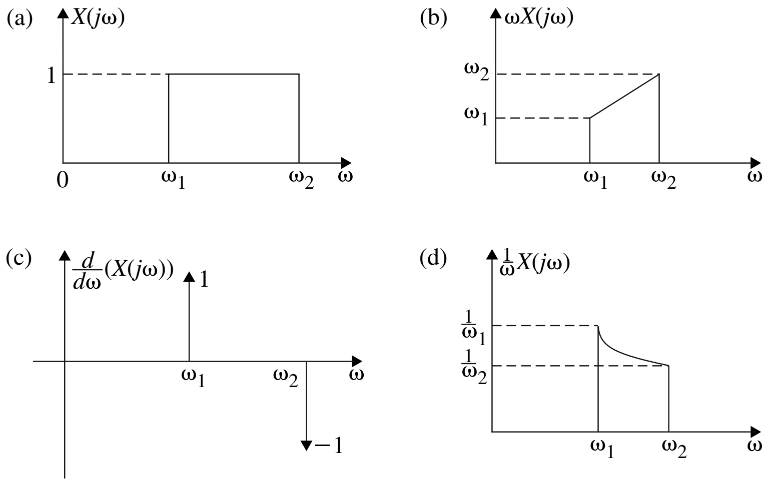


Fig. 3.38 **a** FT of $x(t)$; **b** FT of $\frac{dx(t)}{dt}$; **c** FT of $tx(t)$ and **d** FT of $\int_0^t x(t)dt$

The rectangular wave in Fig. 3.38a when differentiated with respect to ω it becomes +ve and −ve impulses of magnitude 1 at $\omega = \omega_1$ and $\omega = \omega_2$ respectively. This is shown in Fig. 3.38c.

(c)

$$F\left[\int_0^t x(t)dt\right] = \frac{1}{j\omega}X(j\omega) = \left|\frac{1}{\omega}X(j\omega)\right|$$

At $\omega = \omega_1$,

$$X(j\omega_1) = \frac{1}{\omega_1}$$

At $\omega = \omega_2$,

$$X(j\omega_2) = \frac{1}{\omega_2}$$

For $\omega_1 \le \omega \le \omega_2$, it is a drooping curve. This is represented in Fig. 3.38d.

Example 3.16 Find the magnitude spectrum of FT and plot it where

$$H(j\omega) = \frac{(1 + 2e^{-j\omega})}{(1 + \frac{1}{2}e^{-j\omega})}.$$

(*Anna University, April, 2004*)

Solution

$$H(j\omega) = \frac{(1 + 2e^{-j\omega})}{(1 + \frac{1}{2}e^{-j\omega})} = \frac{(1 + 2\cos\omega) - j2\sin\omega}{(1 + \frac{1}{2}\cos\omega) - \frac{j}{2}\sin\omega}$$

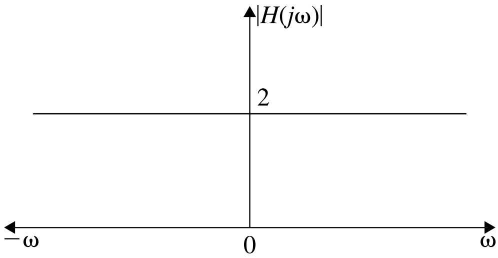
$$|H(j\omega)| = \frac{\sqrt{(1 + 2\cos\omega)^2 + 4\sin^2\omega}}{\sqrt{(1 + \frac{1}{2}\cos\omega)^2 + \frac{1}{4}\sin^2\omega}} = \frac{\sqrt{1 + 4\cos^2\omega + 4\cos\omega + 4\sin^2\omega}}{\sqrt{1 + \frac{1}{4}\cos^2\omega + \cos\omega + \frac{1}{4}\sin^2\omega}}$$

$$= \frac{\sqrt{5 + 4\cos\omega}}{\sqrt{\frac{5}{4} + \cos\omega}} = 2$$

$$|H(j\omega)| = 2$$

$|H(j\omega)|$ is independent of frequency and is shown in Fig. 3.39.

Fig. 3.39 Magnitude spectrum of $H(j\omega)$



Example 3.17 Using the properties of continuous time Fourier transform determine the time domain signal $x(t)$, if the frequency domain signal

$$X(j\omega) = j\frac{d}{d\omega}\left[\frac{e^{j2\omega}}{(1+\frac{j\omega}{3})}\right].$$

(*Anna University, December, 2007*)

Solution From inspection of $X(j\omega)$, the given problem can be solved using differentiation in frequency, time shifting and scaling in the proper order.

First, the time scaling property is applied. Let

$$X_1(j\omega) = \frac{1}{1+j\omega}$$

$$x_1(t) = e^{-t}u(t)$$

$$F\left[x_1[3t]\right] = 3e^{-3t}u(3t)$$

$$F\left[3e^{-3t}u(3t)\right] = \frac{1}{\left[1+\frac{j\omega}{3}\right]}$$

$$F^{-1}\left[\frac{1}{\left(1+\frac{j\omega}{3}\right)}\right] = 3e^{-3t}u(t) \quad [\because u(t) = u(3t)]$$

According to the time shifting property,

$$e^{j2\omega}Y(j\omega) = y(t+2)$$

$$F^{-1}\left[\frac{e^{j2\omega}}{\left(1+\frac{j\omega}{3}\right)}\right] = 3e^{-3(t+2)}u(t+2)$$

According to differentiating property,

$$j\frac{d}{d\omega}X(j\omega) = tx(t).$$

Applying the above property, we have

$$F^{-1}\left[j\frac{d}{d\omega}\frac{e^{j2\omega}}{\left(1+\frac{j\omega}{3}\right)}\right] = 3te^{-3(t+2)}u(t+2)$$

$$\therefore \quad X(j\omega) = \frac{jd}{d\omega}\left[\frac{e^{j2\omega}}{\left(1+\frac{j\omega}{3}\right)}\right]$$

$$x(t) = 3te^{-3(t+2)}u(t+2)$$

Example 3.18 Find the inverse Fourier transform of the following functions:

1. $X(j\omega) = \delta(\omega - \omega_0)$
2. $X(j\omega) = \dfrac{j\omega}{(2+j\omega)^2}$
3. $X(j\omega) = \begin{cases} 1 & |\omega| < 2 \\ 0 & \text{elsewhere} \end{cases}$
4. $X(j\omega) = \dfrac{6}{(\omega^2+9)}$
5. $X(j\omega) = \dfrac{(j\omega+2)}{\left[(j\omega)^2+4j\omega+3\right]}$
6. $X(j\omega) = \dfrac{(j\omega+1)}{\left[(j\omega+2)^2(j\omega+3)\right]}$

Solution

1. $\boldsymbol{X(j\omega) = \delta(\omega - \omega_0)}$
   The IFT of $\delta(\omega) = \frac{1}{2\pi}\delta(\omega)$ is frequency shifted by ω_0.

$$F^{-1}\left[X(j\omega)\right] = e^{j\omega_0 t}\frac{1}{2\pi}$$

$$F^{-1}\left[\delta(\omega - \omega_0)\right] = \frac{1}{2\pi}e^{j\omega_0 t}$$

The above result can also be got from the first principle of inverse Fourier transform

$$F^{-1}\left[\delta(\omega-\omega_0)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}d\omega$$

Using the sampling property of the impulse function which exists only at $\omega=\omega_0$, we get

$$F^{-1}\left[\delta(\omega-\omega_0)\right]=\frac{1}{2\pi}e^{j\omega_0 t}$$

2. $\boldsymbol{X(j\omega)=\frac{j\omega}{(2+j\omega)^2}}$

$$F\left[e^{-2t}\right]=\frac{1}{(2+j\omega)}$$

By applying,

$$F\left[te^{-2t}\right]=\frac{d}{d\omega}\frac{1}{(2+j\omega)}.$$

(Applying frequency differentiation).

$$F\left[te^{-2t}\right]=\frac{1}{(2+j\omega)^2}$$

$$\therefore \quad F^{-1}\left[\frac{1}{(2+j\omega)^2}\right]=te^{-2t}.$$

By applying time differentiation, namely

$$\frac{dx(t)}{dt}=j\omega X(j\omega)$$

$$F^{-1}\left[\frac{j\omega}{(2+j\omega^2)}\right]=\frac{d}{dt}\left(te^{-2t}\right)$$

3. $\boldsymbol{X(j\omega)=\begin{cases}1 & |\omega|<2\\ 0 & \text{otherwise}\end{cases}}$

The frequency spectrum of the above function is shown in Fig. 3.40.

Using the definition of inverse FT, we get

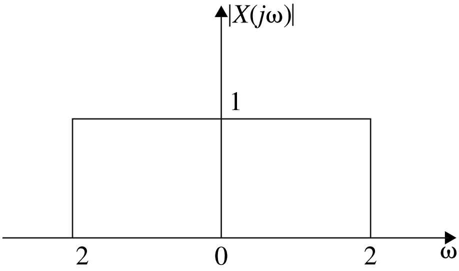


Fig. 3.40 Representation of $X(j\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-2}^{2} X(j\omega)e^{j\omega t}d\omega$$
$$= \frac{1}{2\pi jt}\left[1e^{j\omega t}\right]_{2}^{2}$$
$$= \frac{1}{2\pi jt}\left[e^{j2t} - e^{-j2t}\right]$$
$$= \frac{1}{\pi t}\sin 2t$$
$$x(t) = \frac{2}{\pi}\text{sinc}2t$$

4. $\boldsymbol{X(j\omega) = \frac{6}{(\omega^2+9)}}$

$$X(j\omega) = \frac{-6}{(j\omega+3)(j\omega-3)}$$
$$= \frac{A_1}{j\omega+3} + \frac{A_2}{j\omega-3}$$
$$-6 = A_1(j\omega-3) + A_2(j\omega+3).$$

Let $j\omega = -3$

$$A_1 = 1$$

Let $j\omega = 3$

$$A_2 = -1$$
$$X(j\omega) = \frac{1}{j\omega+3} - \frac{1}{j\omega-3}$$
$$x(t) = F^{-1}\left[X(j\omega)\right] = e^{-3t}u(t) + e^{3t}u(-t)$$

$$x(t) = e^{-3t}u(t) + e^{3t}u(-t)$$

5. $\boldsymbol{X(j\omega) = \frac{(j\omega+2)}{[(j\omega)^2+4j\omega+3]}}$

$$X(j\omega) = \frac{(j\omega + 2)}{(j\omega + 1)(j\omega + 3)}$$
$$= \frac{A_1}{(j\omega + 1)} + \frac{A_2}{(j\omega + 3)}$$
$$(j\omega + 2) = A_1(j\omega + 3) + A_2(j\omega + 1)$$

Let $j\omega = -1$,

$$1 = 2A_1$$
$$A_1 = \frac{1}{2}$$

Let $j\omega = -3$, $A_2 = \frac{1}{2}$

$$X(j\omega) = \frac{1}{2}\left[\frac{1}{j\omega + 1} + \frac{1}{j\omega + 3}\right]$$

$$x(t) = \frac{1}{2}\left[e^{-t} + e^{-3t}\right]u(t)$$

6. $\boldsymbol{X(j\omega) = \frac{(j\omega+1)}{(j\omega+2)^2(j\omega+3)}}$

$$X(j\omega) = \frac{A_1}{(j\omega + 2)^2} + \frac{A_2}{(j\omega + 2)} + \frac{A_3}{(j\omega + 3)}$$
$$(j\omega + 1) = A_1(j\omega + 3) + A_2(j\omega + 2)(j\omega + 3) + A_3(j\omega + 2)^2$$

Let $j\omega = -2$;

$$-1 = A_1$$

Let $j\omega = -3$;

$$-2 = A_3$$
$$(j\omega + 1) = A_1(j\omega + 3) + A_2\left[(j\omega)^2 + 5j\omega + 6\right] + A_3\left[(j\omega)^2 + 4j\omega + 4\right]$$

Compare the coefficients of $j\omega$ on both sides

$$
\begin{aligned}
1 &= A_1 + 5A_2 + 4A_3 \\
&= -1 + 5A_2 - 8 \\
A_2 &= 2 \\
X(j\omega) &= \frac{-1}{(j\omega+2)^2} + \frac{2}{(j\omega+2)} - \frac{2}{(j\omega+3)} \\
x(t) &= F^{-1}[x(j\omega)]
\end{aligned}
$$

$$x(t) = \left[-te^{-2t} + 2e^{-2t} - 2e^{-3t}\right] u(t)$$

Example 3.19 Consider a causal LTI system with frequency response

$$H(j\omega) = \frac{1}{j\omega + 3}.$$

For a particular input $x(t)$, this system is to produce the output

$$y(t) = e^{-3t}u(t) - e^{-4t}u(t).$$

Determine $x(t)$.

(*Anna University, April, 2008*)

Solution

$$
\begin{aligned}
y(t) &= e^{-3t}u(t) - e^{-4t}u(t) \\
Y(j\omega) &= \frac{1}{(j\omega+3)} - \frac{1}{(j\omega+4)} = \frac{1}{(j\omega+3)(j\omega+4)} \\
H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} \\
X(j\omega) &= \frac{Y(j\omega)}{H(j\omega)} = \frac{(j\omega+3)}{(j\omega+3)(j\omega+4)} = \frac{1}{(j\omega+4)} \\
x(t) &= F^{-1}X(j\omega) = e^{-4t}u(t)
\end{aligned}
$$

$$x(t) = e^{-4t}u(t)$$

Example 3.20 Find the Fourier transform of the following signals using the convolution theorem.

$$\begin{aligned}
&1. \qquad x(t) = e^{-2t}u(t) * e^{-5t}u(t) \\
&2. \qquad x(t) = \frac{d}{dt}\left[e^{-2t}u(t) * e^{-5t}u(t)\right] \\
&3. \qquad x(t) = \left[e^{-2t}u(t) * e^{-5t}u(t-5)\right]
\end{aligned}$$

Determine $x(t)$ in all the above cases.

Solution

1. $\boldsymbol{x(t) = e^{-2t}u(t) * e^{-5t}u(t)}$

$$X(j\omega) = F\left[e^{-2t}u(t)\right] F\left[e^{-5t}u(t)\right]$$
$$F\left[e^{-2t}u(t)\right] = \frac{1}{(j\omega+2)}$$

$$F\left[e^{-5t}u(t)\right] = \frac{1}{(j\omega+5)}$$

$$X(j\omega) = \frac{1}{(j\omega+2)(j\omega+5)}$$

$$X(j\omega) = \frac{1}{3}\left[\frac{1}{j\omega+2} - \frac{1}{(j\omega+5)}\right]$$
$$x(t) = F^{-1}[X(j\omega)] = \frac{1}{3}[e^{-2t}u(t) - e^{-5t}u(t)]$$

$$x(t) = \frac{1}{3}\left[e^{-2t} - e^{-5t}\right]u(t)$$

2. $\boldsymbol{x(t) = \frac{d}{dt}\left[e^{-2t}u(t) * e^{-5t}u(t)\right]}$
Let

$$x_1(t) = e^{-2t}u(t) * e^{-5t}u(t)$$
$$X_1(j\omega) = \frac{1}{(j\omega+2)(j\omega+5)}$$

Using the time differentiation property of FT, we get

$$x(t) = \frac{d\,x_1(t)}{dt}$$
$$X(j\omega) = j\omega X_1(j\omega)$$

$$X(j\omega) = \frac{j\omega}{(j\omega+2)(j\omega+5)}$$

Putting into partial fraction, we get

$$X(j\omega) = \frac{A_1}{j\omega+2} + \frac{A_2}{j\omega+5}$$
$$j\omega = A_1(j\omega+5) + A_2(j\omega+2)$$

Let $j\omega = -2$;

$$A_1 = -\frac{2}{3}$$

Let $j\omega = -5$;

$$A_2 = \frac{5}{3}$$
$$X(j\omega) = \frac{1}{3}\left[-\frac{2}{j\omega+2} + \frac{5}{j\omega+5}\right]$$
$$x(t) = F^{-1}[X(j\omega)] = \frac{1}{3}\left[-2e^{-2t} + 5e^{-5t}\right]u(t)$$

$$x(t) = \frac{1}{3}\left[-2e^{-2t} + 5e^{-5t}\right]u(t)$$

3. $\boldsymbol{x(t) = e^{-2t}u(t) * e^{-5t}u(t-5)}$

$$x(t) = x_1(t) * x_2(t)$$
$$X(j\omega) = X_1(j\omega)X_2(j\omega)$$
$$X_1(j\omega) = \frac{1}{(j\omega+2)}$$
$$x_2(t) = e^{-5t}u(t-5) = e^{-25}e^{-5(t-5)}u(t-5)$$
$$X_2(j\omega) = e^{-25}\frac{1}{(j\omega+5)}$$

$$X(j\omega) = e^{-25}\left[\frac{1}{(j\omega+2)(j\omega+5)}\right]$$

$$X(j\omega) = \frac{1}{3}e^{-25}\left[\frac{1}{j\omega + 2} - \frac{1}{j\omega + 5}\right]$$

$$x(t) = \frac{e^{-25}}{3}\left[e^{-2t} - e^{-5t}\right]u(t)$$

Example 3.21 Consider the following signals $x_1(t)$ and $x_2(t)$. Find

$$y(t) = x_1(t) * x_2(t).$$

1.

$$x_1(t) = e^{-2t}u(t)$$
$$x_2(t) = e^{3t}u(-t)$$

2.

$$x_1(t) = e^{2t}u(-t)$$
$$x_2(t) = e^{4t}u(-t)$$

Solution

1. $\boldsymbol{x_1(t) = e^{-2t}u(t)}$ **and** $\boldsymbol{x_2(t) = e^{3t}u(-t)}$

$$X_1(j\omega) = \frac{1}{(j\omega + 2)}$$
$$X_2(j\omega) = -\frac{1}{(j\omega - 3)}$$

$$\begin{aligned}
x_1(t) * x_2(t) &= X_1(j\omega)X_2(j\omega) \\
Y(j\omega) &= \frac{1}{(j\omega + 2)}\frac{(-1)}{(j\omega - 3)} \\
Y(j\omega) &= \frac{A_1}{(j\omega + 2)} + \frac{A_2}{(j\omega - 3)} \\
&= \frac{1}{5}\left[\frac{1}{j\omega + 2} - \frac{1}{j\omega - 3}\right] \\
y(t) &= F^{-1}[Y(j\omega)] = \frac{1}{5}\left[e^{-2t}u(t) + e^{3t}u(-t)\right]
\end{aligned}$$

$$y(t) = \frac{1}{5}\left[e^{-2t}u(t) + e^{3t}u(-t)\right]$$

2. $\boldsymbol{x_1(t) = e^{2t}u(-t)}$ **and** $\boldsymbol{x_2(t) = e^{4t}u(-t)}$

$$\begin{aligned}
X_1(j\omega) &= \frac{-1}{(j\omega-2)} \\
X_2(j\omega) &= \frac{-1}{(j\omega-4)} \\
x_1(t) * x_2(t) &= X_1(j\omega)X_2(j\omega) \\
Y(j\omega) &= \frac{1}{(j\omega-2)(j\omega-4)} \\
&= \frac{A_1}{(j\omega-2)} + \frac{A_2}{(j\omega-4)} \\
&= \frac{1}{2}\left[\frac{-1}{(j\omega-2)} + \frac{1}{(j\omega-4)}\right] \\
y(t) &= F^{-1}[Y(j\omega)] = \frac{1}{2}\left[e^{2t} - e^{4t}\right]u(-t)
\end{aligned}$$

$$y(t) = \frac{1}{2}\left[e^{2t} - e^{4t}\right]u(-t)$$

Example 3.22 Find the Fourier transform of the following functions:

1. $x(t) = e^{j\omega_0 t}\,u(t)$
2. $x(t) = \cos\omega_0 t\,u(t)$
3. $x(t) = \sin\omega_0 t\,u(t)$
4. $x(t) = e^{-at}\cos\omega_0 t\,u(t); \quad a > 0$
5. $x(t) = e^{-at}\sin\omega_0 t\,u(t); \quad a > 0$
6. $x(t) = [u(t+2) - ut - 2]\cos 3t$
7. $x(t) = e^{-2|t|}\cos 5t$
8. $x(t) = e^{-3|t|}\sin 2t$

Solution

1. $\boldsymbol{x(t) = e^{j\omega_0 t}u(t)}$

$$F\,[u(t)] = \frac{1}{j\omega} + \pi\delta(\omega)$$

By using the frequency shifting property, the FT of $x(t)$ is obtained.

$$F\left[e^{j\omega_0 t}u(t)\right] = \frac{1}{j(\omega - \omega_0)} + \pi\delta(\omega - \omega_0)$$

2. $\boldsymbol{x(t) = \cos\omega_0 t u(t)}$

$$\cos\omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$
$$\cos\omega_0 t u(t) = \frac{1}{2}\left[e^{j\omega_0 t}u(t) + e^{-j\omega_0 t}u(t)\right]$$

By using the frequency shifting property, $F[x(t)]$ is obtained.

$$\begin{aligned} X(j\omega) &= F[\cos\omega_0 t u(t)] \\ &= \frac{1}{2}\left[\frac{1}{j(\omega - \omega_0)} + \pi\delta(\omega - \omega_0) + \frac{1}{j(\omega + \omega_0)} + \pi\delta(\omega + \omega_0)\right] \\ X(j\omega) &= \frac{1}{2}\left[\frac{2\omega}{j(\omega - \omega_0^2)} + \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)\right] \end{aligned}$$

$$X(j\omega) = \frac{j\omega}{(\omega_0^2 - \omega^2)} + \frac{1}{2}\pi\delta(\omega - \omega_0) + \frac{1}{2}\pi\delta(\omega + \omega_0)$$

3. $\boldsymbol{x(t) = \sin\omega_0 t u(t)}$

$$\sin\omega_0 t = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$
$$\sin\omega_0 t u(t) = \frac{1}{2j}\left[e^{j\omega_0 t}u(t) - e^{-j\omega_0 t}u(t)\right]$$

By using the frequency shifting property, $F[x(t)]$ is obtained.

$$F[x(t)] = \frac{1}{2j}\left[\frac{1}{j(\omega - \omega_0)} + \pi\delta(\omega - \omega_0) - \frac{1}{j(\omega + \omega_0)} - \pi\delta(\omega + \omega_0)\right]$$

$$X(j\omega) = \left[\frac{\omega_0}{\omega_0^2 - \omega^2} + \frac{\pi}{2j}\delta(\omega - \omega_0) - \frac{\pi}{2j}\delta(\omega + \omega_0)\right]$$

4. $\boldsymbol{x(t) = e^{-at} \cos \omega_0 t u(t)}$

$$\begin{aligned}
\cos \omega_0 t &= \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] \\
X(j\omega) &= \int_0^\infty e^{-at} \cos \omega_0 T e^{-j\omega t}\, dt \\
&= \frac{1}{2}\int_0^\infty e^{-at} e^{j\omega_0 t} e^{-j\omega t}\, dt + \frac{1}{2}\int_0^\infty e^{-at} e^{-j\omega_0 t} e^{-j\omega t}\, dt \\
&= \frac{1}{2}\int_0^\infty e^{-(a-j\omega_0+j\omega)t}\, dt + \frac{1}{2}\int_0^\infty e^{-(a+j\omega_0+j\omega)t}\, dt \\
&= \frac{1}{2}\left[\frac{-1}{(a-j\omega_0+j\omega)} e^{-(a-j\omega_0+j\omega)t} - \frac{e^{-(a+j\omega_0+j\omega)t}}{(a+j\omega_0+j\omega)}\right]_0^\infty \\
&= \frac{1}{2}\left[\frac{1}{(a+j\omega)-j\omega_0} + \frac{1}{(a+j\omega)+j\omega_0}\right] \\
&= \frac{1}{2}\frac{[a+j\omega+j\omega_0+a+j\omega-j\omega_0]}{(a+j\omega)^2+\omega_0^2}
\end{aligned}$$

$$X(j\omega) = \frac{(a+j\omega)}{(a+j\omega)^2+\omega_0^2}$$

Note: The property used to solve this problem is called the "Modulation" property which states that

$$\boldsymbol{x(t)\cos\omega_0 t \xleftrightarrow{\text{FT}} \frac{1}{2}[X(\omega-\omega_0) + X(\omega+\omega_0)]}$$

where $\boldsymbol{x(t)}$ is the modulating signal and $\boldsymbol{\cos\omega_0 t}$ is the carrier signal.

5. $\boldsymbol{x(t) = e^{-at} \sin \omega_0 t u(t)}$

$$\begin{aligned}
\sin \omega_0 t &= \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] \\
X(j\omega) &= \frac{1}{2j}\int_0^\infty e^{-at} e^{j\omega_0 t} e^{-j\omega t}\, dt - \frac{1}{2j}\int_0^\infty e^{-at} e^{-j\omega_0 t} e^{-j\omega t}\, dt \\
&= \frac{1}{2j}\int_0^\infty e^{-(a-j\omega_0+j\omega)}\, dt - \frac{1}{2j}\int_0^\infty e^{-(a+j\omega_0+j\omega)t}\, dt \\
&= \frac{1}{2j}\left[\frac{-e^{-(a-j\omega_0+j\omega)t}}{(a-j\omega_0+j\omega)} + \frac{e^{-(a+j\omega_0+j\omega)t}}{(a+j\omega_0+j\omega)}\right]_0^\infty \\
&= \frac{1}{2j}\left[\frac{1}{(a+j\omega)-j\omega_0} - \frac{1}{(a+j\omega)+j\omega_0}\right] \\
&= \frac{1}{2j}\left[\frac{a+j\omega+j\omega_0-a-j\omega+j\omega_0}{(a+j\omega)^2+\omega_0^2}\right]
\end{aligned}$$

$$X(j\omega) = \frac{\omega_0}{[(a+j\omega)^2 + \omega_0^2]}$$

6. $\boldsymbol{x(t) = [u(t+2) - u(t-2)]\cos 3t}$

$$X_1(t) = u(t+2) - u(t-2) = 1; \quad |t| < 2$$

$$\begin{aligned} X_1(j\omega) &= \int_{-2}^{2} e^{-j\omega t}\,dt \\ &= -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_{-2}^{2} \\ &= -\frac{1}{j\omega}\left[e^{-j2\omega} - e^{j2\omega}\right] \\ &= \frac{2}{\omega}\frac{\left[e^{j2\omega} - e^{-j2\omega}\right]}{2j} \end{aligned}$$

$$\begin{aligned} X_1(j\omega) &= \frac{2}{\omega}\sin 2\,\omega \\ \cos 3t &= \frac{e^{j3t} + e^{-j3t}}{2} \\ F[x(t)\cos\omega_0 t] &= \frac{1}{2}[X(\omega-\omega_0) + X(\omega+\omega_0)] \\ F[\{u(t+2) - u(t-2)\}\cos 3t] &= \frac{[\sin 2(\omega-3)]}{(\omega-3)} + \frac{[\sin 2(\omega+3)]}{(\omega+3)} \end{aligned}$$

$$X(j\omega) = \left[\frac{\sin 2(\omega-3)}{(\omega-3)} + \frac{\sin 2(\omega+3)}{(\omega+3)}\right]$$

7. $\boldsymbol{x(t) = e^{-2|t|}\cos 5t}$

$$\begin{aligned} F[e^{-2|t|}] &= \frac{4}{\omega^2+4} \text{ [see Example 3.1(f)]} \\ F[x(t)\cos\omega_0 t] &= \frac{1}{2}[X(\omega-\omega_0) + X(\omega+\omega_0)] \end{aligned}$$

In the given problem, $\omega_0 = 5$

$$X(j\omega) = \frac{2}{[(\omega-5)^2+4]} + \frac{2}{[(\omega+5)^2+4]}$$

$x(t) = [u(t+2) - u(t-2)]$ is shown in Fig. 3.41.

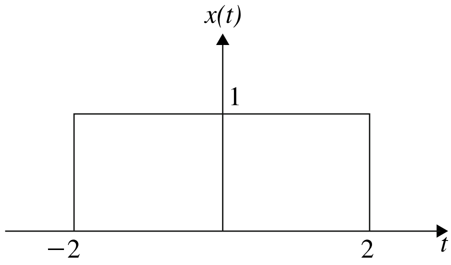


Fig. 3.41 Representation of $x(t) = [u(t+2) - u(t-2)]$

8. $\boldsymbol{x(t) = e^{-3|t|} \sin 2t}$

$$F\left[e^{-3|t|}\right] = \frac{6}{(9+\omega^2)}$$

$$x(t)\sin\,\omega_0 t \overset{\text{FT}}{\longleftrightarrow} \frac{1}{2j}[X(\omega-\omega_0) - X(\omega+\omega_0)]$$

$$F\left[e^{-3|t|}\sin 2t\right] \overset{\text{FT}}{\longleftrightarrow} \frac{1}{2j}\left[\frac{6}{[9+(\omega-2)^2]} - \frac{1}{[9+(\omega+2)^2]}\right]$$

$$x(j\omega) = \frac{-j24}{[9+(\omega-2)^2][9+(\omega+2)^2]}$$

Example 3.23 Consider the following differential equation. Determine the frequency response.

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + 4x(t)$$

Solution Taking FT on both sides of the above differential equation, we get the following algebraic equation. In the above equation, $(j\omega)^2 = \frac{d^2}{dt^2}$; $(j\omega) = \frac{d}{dt}$ are substituted while Fourier transformed.

$$(j\omega)^2 Y(j\omega) + 5(j\omega)Y(j\omega) + 6Y(j\omega) = [(j\omega)+4]X(j\omega)$$

$$\frac{Y(j\omega)}{X(j\omega)} = H(j\omega) = \frac{(j\omega+4)}{[(j\omega)^2+5j\omega+6]}$$

$$H(j\omega) = \frac{(j\omega+4)}{(j\omega+2)(j\omega+3)}$$

$$|H(j\omega)| = \frac{\sqrt{(\omega^2+16)}}{\sqrt{(\omega^2+4)(\omega^2+9)}}$$

$$\underline{|H(j\omega)} = \tan^{-1}\frac{\omega}{4} - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{3}$$

$H(j\omega)$ is the ratio of the Fourier transform of the output variable to the Fourier transform of the input variable. It is called "**Sinusoidal Transfer Function**".

Example 3.24 A certain continuous linear time invariant system is described by the following differential equation.

$$\frac{dy(t)}{dt} + 5y(t) = x(t)$$

Determine $y(t)$, using FT for the following input signals.

(a) $x(t) = e^{-2t}u(t)$
(b) $x(t) = 10u(t)$
(c) $x(t) = \delta(t)$.

Solution

(a) $\boldsymbol{x(t) = e^{-2t}u(t)}$
Taking FT on both sides, we get

$$(j\omega+5)Y(j\omega) = X(j\omega)$$

$$F[e^{-2t}u(t)] = \frac{1}{(j\omega+2)}$$

$$Y(j\omega) = \frac{1}{(j\omega+2)(j\omega+5)}$$

$$= \frac{1}{3}\left[\frac{1}{j\omega+2} - \frac{1}{j\omega+5}\right]$$

$$y(t) = F^{-1}[Y(j\omega)] = \frac{1}{3}\left[e^{-2t} - e^{-5t}\right]u(t)$$

$$y(t) = \frac{1}{3}\left[e^{-2t} - e^{-5t}\right]u(t)$$

(b) $\boldsymbol{x(t) = 10u(t)}$

$$X(j\omega) = F[10u(t)] = \left[10\pi\delta(\omega) + \frac{10}{j\omega}\right]$$

$$Y(j\omega) = \frac{X(j\omega)}{(j\omega+5)}$$

$$= \left[\pi\delta(\omega) + \frac{1}{j\omega}\right]\frac{10}{(j\omega+5)}$$
$$= \frac{10\pi\delta(\omega)}{(j\omega+5)} + \frac{10}{j\omega(j\omega+5)}$$
$$= \frac{10\pi\delta(\omega)}{(j\omega+5)} + \frac{2}{j\omega} - \frac{2}{(j\omega+5)}$$

Applying the property $X(j\omega)\delta(\omega) = X(0)\delta(\omega)$ in the above equation, we get

$$Y(j\omega) = \frac{10}{5}\pi\delta(\omega) + \frac{2}{j\omega} - \frac{2}{(j\omega+5)}$$
$$= 2\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] - \frac{2}{(j\omega+5)}$$
$$y(t) = F^{-1}Y(j\omega) = 2\left[u(t) - e^{-5t}u(t)\right]$$

$$y(t) = 2\left[1 - e^{-5t}\right]u(t)$$

Note

$$\boldsymbol{F^{-1}\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] = u(t).}$$

The above response is called "Step Response" because the input $u(t)$ is a step signal.

(c) $\boldsymbol{x(t) = \delta(t)}$

$$X(j\omega) = 1$$
$$Y(j\omega) = \frac{1}{j\omega+5}$$
$$y(t) = F^{-1}[Y(j\omega)] = e^{-5t}\,u(t)$$

$$y(t) = e^{-5t}\,u(t)$$

The above response is called "Impulse Response of the System" because the input $\delta(t)$ is an impulse.

Example 3.25 Consider an LTI system with the differential equation.

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Find the frequency response and impulse response.

(*Anna University, December, 2006*)

Solution Taking FT on both sides of the above equation, we get

$$[(j\omega)^2 + 4j\omega + 3]Y(j\omega) = (j\omega + 2)X(j\omega)$$

$$\begin{aligned} H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} = \frac{(j\omega + 2)}{[(j\omega)^2 + 4j\omega + 3]} \\ &= \frac{(j\omega + 2)}{(j\omega + 1)(j\omega + 3)} \end{aligned}$$

$$|H(j\omega)| = \frac{\sqrt{(\omega^2 + 4)}}{\sqrt{(\omega^2 + 1)(\omega^2 + 9)}}$$

$$\underline{|H(j\omega)} = \tan^{-1}\frac{\omega}{2} - \tan^{-1}\omega - tan^{-1}\frac{\omega}{3}$$

The above expressions give the magnitude and phase of the frequency response.

To find the impulse response

$$\begin{aligned} x(t) &= \delta(t) \\ F[x(t)] &= F[\delta(t)] = 1 \\ Y(j\omega) &= \frac{(j\omega + 2)}{(j\omega + 1)(j\omega + 3)} \\ &= \frac{A_1}{(j\omega + 1)} + \frac{A_2}{(j\omega + 3)} \\ (j\omega + 2) &= A_1(j\omega + 3) + A_2(j\omega + 1) \end{aligned}$$

Let $j\omega = -1$;

$$1 = 2A_1 \quad \text{or} \quad A_1 = \frac{1}{2}$$

Let $j\omega = -3$;

$$-1 = -2A_2 \quad \text{or} \quad A_2 = \frac{1}{2}$$

$$Y(j\omega) = \frac{1}{2}\left[\frac{1}{(j\omega + 1)} + \frac{1}{(j\omega + 3)}\right]$$

Taking inverse FT, we get

$$y(t) = F^{-1}[Y(j\omega)] = \frac{1}{2}\left[e^{-t} + e^{-3t}\right] u(t)$$

$$y(t) = \frac{1}{2}\left[e^{-t} + e^{-3t}\right] u(t)$$

Example 3.26 An LTI continuous time system is described by the following differential equation.

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = x(t)$$

Determine the impulse response of the system using FT and inverse FT.

Solution Taking FT on both sides, we get the following equation:

$$\left[(j\omega)^2 + 2j\omega + 2\right] Y(j\omega) = X(j\omega)$$

For an impulse input $x(t) = \delta(t)$

$$\begin{aligned}
X(j\omega) &= 1 \\
Y(j\omega) &= \frac{1}{(j\omega)^2 + 2j\omega + 2} \\
&= \frac{1}{(j\omega + 1 + j)(j\omega + 1 - j)} \\
&= \frac{A_1}{(j\omega + 1 + j)} + \frac{A_2}{(j\omega + 1 - j)} \\
1 &= A_1(j\omega + 1 - j) + A_2(j\omega + 1 + j)
\end{aligned}$$

Let $j\omega = -(1 + j)$

$$1 = A_1(-1 - j - 1 - j)$$

$$A_1 = \frac{-1}{2j}; \qquad A_2 = A_1^* = \frac{1}{2j}$$

$$Y(j\omega) = \frac{1}{2j}\left[\frac{-1}{j\omega + (1 + j)} + \frac{1}{j\omega + (1 - j)}\right]$$

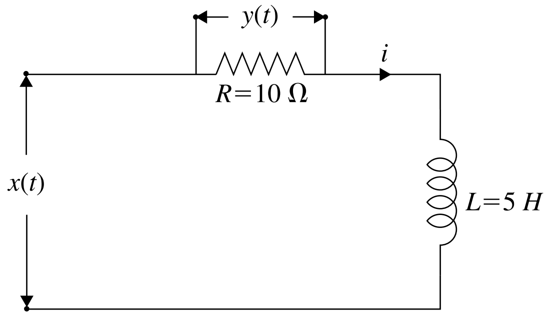


Fig. 3.42 Time response of R-L circuit

Taking inverse FT, we get

$$y(t) = \frac{1}{2j}\left[-e^{-(1+j)t} + e^{-(1-j)t}\right]$$
$$= e^{-t}\left[\frac{e^{jt} - e^{-jt}}{2j}\right]$$

$$y(t) = e^{-t}\sin t$$

Example 3.27 Find the unit step response of the circuit shown in Fig. 3.38. Use Fourier transform method.

(*Anna University, December, 2007*)

Solution For the circuit shown in Fig. 3.42 the following equation is written

$$L\frac{di(t)}{dt} + Ri(t) = x(t)$$
$$5\frac{di(t)}{dt} + 10i(t) = x(t)$$

Taking FT on both sides, we get

$$[5j\omega + 10]I(j\omega) = X(j\omega)$$
$$I(j\omega) = \frac{0.2X(j\omega)}{(j\omega + 2)}$$

For a step input

$$X(j\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$
$$I(j\omega) = \frac{0.2\pi\delta(\omega)}{(j\omega+2)} + \frac{0.2}{j\omega(j\omega+2)}$$

Applying the property $X(j\omega)\delta(\omega) = X(0)\,\delta(\omega)$ the above equation is written as

$$\begin{aligned} I(j\omega) &= 0.1\pi\delta(\omega) + 0.1\left[\frac{1}{j\omega} - \frac{1}{j\omega+2}\right] \\ &= 0.1\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] - \frac{0.1}{j\omega+2} \\ i(t) &= 0.1\left[u(t) - e^{-2t}u(t)\right] \\ y(t) &= i(t)R \end{aligned}$$

$$y(t) = \left[1 - e^{-2t}\right] u(t).$$

Summary

1. Periodic signals are represented by Fourier series as a sum of complex sinusoids or exponentials. However, FS is not applicable to aperiodic signals. Fourier transform gives spectral representation to aperiodic signal. Thus, FT is applicable to periodic and non-periodic signals as well to transform time domain signal $x(t)$ to frequency domain signal $X(j\omega)$. Here the frequency domain representation is continuous.
2. It is possible to transform time domain specifications to frequency domain specifications and *vice versa*. The former is called Fourier transform and the latter is called inverse Fourier transform which are denoted as $F[x(t)]$ and $F^{-1}[X(j\omega)]$ respectively and they are called Fourier transform pair.
3. Fourier transform does not exist for some useful signals. For example for $x(t) = e^{at}u(t)$ FT does not converge.
4. Fourier and Laplace were contemporaries and great mathematicians who were encouraged by the French ruler Napoleon Bonaparte. Laplace, by introducing an exponential decay in the everlasting exponential made many functions converge while FT failed in these cases. Further, Laplace transform is more powerful especially in getting the solution of differential equations compared to FT.
5. FT is a special case of LT which is obtained in many cases by replacing s by $j\omega$. But this is not always true. For example, in the case of a step signal, this is not applicable.
6. Fourier transform has many useful properties. By applying these properties, one can easily get the FT pair of even complex signals. They are powerful tools for manipulating signals in time and frequency domains.

Exercises

I. Short Answer Type Questions

1. **What do you understand by Fourier transform pair?**
When the time function $x(t)$ is transformed to frequency function $X(j\omega)$, the function $x(t)$ is said to be Fourier transformed. When the frequency function $X(j\omega)$ is transformed to $x(t)$ then the function $X(j\omega)$ is said to be inverse Fourier transformed. These transformations are respectively defined as follows:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

The above two equations are called FT pair.
2. **How Fourier transform is different from Fourier series?**
Fourier series is applicable to periodic signals. Fourier transform is applicable periodic and aperiodic signals as well.
3. **How FT is developed from Fourier series?**
When the aperiodic signals is considered as a periodic signal with its fundamental period tending to infinite, the fundamental frequency decreases and the higher harmonics become closer. The frequency components form a continuum and the Fourier series sum becomes Fourier integral which is defined as Fourier transform.
4. **How Parseval's Energy theorem is defined for the frequency domain signal?**
According to Parseval's theorem (French mathematician of late eighteenth and early nineteenth centuries) the energy of the frequency domain is defined as

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$

5. **What is the connection between Fourier transform and Laplace transform?**
The connection between Fourier transform and Laplace transform is that the Fourier transform is the Laplace transform with $s = j\omega$. The Laplace transform of $x(t) = e^{-at}u(t)$ is $X(s) = \frac{1}{(s+a)}$ and its Fourier transform is $X(j\omega) = \frac{1}{(j\omega+a)}$. However, this is not generally true of signals which are not absolutely integrable. The Laplace transform of a step signal is $X(s) = \frac{1}{s}$. The Fourier transform of the step signal is $X(j\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$ and not simply $\frac{1}{j\omega}$.
6. **What do you understand by frequency response?**
If $y(t)$ is the output, $x(t)$ the input and $h(t)$ is the impulse response, then they are related as

$$y(t) = x(t) * h(t)$$

By using convolution property, we get

$$Y(j\omega) = X(j\omega)H(j\omega)$$
$$\frac{Y(j\omega)}{X(j\omega)} = H(j\omega)$$

The function $H(j\omega)$ is called the frequency response.

7. **What is the condition required for the convergence of Fourier transform?**
If the signal $x(t)$ has finite energy or if it is square integrable such that

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$$

then the Fourier transform $X(j\omega)$ converges.

8. **What is the Fourier transform of**

$$\boldsymbol{x(t) = \frac{d^2}{dt^2}x(t+1)}$$

$$F[x(t)] = (j\omega)^2 e^{j\omega} X(j\omega)$$

9. **What is the FT of $\boldsymbol{x(t) = [\delta(t+5) - \delta(t-5)]}$**?

$$X(j\omega) = 2j \sin 5\omega$$

10. **Find the FT of $\boldsymbol{x(t) = 2[u(t+6) - u(t-6)]}$**?

$$X(j\omega) = \frac{4}{\omega} \sin 6\omega = 24\text{sinc}6\omega$$

II. Long Answer Type Questions

1. **Consider the following continuous time signal.**

$$\boldsymbol{x(t) = e^{-5|t|}}$$

Find the FT. Hence determine the FT of $\boldsymbol{tx(t)}$.

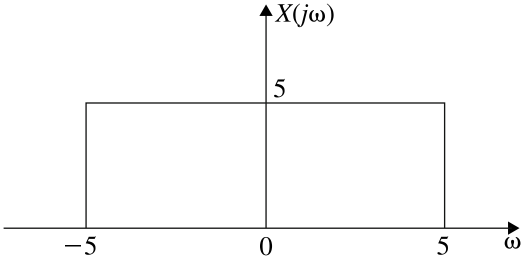


Fig. 3.43 FT of $x(t) = e^{-5|t|}$

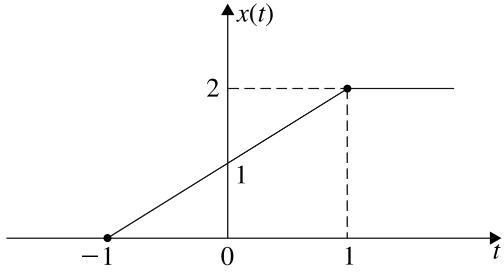


Fig. 3.44 Representation of $x(t)$ for question 3

$$X(j\omega) = \frac{10}{(25 + \omega^2)}$$

$$F\left[te^{-5|t|}\right] = \frac{-j20\omega}{(25 + \omega^2)^2}$$

2. **For the signal $X(j\omega)$ shown in Fig. 3.43, determine $x(t)$?**

$$\boldsymbol{x(t) = 5\frac{\sin 5t}{\pi t}}$$

3. **Consider the signal shown in Fig. 3.44. Find $X(j\omega)$. What is the FT for $x(t-1)$?**

$$X(j\omega) = 2\frac{\sin\omega}{j\omega^2} + e^{-j\omega}$$

$$F[x(t-1)] = \frac{2\sin\omega}{j\omega^2}$$

4. **Using Parseval's theorem evaluate energy in the frequency domain.**

$$x(t) = e^{-4|t|}$$

$$p = \frac{1}{4}$$

5.

$$\boldsymbol{x(t) = e^{-2t}u(t)}$$

and

$$\begin{aligned} h(t) &= e^{-4t}u(t) \\ y(t) &= x(t) * h(t) \end{aligned}$$

Using time convolution property find $Y(j\omega)$ and $y(t)$?

$$\begin{aligned} Y(j\omega) &= \frac{1}{(j\omega + 2)(j\omega + 4)} \\ y(t) &= \frac{1}{2}\left[e^{-2t} - e^{-4t}\right]u(t) \end{aligned}$$

6.

$$\begin{aligned} x(t) &= e^{-2t}u(t) \\ h(t) &= e^{-2t}u(t) \\ y(t) &= x(t) * h(t) \end{aligned}$$

Find $Y(j\omega)$ and hence $y(t)$?

$$\begin{aligned} Y(j\omega) &= \frac{1}{(j\omega + 2)^2} \\ y(t) &= te^{-2t}u(t) \end{aligned}$$

7. **A certain LTIC system is described by the following differential equation.**

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

Determine the Frequency response and the Impulse response?

$$\begin{aligned} H(j\omega) &= \frac{1}{(j\omega + 2)} \\ h(t) &= e^{-2t}u(t) \end{aligned}$$

8. **Consider the following differential equation**

$$\frac{d^2y(t)}{dt^2} + 8\frac{dy(t)}{dt} + 15y(t) = \frac{dx(t)}{dt} + 4x(t)$$

(a) Find the frequency response.
(b) Find the impulse response.
(c) Find the response $y(t)$ due to the input $x(t) = e^{-3t}u(t)$.

(a) $$H(j\omega) = \frac{(j\omega + 4)}{(j\omega + 3)(j\omega + 5)}$$

(b) $$h(t) = \frac{1}{2}\left[e^{-3t} + e^{-5t}\right]u(t)$$

(c) $$y(t) = \frac{1}{4}\left[2te^{-3t} + e^{-3t} - e^{-5t}\right]u(t).$$

Chapter 4
The Laplace Transform Method for the Analysis of Continuous-Time Signals and Systems

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Chapter Objectives

- To develop a new transform method, the Laplace transform (LT) which is applicable for the analysis of continuous-time signals and systems.
- To determine the range of signals to which the LT is applicable.
- To derive the properties of LT.
- To determine the LT of typical Continuous-Time (CT) signals.
- To develop inverse LT method and illustrate it with examples.
- To solve differential equations with and without initial conditions using LT and inverse LT and also by classical method.
- To realize the structure of linear time invariant continuous-time systems using LT.

4.1 Introduction

The Continuous-Time Fourier Transform (CTFT) is a powerful tool for the analysis of CT signals and systems. However, the method has its limitation in that some useful signals do not have CTFT because these signals do not converge. Marquis Pierre Simon de Laplace (1749–1827), the great French mathematician and Astronomer and the contemporary of Fourier (1768–1830), Louis de Lagrange and the French ruler Napoleon, developed a new transform technique which overcame the problem of convergence in CTFT. Laplace, first presented the transform and its applications to solve linear differential equations in a paper published in the year 1779 when he was just 30 years of age. For his excellent contributions to probability theory, astronomy, special functions and celestial mechanics, Laplace was honored by Napoleon, as a

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https://doi.org/10.1007/978-3-031-28280-5_4

policy of honoring and promoting scientists of high caliber, by appointing him as a minister in the French Government. However, Laplace a born genius, showed more interest in his research activities and totally neglected the administrative work in the government. It was no surprise that soon Laplace was sacked from the ministerial post by his admirer, Napoleon.

The CTFT expresses signals as linear combinations of complex sinusoids. Some useful signals, when expressed as a combination of complex sinusoids, do not converge and they do not have Fourier transform (FT). However, Laplace made a small modification in his transform technique from time domain to frequency domain by expressing time signals as linear combinations of complex exponential instead of complex sinusoids. LT is more general since complex sinusoids are a special case of complex exponentials. Thus, LT can describe functions that FT cannot describe. Both the FT and LT using mathematical operations, convert the time signal $x(t)$ to frequency function $X(j\omega)$ and $X(s)$ respectively, where $s = \sigma + j\omega$. By introducing σ in LT method, most of the signals become damped waves and convergence becomes possible. However, it is to be noted that there exists a class of signals which do not converge in LT also and, for these signals, LT does not exist. The LT, even though a very powerful tool in the analysis and design of linear time invariant signals and systems today, did not catch on until nearly a century later. We discuss the development of the LT in the following sections.

4.2 Definition and Derivations of the LT

The time signal $x(t)$ is expressed as a linear combination of complex sinusoids of the form $e^{j\omega t}$ by the FT. Here $j\omega$ takes only imaginary value of ω which is associated with the frequency f as $\omega = 2\pi f$. Thus, some of the useful time functions such as $x(t) = e^{at}$ do not coverage as per the FT. By changing the complex sinusoid to complex exponential of the form e^{st}, the FT can be generated and is termed as the LT and is defined as

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{4.1}$$

The complex variable s has a real part and an imaginary part and is expressed as

$$s = \sigma + j\omega \tag{4.2}$$

If the real part $\sigma = 0$, then Eq. (4.1) becomes a special case and it becomes the FT. By substituting $s = (\sigma + j\omega)$, Eq. (4.1) can be written as follows:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt \tag{4.3}$$

In Eq. (4.3), the real exponential convergence factor $e^{-\sigma t}$ enables some of the time functions $x(t)$ to converge in the complex s plane. Equation (4.1) is called the two-sided (or bilateral) LT. The signal $x(t)$ is obtained from $X(s)$ by taking the inverse LT which is derived as

$$x(t) = L^{-1}X(s) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \tag{4.4}$$

Equations (4.1) and (4.4) are called two-sided or bilateral LT pair. The symbol L^{-1} is used when $X(s)$ is the inverse Laplace transformed. The following notations are used to represent LT and inverse LT:

$$X(s) = L[x(t)]$$

or

$$\begin{aligned} x(t) &\overset{L}{\longleftrightarrow} X(s) \\ x(t) &= L^{-1}[X(s)] \\ X(s) &= \overset{L^{-1}}{\longleftrightarrow} x(t) \end{aligned} \tag{4.5}$$

It is to be noted that the time function is represented by lower case letter and the *s* function by upper case letter.

4.2.1 LT of Causal and Non-causal Systems

In Eq. (4.1), the transformation of $x(t)$ to $X(s)$ is done for the following conditions:

- $x(t)$ is anti-causal where $t < 0$,
- $x(t)$ is an impulse where $t = 0$,
- $x(t)$ is causal where $t > 0$.

The unilateral LT is a special case of LT and is defined as follows:

$$X(s) = \int_0^\infty x(t)e^{-st}dt \tag{4.6}$$

It is to be noted here that Eq. (4.6) is valid only for causal signals and systems. For non-casual signals and systems the limits if integration have to be changed. The following two examples illustrate the method to determine the LT for casual and non-causal signals.

Example 4.1 For the following signal determine the LT,

$$x(t) = e^{-at}u(t)$$

Solution The given signal $x(t)$ is a causal signal. The limit of integration is therefore from 0 to ∞. Hence

$$\begin{aligned} X(s) &= \int_0^\infty e^{-at} e^{-st} dt \\ &= \int_0^\infty e^{-(s+a)t} dt \\ &= -\frac{1}{(s+a)} \left[e^{-(s+a)t}\right]_0^\infty = -\frac{1}{(s+a)} \left[e^{-(s+a)\infty} - e^{-(s+a)0}\right] \end{aligned}$$

$$X(s) = \frac{1}{(s+a)}$$

The above integration converges when the upper limit ∞ is applied iff $(s+a) > 0$ or $s > -a$. If $(s+a) < 0$, then $e^{(s+a)\infty}$ does not converge. In such a case LT does not exist.

Example 4.2 Consider the following signal:

$$x(t) = e^{-at} u(-t)$$

Determine the LT.

Solution The given signal $x(t)$ is a non-causal signal. Hence, the limit of integration is from $-\infty$ to 0.

$$\begin{aligned} X(s) &= \int_{-\infty}^0 x(t) e^{-st} dt = \int_{-\infty}^0 e^{-at} e^{-st} dt \\ &= \int_{-\infty}^0 e^{-(s+a)} dt \\ &= \frac{-1}{(s+a)} \left[e^{-(s+a)t}\right]_{-\infty}^0 \end{aligned}$$

$$X(s) = \frac{-1}{(s+a)}$$

The above integration converges when the lower limit $-\infty$ is applied iff $(s+a) < 0$ or $s < -a$. The above two examples illustrate that for the same time signal $x(t)$, the LT is also same with a change of sign. However, the mode of convergence is different which is an important thing to note. This will be discussed in detail in the sections to follow.

4.3 The Existence of LT

Consider the one-sided LT given below.

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

Substituting $s = \sigma + j\omega$ in the above equation, we get

$$X(s) = \int_0^\infty \left[x(t)e^{-\sigma t}\right] e^{-j\omega t}dt$$

Since $|e^{-j\omega t}| = 1$, the above integral can be written as

$$X(s) = \int_0^\infty \left[x(t)e^{-\sigma t}\right] dt \tag{4.7}$$

The integral in Eq. (4.7) converges if

$$\int_0^\infty \left[x(t)e^{-\sigma t}\right] dt < \infty \tag{4.8}$$

In other words, the LT of (4.7) exists if the integral of equation (4.8) is finite for some value of $\sigma > \sigma_0$ or Re(s) which is σ should be greater than σ_0 which is expressed as

$$\sigma > \sigma_0$$

4.4 The Region of Convergence

One of the limitations of CTFT as mentioned earlier is that some useful functions whether causal or non-causal do not have FT. By making the complex variable s as expressed in Eq. (4.2) and defining LT as in Eq. (4.1), it is possible to overcome this limitation of non-convergence of FT. For example, consider the following causal signal:

$$x(t) = Ae^{at}u(t) \qquad a > 0 \tag{4.9}$$

The plot of equation (4.9) as a function of time is shown in Fig. 4.1. From Fig. 4.1, it is evident that $x(t)$ increases without bound as t increases. It can be easily shown that FT does not exist for the above $x(t)$. However, the LT exists for the above $x(t)$ with certain constraint and it is derived as follows. Substituting $x(t) = Ae^{at}$ in (4.1), the following equation is obtained:

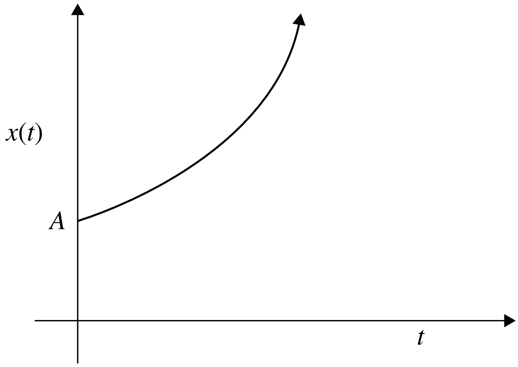


Fig. 4.1 Plot of $x(t) = Ae^{at}u(t)$

$$X(s) = \int_{-\infty}^{\infty} Ae^{at}e^{-st}u(t)dt \tag{4.10}$$

For a causal signal (also called right-sided signal), changing the limit of integration, we get

$$X(s) = \int_{0}^{\infty} Ae^{at}e^{-st}dt \tag{4.11}$$

$$= A\int_{0}^{\infty} e^{-(s-a)t}dt \tag{4.12}$$

$$= \frac{-A}{(s-a)}\left[e^{-(s-a)t}\right]_0^{\infty} \tag{4.13}$$

$$X(s) = \frac{A}{(s-a)} \tag{4.14}$$

Equation (4.13) converges iff $(s - a) > 0$. In other words Re $s > a$. In that case when the upper limit of $t = \infty$ is applied, $X(s) = 0$ and when the lower limit of $t = 0$ is applied, $X(s)$ is finite. Thus, Eq. (4.13) is simplified and given in Eq. (4.14). The LT of $x(t)$ of (4.9) exists or Eq. (4.13) converges iff $\sigma > a$ in the complex s-plane. This is called the region of convergence.

The region of convergence which is denoted as ROC is therefore defined as the set of values of s of the real part of s for which part the integral of equation (4.1) converges.

The ROC of $x(t)$ in Eq. (4.9) is illustrated in Fig. 4.2. It is to be noted here that $X(s)$ in Eq. (4.14) becomes infinity at $s = a$. Therefore, the points in the s-plane at which the function $X(s)$ becomes infinity are called poles and are marked by a small cross $\times$. Now consider a function $X(s) = (s + a)$. The function $X(s)$ becomes zero

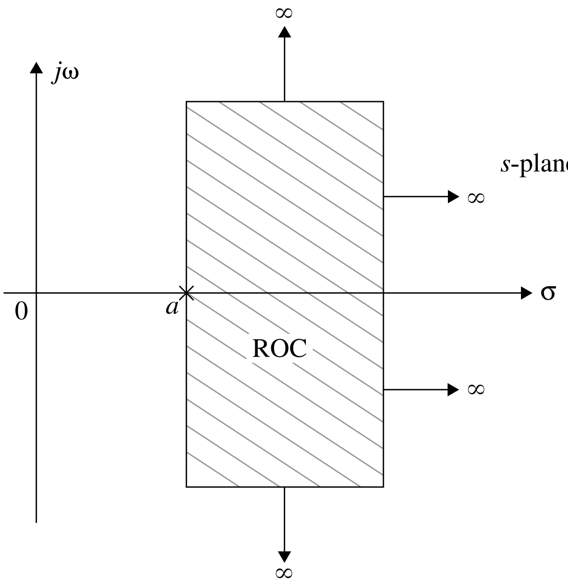


Fig. 4.2 Pole-zero plot and ROC of $X(s) = \frac{A}{(s-a)}$

at $s = -a$. Therefore, the point in the s-plane at which the function $X(s)$ becomes zero are called zeros and are marked by a small circle O.

Now consider the following non-causal signal or otherwise called left-sided signal shown in Fig. 4.3.

$$x(t) = Ae^{-at}u(-t) \tag{4.15}$$

The LT of the above signal is obtained from

$$\begin{aligned} X(s) &= \int_{-\infty}^{0} x(t)e^{-st}dt \\ &= \int_{-\infty}^{0} Ae^{-at}e^{-st}dt \qquad (4.16) \\ &= \int_{-\infty}^{0} Ae^{-(s+a)t}dt \\ &= \frac{-A}{(s+a)}\left[e^{-(s+a)t}\right]_{-\infty}^{0} \qquad (4.17) \end{aligned}$$

It is evident from Eq. (4.17) that the integral given in Eq. (4.16) will converge iff $(s+a) < 0$ when the lower limit of $t = -\infty$ is applied to (4.17). Thus, $X(s)$ is obtained as

$$X(s) = -\frac{A}{(s+a)}$$

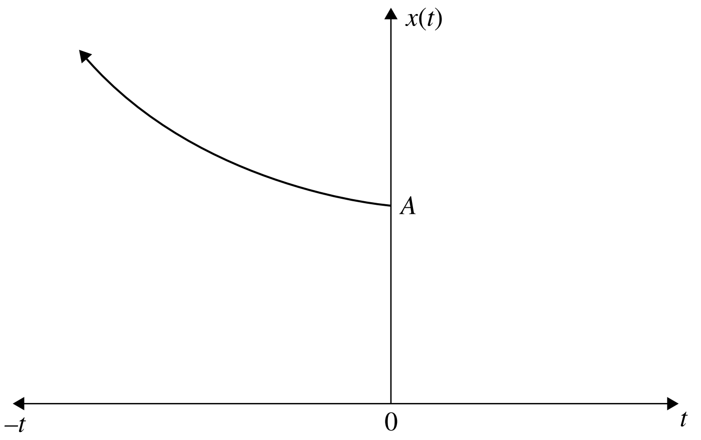


Fig. 4.3 Plot of $x(t) = Ae^{-at}u(-t)$

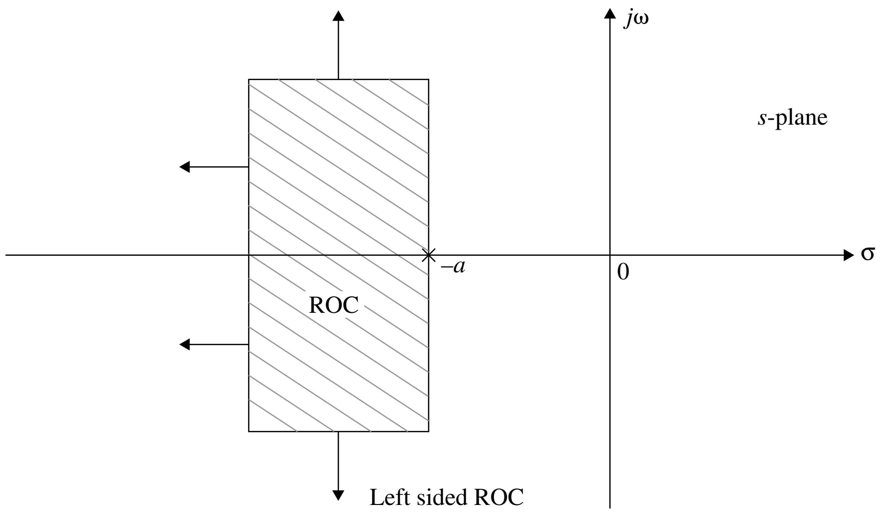


Fig. 4.4 ROC of $X(s) = \dfrac{A}{(s+a)}$

The ROC for the left-sided signal is Re $s < -a$. The ROC is shown in Fig. 4.4.

From the above examples illustrated, for the same $X(s)$, different time signals $x(t)$ exist and therefore the inverse LT is not unique. Hence, it is necessary to specify the ROC while determining LT and inverse LT. However, for unilateral LT, there exists one to one correspondence between the LT pair. For the bilateral or two-sided LT it is essential to specify the ROC to avoid any ambiguity.

4.4.1 Properties of ROCs for LT

Property 1: The ROC of $X(s)$ consists of parallel strips to the imaginary axis.
Property 2: The ROC of LT does not include any pole of $X(s)$.
Property 3: If $x(t)$ is a finite duration signal, and is absolutely integrable then the ROC of $X(s)$ is the entire s-plane.
Property 4: For the right-sided (causal) signal if the $\mathrm{Re}(s) = \sigma_0$ and is in ROC, then for all the values of s for which $\mathrm{Re}(s) > \sigma_0$ is also in ROC.
Property 5: If $x(t)$ is a left-sided (non-causal) signal and if $\mathrm{Re}(s) = \sigma_0$ is in ROC, then for all the values of s for which $\mathrm{Re}(s) < \sigma_0$ is also in ROC.
Property 6: If $x(t)$ is two-sided signal and if $\mathrm{Re}(s) = \sigma_0$ and is in ROC, then the ROC of $X(s)$ will consist of a strip in the s-plane which will include $\mathrm{Re}(s) = \sigma_0$.

The following examples illustrate the above properties of ROC and pole-zero locations of $X(s)$ in the s-plane.

Example 4.3 Determine the LT of the following signal. Mark the poles and ROC in the s-plane. $x(t) = Ae^{-at}u(t) + Be^{-bt}u(-t)$ where $a > 0$, $b > 0$ and $|a| > |b|$.

Solution

1. The given signal $x(t)$ consists of causal and anti-causal signals and can be written as

$$x(t) = x_1(t) + x_2(t)$$

where

$$x_1(t) = Ae^{-at}u(t)$$
$$x_2(t) = Be^{-bt}u(-t)$$

2. $X_1(s)$ is found as follows for a right-sided signal.

$$\begin{aligned} X_1(s) &= \int_0^{\infty} Ae^{-at}e^{-st}dt \\ &= A\int_0^{\infty} e^{-(s+a)t}dt \\ &= \frac{-A}{(s+a)}\left[e^{-(s-a)t}\right]_0^{-\infty} \\ &= \frac{A}{(s+a)} \end{aligned}$$

The ROC is $\mathrm{Re}(s) > -a$.

3. $X_2(s)$ is found as follows for a left-sided signal.

$$\begin{aligned}X_2(s) &= \int_{-\infty}^{0} Be^{-bt}e^{-st}dt \\ &= B\int_{-\infty}^{0} e^{-(s+b)t}dt \\ &= \frac{-B}{(s+b)}\left[e^{-(s+b)t}\right]_{-\infty}^{0} \\ &= -\frac{B}{(s+b)}[1-0] \\ &= \frac{-B}{(s+b)}\end{aligned}$$

The ROC is $\text{Re}(s) < -b$.

4. $$X(s) = X_1(s) + X_2(s) = \frac{A}{(s+a)} - \frac{B}{(s+b)}$$

5. The poles and ROC are marked as shown in Fig. 4.5b. In Fig. 4.5b, $|a| > |b|$. Vertical lines passing through $-a$ and $-b$ are drawn. For $X_1(s)$, the ROC is right-sided and for $X_2(s)$ the ROC is left-sided. A strip where $-a <\text{Re } s < -b$ is drawn and hatched and the ROC is identified.
6. Consider the case where $|b| > |a|$. The poles are located as shown in Fig. 4.5c. Vertical line passing through $-a$ and $-b$ are drawn. For $X_1(s)$, the ROC is right-sided and a strip where $\text{Re}(s) > -a$ is drawn and hatched. For $X_2(s)$, the ROC is left-sided. A vertical strip to the left of $-b$ is formed and hatched. **It is to be noted that the ROC s of $x_1(t)$ and $x_2(t)$ do not overlap and hence $x(t)$ does not have LT.**

Example 4.4 Determine the LT of

$$x(t) = e^{-2t}u(t) + e^{-3t}u(t)$$

and sketch the ROC in the s-plane.

(*Anna University, May, 2007*)

Solution

1. $x(t)$ is completely a right-sided signal and hence the limit of the LT integration is from $t = 0$ to $t = \infty$. Thus, the following equation is written for $X(s)$.

$$\begin{aligned}X(s) &= \int_{0}^{\infty} e^{-2t}e^{-st}dt + \int_{0}^{\infty} e^{-3t}e^{-st}dt \\ &= \int_{0}^{\infty} e^{-(s+2)t}dt + \int_{0}^{\infty} e^{-(s+3)t}dt \\ &= \frac{1}{(s+2)} + \frac{1}{(s+3)} = \frac{(2s+5)}{(s+2)(s+3)}\end{aligned}$$

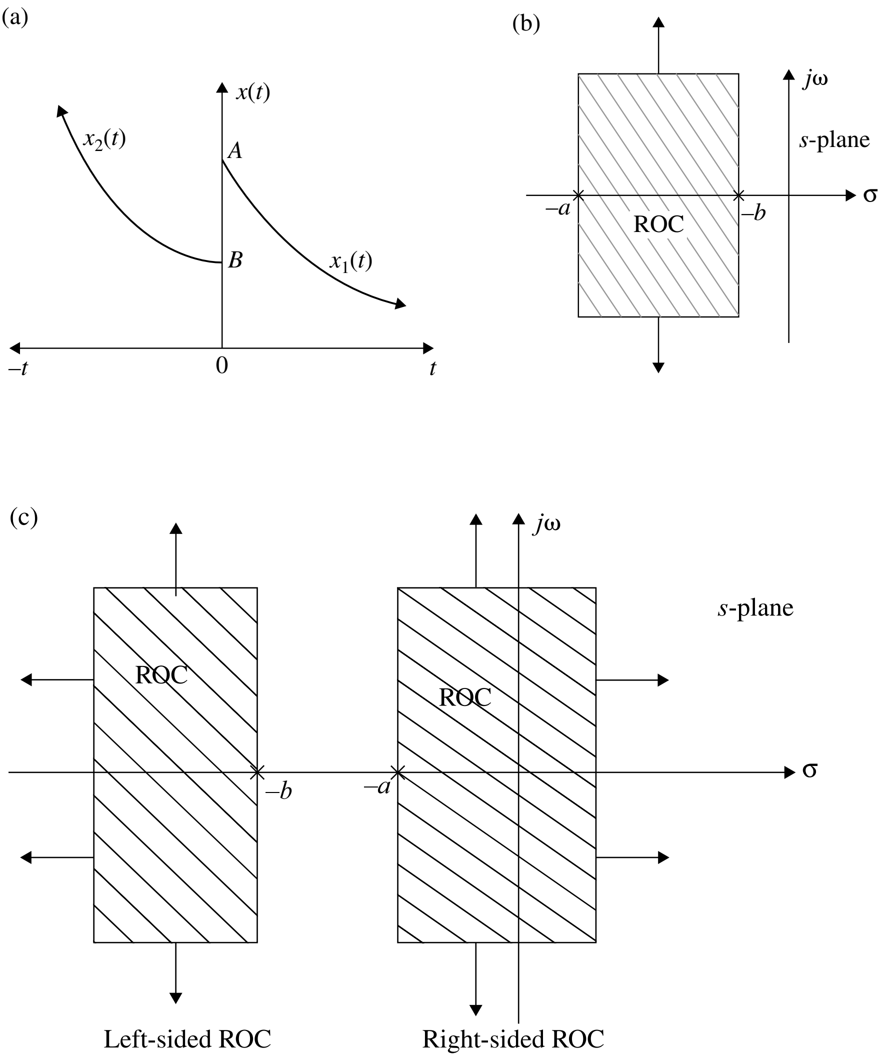


Fig. 4.5 **a** Representation of $x(t)$. **b** ROC and poles of $X(s)$ $|a| > |b|$. **c** Poles and ROC of $X(s)$ for $|b| > |a|$

$$X(s) = \frac{2(s+2.5)}{(s+2)(s+3)}$$

2. The poles are at $s = -2$ and $s = -3$ and a zero is at $s = -2.5$ and are marked in Fig. 4.6.
3. For the pole $\frac{1}{s+2}$, the ROC is right-sided to the vertical line passing through $\sigma = -2$. For the pole $\frac{1}{s+3}$, the ROC is also right-sided passing through $\sigma = -3$.

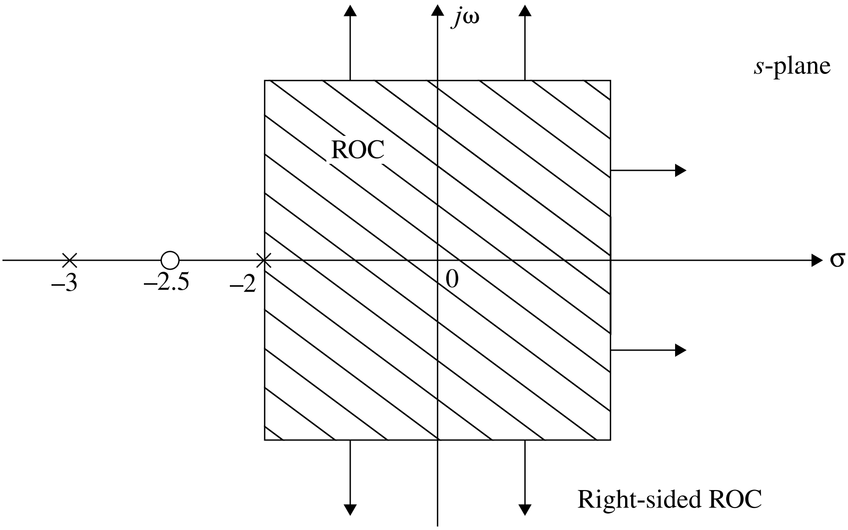


Fig. 4.6 Poles and zeros and ROC of $X(s) = \dfrac{2(s+2.5)}{(s+2)(s+3)}$

If ROC where $\sigma > -2$ is satisfied then ROC where $\sigma > -3$ is automatically satisfied. Further, no pole of $X(s)$ will be inside the ROC.

4. A strip to the right of $\sigma = -2$ is created and shaded. The strip is enlarged to ∞ in the direction of real and imaginary axis.
5. **Thus, the ROC of a causal signal is to the right of the right most pole of $X(s)$.**

Example 4.5 Determine the LT of

$$x(t) = e^{-2t}u(-t) + e^{-3t}u(-t)$$

Locate the poles and zero of $X(s)$ and also the ROC in the s-plane.

Solution

1. The given signal is fully a left-sided signal and hence the limit of LT integration is from $-\infty$ to 0. The LT of $x(t)$ is obtained as follows:

$$X(s) = \int_{-\infty}^{0} e^{-2t}e^{-st}dt + \int_{-\infty}^{0} e^{-3t}e^{-st}dt = \int_{-\infty}^{0} e^{-(s+2)t}dt + \int_{-\infty}^{0} e^{-(s+3)t}dt$$

$$= \frac{-1}{(s+2)}\left[e^{-(s+2)t}\right]_{-\infty}^{0} - \frac{1}{(s+3)}\left[e^{-(s+3)t}\right]_{-\infty}^{0}$$

$$X(s) = -\frac{1}{(s+2)} - \frac{1}{(s+3)} \qquad \text{ROC Re } s < -3$$

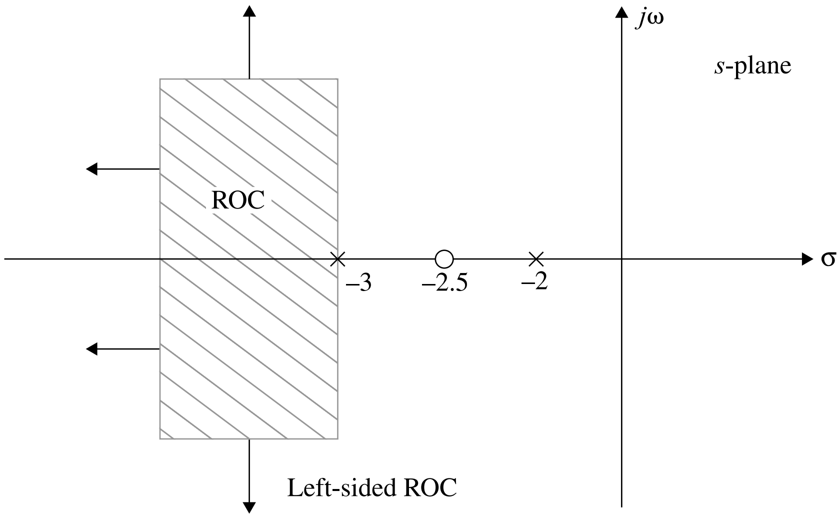


Fig. 4.7 Poles and zeros and ROC of $X(s) = \frac{-2(s+2.5)}{(s+2)(s+3)}$

$$X(s) = \frac{-2(s+2.5)}{(s+2)(s+3)}$$

2. The poles are at $s = -2$ and $s = -3$ and a zero is at $s = -2.5$ and are marked in Fig. 4.7.
3. For the pole $\frac{1}{(s+2)}$, the ROC is left-sided to the vertical line passing through $\sigma = -2$. For the pole $\frac{1}{s+3}$, the ROC is also left-sided to the vertical line passing through $\sigma = -3$. If ROC where $\sigma = -3$ is satisfied then ROC where $\sigma = -2$ is also satisfied. Further, no pole of $X(s)$ will be inside the ROC.
4. A vertical strip to the left of $\sigma = -3$ is created and shaded. The strip is enlarged to ∞ in the direction of real and imaginary axis.
5. **Thus, the ROC of a non-causal signal is to the left of the left most pole of $X(s)$.**

Example 4.6 Consider the following signal:

$$x(t) = e^{-2t}u(-t) + e^{-3t}u(t)$$

Determine the LT and locate the poles and zeros and the ROC in the s-plane.

Solution

1. The given signal is a combination of left- and right-sided. The integration is performed as given below:

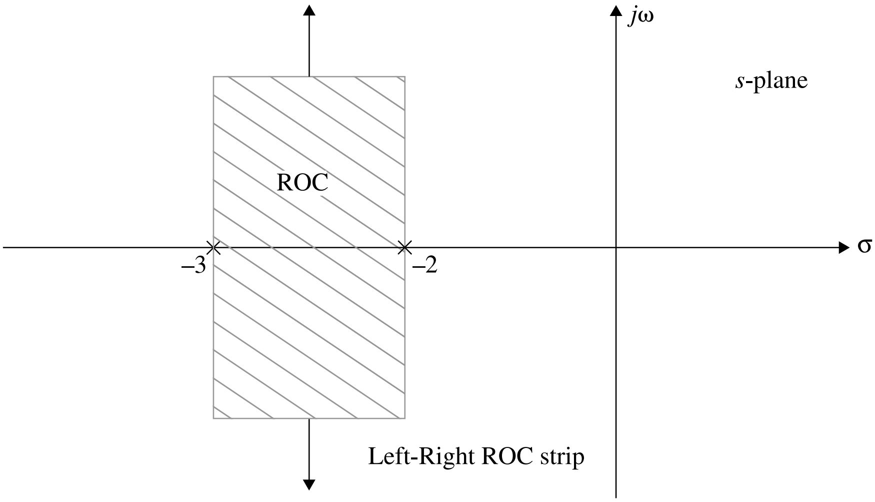


Fig. 4.8 Poles and zeros of $X(s) = \dfrac{-1}{(s+2)(s+3)}$ and the ROC

$$X(s) = \int_{-\infty}^{0} e^{-2t}e^{-st}dt + \int_{0}^{\infty} e^{-3t}e^{-st}dt$$
$$= \int_{-\infty}^{0} e^{-(s+2)t}dt + \int_{0}^{\infty} e^{-(s+3)t}dt$$
$$= \frac{1}{(s+2)}\left[e^{-(s+2)t}\right]_{-\infty}^{0} - \frac{1}{(s+3)}\left[e^{-(s+3)t}\right]_{0}^{\infty} = -\frac{1}{(s+2)} + \frac{1}{(s+3)}$$

$$X(s) = \frac{-1}{(s+2)(s+3)} \quad \text{ROC} - 3 < \text{Re}\, s < -2$$

2. The pole locations are shown in Fig. 4.8. For the left-sided signal the ROC is Re $s < -2$ and for the right-sided signal the ROC is Re $s > -3$. The resultant ROC is a strip in between the vertical lines passing through $\sigma = -2$ and $\sigma = -3$. The strip is shaded as shown in Fig. 4.8. It is enlarged in the vertical direction. The poles are at $s = -2$ and $s = -3$. There is no zero for this function.

Example 4.7 Determine the LT and locate the poles and zeros and ROC in the s-plane for the following signal:

$$x(t) = Au(t)$$

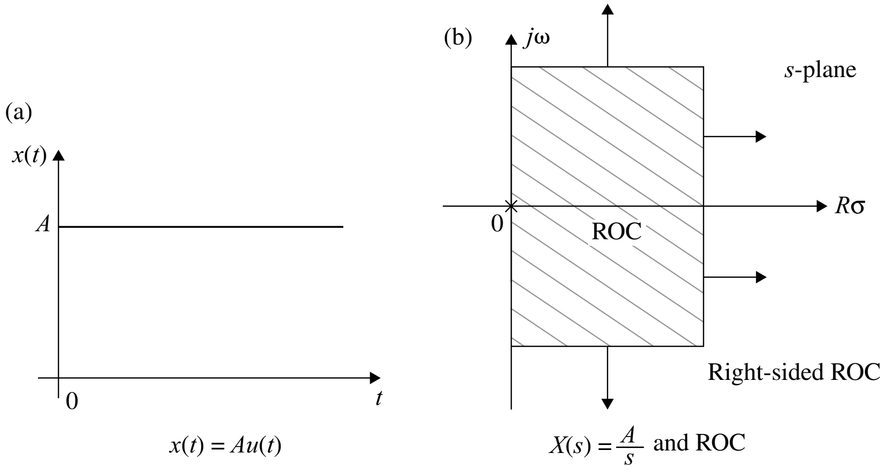


Fig. 4.9 Representation of $x(t)$ and ROC

Solution

1. The given signal is right-sided signal. Its LT is obtained as follows:

$$X(s) = \int_0^{\infty} Ae^{-st}dt$$
$$= \frac{-A}{s}\left[e^{-st}\right]_0^{\infty}$$

$$X(s) = \frac{A}{s} \qquad \text{ROC Re } s > 0.$$

2. For the given signal, a pole at the origin exists and it is marked in Fig. 4.9b.
3. The LT converges only if $\sigma > 0$. Thus, the ROC is the entire right half of s-plane.

4.5 The Unilateral Laplace Transform

The unilateral LT is a special case of bilateral LT and is defined as

$$X(s) = \int_0^{\infty} x(t)e^{-st}dt \tag{4.18}$$

The unilateral LT has the following features:

1. The unilateral LT simplifies the system analysis considerably.

2. The signals are restricted to causal signals.
3. There is one to one correspondence between LT and inverse LT.
4. In view of the above advantages, Laplace transform means unilateral LT as defined in Eq. (4.18) unless otherwise it is specifically mentioned that the signal is anti-causal.

Before we go for the determination of LT of some of the commonly used signals, we give below some of the properties of LT which will be useful to determine $X(s)$ from $x(t)$ and *vice versa* in a simplified way.

4.6 Properties of Laplace Transform

4.6.1 Linearity

$$x_1(t) \overset{L}{\longleftrightarrow} X_1(s)$$

$$x_2(t) \overset{L}{\longleftrightarrow} X_2(s)$$

$$[a_1 x_1(t) + a_2 x_2(t)] \overset{L}{\longleftrightarrow} [a_1 X_1(s) + a_2 X_2(s)] \tag{4.19}$$

4.6.2 Time Shifting

Let $x(t)$ be time shifted to the right (time delay) by a real constant t_0. The delayed time function is written as $x(t - t_0)$. As per the time shifting property,

$$\begin{aligned} x(t) &\overset{L}{\longleftrightarrow} X(s) \\ x(t - t_0) &\overset{L}{\longleftrightarrow} X(s)e^{-st_0} \end{aligned} \tag{4.20}$$

Proof By definition of LT,

$$L[x(t - t_0)] = \int_0^\infty x(t - t_0)e^{-st} dt \tag{4.21}$$

Let

$$\begin{aligned} t - t_0 &= \lambda \\ dt &= d\lambda \end{aligned}$$

For the integration of equation (4.21), the lower and upper limits are determined as follows.

When $t = 0$, $\lambda = -t_0$ and when $t = \infty$, then $\lambda = \infty$. Thus, Eq. (4.21) is written as follows:

$$L[x(t - t_0)] = \int_{-t_0}^{\infty} x(\lambda)e^{-s(\lambda+t_0)}d\lambda \tag{4.22}$$

For a causal signal, $x(t) = 0$ for $t < 0$ and the lower limit of integration is zero. Now Eq. (4.22) is written as follows:

$$\begin{aligned} L[x(\lambda)] &= e^{-st_0}\int_{0}^{\infty} x(\lambda)e^{-s\lambda}d\lambda \\ &= e^{-st_0}X(s) \end{aligned}$$

Thus,

$$x(t - t_0) \stackrel{L}{\longleftrightarrow} X(s)e^{-st_0} \qquad t_0 > 0 \tag{4.23}$$

4.6.3 Frequency Shifting

According to frequency shifting property, if

$$\begin{aligned} x(t) &\stackrel{L}{\longleftrightarrow} X(s) \\ x(t)e^{s_0 t} &\stackrel{L}{\longleftrightarrow} X(s - s_0) \end{aligned}$$

Proof

$$L[x(t)e^{s_0 t}] = \int_{0}^{\infty} x(t)e^{s_0 t}e^{-st}dt$$

$$\begin{aligned} L[x(t)e^{s_0 t}] &= \int_{0}^{\infty} x(t)e^{-(s-s_0)t}dt \\ &= X(s - s_0) \end{aligned}$$

$$x(t)e^{s_0 t} \stackrel{L}{\longleftrightarrow} X(s - s_0) \tag{4.24}$$

4.6.4 Time Scaling

The time scaling property states that if

$$x(t) \overset{L}{\longleftrightarrow} X(s)$$
$$x(at) \overset{L}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

Proof

$$L[x(at)] = \int_0^\infty x(at)e^{-st}dt \tag{4.25}$$

Let

$$\lambda = at \quad \text{and} \quad d\lambda = adt$$

For the lower limit of integration of equation (4.25), when $t = 0$, $\lambda = 0$ and for the upper limit of integration when $t = \infty$, then $\lambda = \infty$. Hence, Eq. (4.25) is written as follows:

$$\begin{aligned} L[x(at)] &= \int_0^\infty x(\lambda)e^{-\frac{\lambda s}{a}} \frac{1}{a} d\lambda \\ &= \frac{1}{|a|} \int_0^\infty x(\lambda)e^{-\frac{s}{a}\lambda} d\lambda \\ &= \frac{1}{a} X\left(\frac{s}{a}\right) \end{aligned}$$

$$x(at) \overset{L}{\longleftrightarrow} \frac{1}{a} X\left(\frac{s}{a}\right) \tag{4.26}$$

4.6.5 Frequency Scaling

According to frequency scaling property, if

$$x(t) \overset{L}{\longleftrightarrow} X(s)$$
$$\frac{1}{a} x\left(\frac{t}{a}\right) \overset{L}{\longleftrightarrow} X(as)$$

Proof According to time scaling property,

$$x(at) \overset{L}{\longleftrightarrow} \frac{1}{a} X\left(\frac{s}{a}\right)$$

Let

$$b = \frac{1}{a}$$
$$x\left(\frac{t}{b}\right) \stackrel{L}{\longleftrightarrow} bX(bs)$$

Replacing b by a, we get

$$\frac{1}{a}x\left(\frac{t}{a}\right) \stackrel{L}{\longleftrightarrow} X(as) \tag{4.27}$$

4.6.6 Time Differentiation

$$x(t) \stackrel{L}{\longleftrightarrow} X(s)$$
$$\frac{dx(t)}{dt} \stackrel{L}{\longleftrightarrow} sX(s) - x(0^-)$$
$$\frac{d^2x(t)}{dt^2} \stackrel{L}{\longleftrightarrow} s^2X(s) - sx(0^-) - \frac{d}{dt}x(0^-)$$

Proof

$$X(s) = \int_0^\infty x(t)e^{-st}dt \tag{4.28}$$

The above integral is evaluated by parts using

$$\int udv = uv - \int vdu$$

Let $u = x(t)$ and $dv = e^{-st}dt$; $du = \frac{d}{dt}x(t)dt$ and $v = -\frac{1}{s}e^{-st}$

$$\int_0^\infty x(t)e^{-st}dt = \left[\frac{-1}{s}x(t)e^{-st}\right]_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\frac{d}{dt}x(t)dt$$

or

$$X(s) = \frac{1}{s}x(0) + \frac{1}{s}\int_0^\infty e^{-st}\frac{d}{dt}x(t)dt.$$

But

$$L\left[\frac{d}{dt}(x(t))\right] = \int_0^\infty \frac{d}{dt}(x(t))e^{-st}dt$$

$$\therefore \qquad L\frac{d}{dt}(x(t)) \overset{L}{\longleftrightarrow} sX(s) - x(0^-) \tag{4.29}$$

The time differentiation twice is proved as follows:

$$\frac{d^2}{dt^2}(x(t)) = \frac{d}{dt}\left(\frac{d}{dt}(x(t))\right)$$

Using the property

$$\frac{d}{dt}(x(t)) \overset{L}{\longleftrightarrow} sX(s) - x(0^-)$$

we get

$$L\left[\frac{d^2(x(t))}{dt^2}\right] = sL\left[\frac{d}{dt}(x(t))\right] - \frac{d}{dt}(x(0^-))\bigg|_{t=0}$$

$$\frac{d^2(x(t))}{dt^2} \overset{L}{\longleftrightarrow} s^2X(s) - sx(0^-) - \frac{d}{dt}(x(0^-))$$

In general

$$\begin{gathered}\frac{d^n x(t)}{dt^n} \overset{L}{\longleftrightarrow} s^n X(s) - s^{n-1}x(0^-) - s^{n-2}x(0^-)\cdots x^{n-1}(0^-)\\ \text{OR}\\ \frac{d^n x(t)}{dt^n} \overset{L}{\longleftrightarrow} s^n X(s) - \sum_{k=1}^{n} s^n x^{k-1}(0^-)\end{gathered} \tag{4.30}$$

4.6.7 Time Integration

The time integration property states that if

$$\begin{aligned} x(t) &\overset{L}{\longleftrightarrow} X(s)\\ \int_0^t x(\tau)d\tau &\overset{L}{\longleftrightarrow} \frac{X(s)}{s}\end{aligned}$$

Proof We define

$$f(t) = \int_0^t x(\tau)d\tau.$$

Differentiating the above equation, we get

$$\frac{df(t)}{dt} = x(t) \quad \text{and} \quad x(0^-) = 0$$

if

$$f(t) \overset{L}{\longleftrightarrow} F(s)$$
$$X(s) = L\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0^-) = sF(s) \quad \text{if} \quad f(0^-) = 0$$
$$F(s) = \frac{X(s)}{s}$$

$$\int_0^t x(\tau)d\tau \overset{L}{\longleftrightarrow} \frac{X(s)}{s} \tag{4.31}$$

4.6.8 Time Convolution

The time convolution property states that if

$$x_1(t) \overset{L}{\longleftrightarrow} X_1(s)$$
$$x_2(t) \overset{L}{\longleftrightarrow} X_2(s)$$
$$x_1(t) * x_2(t) \overset{L}{\longleftrightarrow} X_1(s)X_2(s) \tag{4.32}$$

Proof

$$\begin{aligned} L[x_1(t) * x_2(t)] &= \int_{-\infty}^{\infty} e^{-st}\left[\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right]dt \\ &= \int_{-\infty}^{\infty} x_1(\tau)\left[\int_{-\infty}^{\infty} e^{-st}x_2(t-\tau)dt\right]d\tau \end{aligned}$$

The inner integral is the LT of $x_2(t-\tau)$ with a time delay τ. Substituting

$$\int_{-\infty}^{\infty} e^{-st}x_2(t-\tau)dt = X_2(s)e^{-\tau s}$$

in the above equation, we get

$$\begin{aligned} L[x_1(t) * x_2(t)] &= \int_{-\infty}^{\infty} x_1(\tau) X_2(s) e^{-\tau s} d\tau \\ &= X_2(s) \int_{-\infty}^{\infty} x_1(\tau) e^{-\tau s} d\tau \\ &= X_2(s) X_1(s) \end{aligned}$$

$$[x_1(t) * x_2(t)] \overset{L}{\longleftrightarrow} X_1(s) X_2(s)$$

4.6.9 Complex Frequency Differentiation

According to this property,

$$-\,tx(t) \overset{L}{\longleftrightarrow} \frac{d}{ds}(X(s)) \tag{4.33}$$

Proof By definition of LT,

$$X(s) = \int_0^{\infty} x(t) e^{-st} dt$$

Differentiating both sides with respect to s,

$$\begin{aligned} \frac{d}{ds}(X(s)) &= \frac{d}{ds} \int_0^{\infty} x(t) e^{-st} dt \\ &= -\int_0^{\infty} tx(t) e^{-st} dt \\ &= -L[tx(t)] \end{aligned}$$

$$\therefore \qquad -tx(t) \overset{L}{\longleftrightarrow} \frac{d}{ds}(X(s))$$

4.6.10 Complex Frequency Shifting

According to this property,

$$[e^{s_0 t}x(t)] \overset{L}{\longleftrightarrow} X(s-s_0) \tag{4.34}$$

$$L[e^{s_0 t}x(t)] = \int_0^{\infty} e^{s_0 t}x(t)e^{-st}dt \qquad \text{where } s_0 \text{is a constant}$$

$$= \int_0^{\infty} x(t)e^{-(s-s_0 t)}dt = X(s-s_0)$$

$$[e^{s_0 t}x(t)] \overset{L}{\longleftrightarrow} X(s-s_0)$$

4.6.11 Conjugation Property

According to this property if $x(t) \overset{L}{\longleftrightarrow} X(s)$ then

$$x^*(t) \overset{L}{\longleftrightarrow} X^*(-s) \tag{4.35}$$

Proof By definition of LT,

$$\begin{aligned} L[x^*(t)] &= \int_0^{\infty} x^*(t)e^{-st}dt \\ &= \int_0^{\infty} [x(t)e^{-(-s)t}dt]^* \\ &= X^*(-s) \end{aligned}$$

$$x^*(t) \overset{L}{\longleftrightarrow} X^*(-s)$$

4.6.12 Initial Value Theorem

According to this theorem,

$$\underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s) \tag{4.36}$$

Proof

$$L\left[\frac{d}{dt}x(t)\right] = \int_0^\infty \frac{d}{dt}(x(t))e^{-st}dt = sX(s) - x(0)$$

Let $s \to \infty$; then

$$\underset{s\to\infty}{Lt}\int_0^\infty \frac{d}{dt}(x(t))e^{-st}dt = \underset{s\to\infty}{Lt}[sX(s) - x(0)]$$
$$0 = \underset{s\to\infty}{Lt}[sX(s) - x(0)]$$

Since $x(0) = \underset{t\to 0}{Lt}\, x(t)$

$$\underset{t\to 0}{Lt}\, x(t) = \underset{s\to\infty}{Lt}\, sX(s)$$

4.6.13 Final Value Theorem

According to this theorem,

$$\underset{t\to\infty}{Lt}\, x(t) = \underset{s\to 0}{Lt}\, sX(s) \tag{4.37}$$

Proof The LT of $\frac{d}{dt}(x(t))$ could be written as

$$\int_0^\infty \frac{d}{dt}(x(t))e^{-st}dt = [sX(s) - x(0)]$$

Taking $\underset{s\to 0}{Lt}$ on both sides of the above equation, we get

$$\int_0^\infty \frac{d}{dt}(x(t))dt = \underset{s\to 0}{Lt}[sX(s) - x(0)]$$
$$\underset{t\to\infty}{Lt}[x(t) - x(0)] = \underset{s\to 0}{Lt}[sX(s) - x(0)]$$

$$\underset{t\to\infty}{Lt}\, x(t) = \underset{s\to 0}{Lt}\, sX(s)$$

The above theorem is valid if $X(s)$ has no poles in RHP of s-plane Table 4.1 gives the summary of properties of LT.

The following examples illustrate the method of determining LT.

Example 4.8 Determine the LT of unit impulse function $\delta(t)$ shown in Fig. 4.10.

Table 4.1 Summary of properties of LT

S.No	Property	Time function $x(t)$	Frequency function $X(s)$
1.	Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1X_1(s) + a_2X_2(s)$
2.	Time shifting	$x(t - t_0)$	$X(s)e^{-st_0}$
3.	Frequency shifting	$x(t)e^{at}$	$X(s - a)$
4.	Time scaling	$x(at)$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
5.	Frequency scaling	$\frac{1}{a}x\left(\frac{t}{a}\right)$	$X(as)$
6.	Time differentiation	$\frac{d}{dt}(x(t))$	$sX(s) - x(0^-)$
		$\frac{d^2}{dt^2}(x(t))$	$s^2X(s) - sx(0^-) - \dot{x}(0^-)$
		$\frac{d^n}{dt^n}(x(t))$	$s^nX(s)$
			$-\sum_{k=1}^{n} s^n x^{(k-1)}(0^-)$
7.	Time integration	$\int_0^t x(\tau)d\tau$	$\frac{X(s)}{s}$
8.	Time convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$
9.	Complex frequency differentiation	$-tx(t)$	$\frac{d}{ds}(X(s))$
		$t^nx(t)$	$(-1)^n\frac{d^n}{ds^n}X(s)$
10.	Complex frequency shifting	$e^{-at}x(t)$	$X(s + a)$
11.	Conjugation	$x^*(t)$	$X^*(-s)$
12.	Initial value theorem	$\underset{t\to 0}{Lt}\ x(t)$	$\underset{s\to\infty}{Lt}\ sX(s)$
13.	Final value theorem	$\underset{t\to\infty}{Lt}\ x(t)$	$\underset{s\to 0}{Lt}\ sX(s)$
14.	Shift theorem	$x(t - a)$	$X(s)e^{-as}$

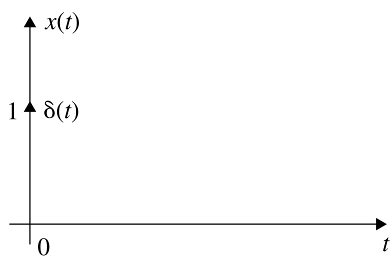
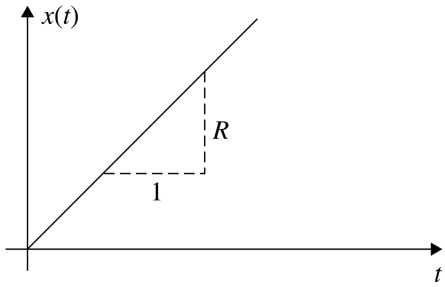


Fig. 4.10 The unit impulse (or delta) function

Fig. 4.11 Ramp (or velocity) function



Solution The unit impulse function is represented as

$$\begin{aligned} \delta(t) &= 1 \quad \text{for} \quad t = 0 \\ &= 0 \quad \text{otherwise} \\ L[\delta(t)] &= \int_{0^-}^{\infty} \delta(t) e^{-st} dt \\ &= \int_{0^-}^{0^+} e^{-st} dt \\ &= 1 \end{aligned}$$

$$\delta(t) \overset{L}{\longleftrightarrow} 1 \qquad \text{ROC : all } s \tag{4.38}$$

Example 4.9 Determine the LT of a ramp function of slope R which is shown in Fig. 4.11.

Solution The ramp function of slope R is represented in Fig. 4.11 and it is mathematically expressed as

$$x(t) = Rt\, u(t) \quad t \geq 0$$

Taking LT, the following equation is written:

$$L[Rt] = \int_0^{\infty} Rt\, e^{-st} dt$$

The above integration is solved by the well-known integration by parts using the following relationship

$$\int u dv = uv - \int v du$$

Let $u = Rt$ and $du = Rdt$; $dv = e^{-st}dt$ and $v = \int e^{-st}dt = -\frac{e^{-st}}{s}$

$$\therefore \quad L[Rt] = R\left[\frac{te^{-st}}{(-s)}\right]_0^\infty - R\int_0^\infty \frac{e^{-st}}{(-s)}dt$$

$$= R[0-0] + R\left[\frac{e^{-st}}{-s^2}\right]_0^\infty$$

$$= \frac{R}{s^2} \tag{4.39}$$

$$L(Rt) \overset{L}{\longleftrightarrow} \frac{R}{s^2}$$

ROC: The entire right half s-plane (RHP) except the origin.

The LT of unit ramp ($R = 1$) is,

$$L(t) \longleftrightarrow \frac{1}{s^2}$$

Example 4.10 Determine the LT of the acceleration function shown in Fig. 4.12.

Solution The acceleration function is expressed by the following equation:

$$x(t) = \frac{1}{2}at^2u(t) \quad t \geq 0.$$

Taking LT for the above function, we get

$$L\left[\frac{1}{2}at^2\right] = \int_0^\infty \frac{1}{2}at^2e^{-st}dt$$

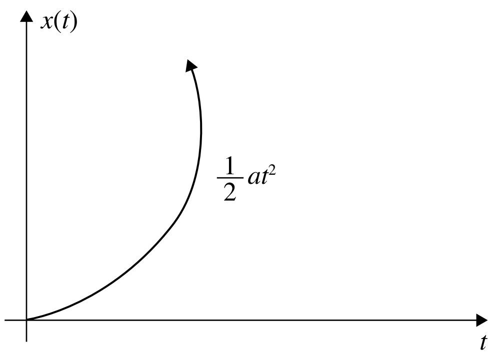


Fig. 4.12 Acceleration function

The above integration is solved using integration by parts as described below:

$$u = \tfrac{1}{2}at^2 \quad \text{and} \quad du = at$$
$$dv = \int e^{-st}dt \quad \text{and} \quad v = \frac{e^{-st}}{(-s)}$$

$$L\left[\frac{1}{2}at^2\right] = uv - \int_0^\infty vdu = \left[\frac{1}{2}at^2\frac{e^{-st}}{(-s)}\right]_0^\infty - \int_0^\infty \frac{ate^{-st}}{(-s)}dt$$
$$= 0 + 0 + \frac{a}{s}\int_0^\infty te^{-st}dt.$$

The integration in the right-hand side of the equation is nothing but a ramp signal whose LT is $\frac{1}{s^2}$. Hence

$$L\left[\frac{1}{2}at^2\right] = \frac{a}{s^3} \tag{4.40}$$

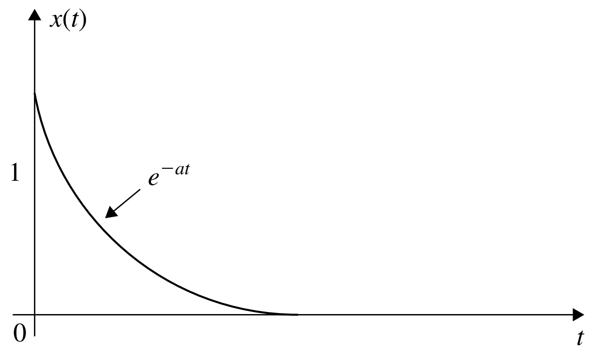
The ROC is the entire RHP except the origin of the s-plane.

Example 4.11 Determine the LT of an exponential decay which is shown in Fig. 4.13.

Solution The exponential decay is represented by

$$x(t) = e^{-at}u(t) \quad t \geq 0.$$

Fig. 4.13 Exponential decay



Taking LT for the above function, we get

$$L[e^{-at}u(t)] = \int_0^\infty e^{-at}e^{-st}dt$$
$$= \int_0^\infty e^{-(s+a)t}dt$$

$$L[e^{-at}u(t)] = -\frac{1}{(s+a)}\left[e^{-(s+a)t}\right]_0^\infty$$
$$= \frac{1}{(s+a)} \quad \text{with ROC: Re } s > -a$$

$$L[e^{-at}u(t)] = \frac{1}{(s+a)} \tag{4.41}$$

Example 4.12 Determine the LT of a sine function which is shown in Fig. 4.14.

Solution A sinusoidal function shown in Fig. 4.14 is mathematically expressed as follows:

$$x(t) = A\sin\omega_0 t\, u(t) \quad t \geq 0$$

The given sinusoidal function is written as follows using Euler's identity.

$$\sin\omega_0 t = \frac{1}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t})$$
$$L[A\sin\omega_0 t] = \frac{A}{2j}[L(e^{j\omega_0 t}) - Le^{-j\omega_0 t}]$$

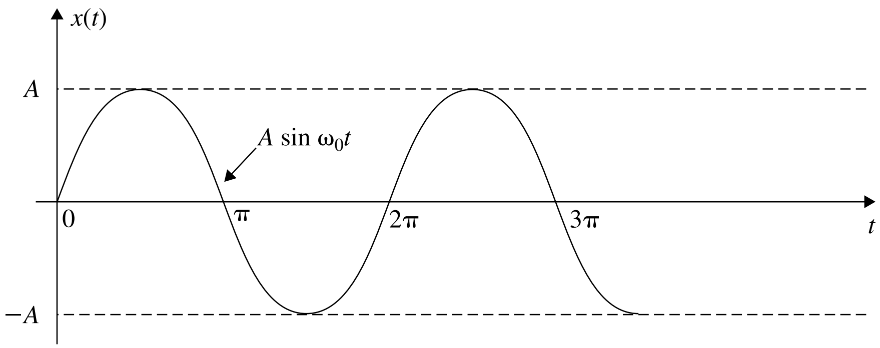


Fig. 4.14 A sine function

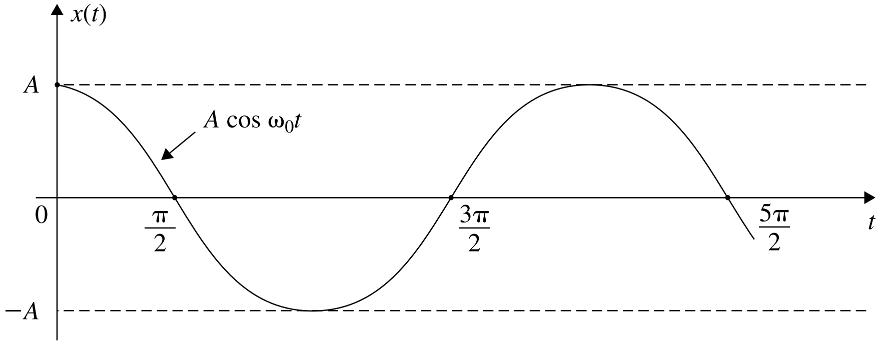


Fig. 4.15 A cosine function

From Eq. (4.41) the above equation is written as

$$L[A\sin\omega_0 t] = \frac{A}{2j}\left[\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0}\right] = \frac{A}{2j}\frac{2j\omega_0}{(s^2 + \omega_0^2)}$$

$$L[A\sin\omega_0 t] = \frac{A\omega_0}{(s^2 + \omega_0^2)} \quad \text{ROC: Re } s > 0. \qquad (4.42)$$

Example 4.13 Determine the LT of a cosine function which is shown in Fig. 4.15.

Solution A cosine function shown in Fig. 4.15 is mathematically expressed as follows:

$$x(t) = A\cos\omega_0 t u(t) \quad t \leq 0.$$

Using Euler's identity, the above equation is written as follows:

$$A\cos\omega_0 t = \frac{A}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

Taking LT for $x(t)$, the following equation is written

$$L[A\cos\omega_0 t u(t)] = \frac{A}{2}[Le^{j\omega_0 t}u(t) + Le^{-j\omega_0 t}u(t)]$$

Using the results obtained in Eq. (4.41), we get

$$L[A\cos\omega_0 t u(t)] = \frac{A}{2}\left[\frac{1}{(s + j\omega_0)} + \frac{1}{(s - j\omega_0)}\right]$$
$$= \frac{As}{(s^2 + \omega_0^2)}$$

$$L[A\cos\omega_0 t u(t)] = \frac{As}{(s^2+\omega_0^2)} \qquad \text{ROC: Re}s > 0. \tag{4.43}$$

Example 4.14 Determine the LT of hyperbolic sine function

$$x(t) = \sin h\omega_0 t.$$

Solution

$$\sin h\omega_0 t = \frac{1}{2}[e^{\omega_0 t} - e^{-\omega_0 t}]$$
$$L[\sin h\omega_0 t] = \frac{1}{2}L[e^{\omega_0 t}] - \frac{1}{2}L[e^{-\omega_0 t}]$$

Using the results obtained in (4.41), we get

$$L[\sin h\omega_0 t] = \frac{1}{2(s-\omega_0)} - \frac{1}{2(s+\omega_0)}$$

$$L[\sin h\omega_0 t] = \frac{\omega_0}{s^2-\omega_0^2} \quad \text{ROC: Re } s > \omega_0. \tag{4.44}$$

Example 4.15 Determine the Laplace transform of hyperbolic cosine function:

$$x(t) = \cos h\omega_0 t.$$

Solution

$$\cos h\omega_0 t = \frac{1}{2}[e^{\omega_0 t} + e^{-\omega_0 t}]$$

Taking LT on both sides, we get

$$\begin{aligned} L[\cos h\omega_0 t] &= \frac{1}{2}L[e^{\omega_0 t}] + \frac{1}{2}L[e^{-\omega_0 t}] \\ &= \frac{1}{2}\left[\frac{1}{s-\omega_0} + \frac{1}{s+\omega_0}\right] \\ &= \frac{s}{(s^2-\omega_0^2)} \end{aligned}$$

$$L[\cos h\omega_0 t] = \frac{s}{(s^2-\omega_0^2)} \quad \text{ROC: Re } s > \omega_0. \tag{4.45}$$

Example 4.16 Determine the LT of

$$x(t) = t^n u(t).$$

Solution Using the definition of LT for the given function, we get

$$L[x(t)] = \int_0^\infty t^n e^{-st} dt$$

Let

$$u = t^n \quad \text{and} \quad du = nt^{n-1} dt$$
$$dv = \int e^{-st} dt \quad \text{and} \quad v = \frac{e^{-st}}{(-s)}$$

Using the property

$$\int u dv = uv - \int v du$$

we get

$$L[t^n] = \left[t^n \frac{e^{-st}}{(-s)} \right]_0^\infty - \int_0^\infty \frac{e^{-st}}{(-s)} nt^{n-1} dt$$
$$= 0 + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt.$$

It can be shown that

$$\int_0^\infty t^{n-1} e^{-st} dt = \frac{(n-1)}{s} \int_0^\infty t^{n-2} e^{-st} dt.$$

Thus, $L[t^n]$ is written as

$$L[t^n] = \frac{n}{s} \frac{(n-1)}{s} \frac{(n-2)}{s} \cdots \frac{2}{s} \frac{1}{s}$$
$$= \frac{n(n-1)(n-2)\ldots 2}{s^n} \frac{1}{s} = \frac{\angle n}{s^{n+1}}$$

$$L[t^n] = \frac{\angle n}{s^{n+1}} \quad \text{ROC: Re } s > 0. \tag{4.46}$$

Example 4.17 Using the complex shifting property of LT, determine the LT of

$$x(t) = e^{-at} \sin \omega_0 t.$$

Table 4.2 Laplace transform tables

S.no	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{\angle n}{s^{n+1}}$
5	$e^{at} u(t)$	$\frac{1}{(s-a)}$
6	$e^{-at} u(t)$	$\frac{1}{(s+a)}$
7	$\cos at\, u(t)$	$\frac{s}{(s^2+a^2)}$
8	$\sin at\, u(t)$	$\frac{a}{(s^2+a^2)}$
9	$e^{-bt} \cos at\, u(t)$	$\frac{(s+b)}{(s+b)^2+a^2}$
10	$e^{-bt} \sin at\, u(t)$	$\frac{a}{(s+b)^2+a^2}$
11	$\delta(t-a)$	e^{-as}
12	$u(t-a)$	$\frac{e^{-as}}{s}$
13	$t \sin at\, u(t)$	$\frac{2as}{(s^2+a^2)^2}$
14	$\sin h\, at$	$\frac{a}{(s^2+a^2)}$
15	$\cos h\, at$	$\frac{s}{s^2+a^2}$
16	$\sin(at+\theta)$	$\frac{s \sin\theta + a\cos\theta}{(s^2+a^2)}$
17	$\cos(at+\theta)$	$\frac{s \cos\theta - a\sin\theta}{(s^2+a^2)}$

Solution

$$L[\sin \omega_0 t] = \frac{\omega_0}{(s^2+\omega_0^2)}$$

From Table 4.2, the complex shifting property is

$$L[e^{-at} x(t)] = X(s+a)$$

Applying the above property, we get

$$L[e^{-at}\sin\omega_0 t] = \frac{\omega_0}{(s+a)^2+\omega_0^2} \tag{4.47}$$

ROC: Re $s > -a$.

Example 4.18 By applying the complex differentiation property, determine the LT of

$$x(t) = t\sin\omega_0 t.$$

Solution

$$L[\sin\omega_0 t] = \frac{\omega_0}{(s^2+\omega_0^2)}$$

According to the complex differentiation property

$$L[-tx(t)] = \frac{d}{ds}X(s)$$

$$\therefore \qquad L[\sin\omega_0 t] = \frac{d}{ds}\frac{\omega_0}{(s^2+\omega_0^2)}$$

$$L[t\sin\omega_0 t] = \frac{2\omega_0 s}{(s^2+\omega_0^2)^2} \tag{4.48}$$

Example 4.19 Determine the LT of

$$x(t) = \cos at\sin bt.$$

Solution The given $x(t)$ is written in the following form:

$$x(t) = \frac{1}{2}[\sin(a+b)t - \sin(a-b)t]$$

$$L[\cos at\sin bt] = \frac{1}{2}[L\sin(a+b)t - L\sin(a-b)t]$$

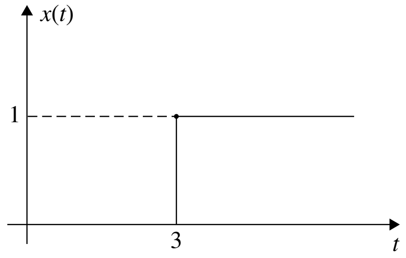
$$L[\cos at\sin bt] = \frac{1}{2}\left[\frac{(a+b)}{s^2+(a+b)^2} - \frac{(a-b)}{s^2+(a-b)^2}\right] \tag{4.49}$$

Example 4.20 Consider the following time function $x(t) = u(t-3)$. Determine the LT using shift theorem.

Solution From Fig. 4.16, for step input the LT is

$$L[u(t)] = \frac{1}{s}$$

Fig. 4.16 $x(t) = u(t-3)$



When the signal is shifted by $t = 3$, using time shifting property

$$L[u(t-3)] = \frac{1}{s}e^{-3s}$$

Table 4.2 gives the LT of some time functions.

Example 4.21 Determine the LT for the following time function:

$$x(t) = \sin(at + \theta)$$

Solution The given $x(t)$ can be expanded and written as follows:

$$\begin{aligned} x(t) &= \sin(at + \theta) \\ &= \sin at \cos\theta + \cos at \sin\theta \\ L[\sin(at+\theta)] &= L[\sin at \cos\theta] + L[\cos at \sin\theta] \end{aligned}$$

Substituting for $L[\sin at]$ and $L[\cos at]$ from Table 4.2, we get

$$L[\sin(at+\theta)] = \frac{a\cos\theta}{(s^2+a^2)} + \frac{s\sin\theta}{(s^2+a^2)} \tag{4.50}$$

Example 4.22 Determine the LT for the following time function:

$$x(t) = \cos(at + \theta)$$

Solution $x(t)$ can be expanded and written as follows:

$$\begin{aligned} x(t) &= \cos at \cos\theta - \sin at \sin\theta \\ L[\cos(at+\theta)] &= \cos\theta L[\cos at] - \sin\theta L[\sin at] \\ &= \frac{s\cos\theta}{(s^2+a^2)} - \frac{a\sin\theta}{(s^2+a^2)} \end{aligned}$$

$$L[\cos(at+\theta)] = \frac{(s\cos\theta - a\sin\theta)}{(s^2+a^2)} \tag{4.51}$$

Example 4.23 Determine the LT for the following time function:

$$x(t) = \delta(t-2) - \delta(t-5).$$

Solution The given time function consists of two impulses occurring at $t = 2$ and $t = 5$. By applying shift theorem, we get

$$L[\delta(t-2)] = e^{-2s}$$
$$L[\delta(t-5)] = e^{-5s}$$

$$L[\delta(t-2) - \delta(t-5)] = e^{-2s} - e^{-5s}$$

Example 4.24 Determine the LT for the following time function:

$$x(t) = u(t-2) - u(t-5).$$

Solution The given time function $x(t)$ consists of two step functions shifted by $t = 2$ and $t = 5$. By applying shift theorem, we get

$$L[u(t-2)] = \frac{e^{-2s}}{s}$$
$$L[u(t-5)] = \frac{e^{-5s}}{s}$$

$$\therefore \quad L[u(t-2) - u(t-5)] = \frac{1}{s}[e^{-2s} - e^{-5s}]$$

Example 4.25 Consider the following function:

$$X(s) = \frac{(5s+4)(s+6)}{s(s+2)(3s+1)}$$

Find the initial and final values of $x(t)$.

Solution The initial value is given by

$$\underset{t\to 0}{Lt}\, x(t) = x(0) = \underset{s\to\infty}{Lt}\, sX(s)$$
$$= \underset{s\to\infty}{Lt} \frac{s(5+\frac{4}{s})(1+\frac{6}{s})}{s(1+\frac{2}{s})(3+\frac{1}{s})}$$
$$= \frac{5\times 1}{1\times 3} = \frac{5}{3}$$

$$x(0) = \frac{5}{3}$$

The final value of $x(t)$ is given by

$$\underset{t\to\infty}{x(t)} = x(\infty) = \underset{s\to 0}{Lt}\, sX(s)$$
$$= \underset{s\to 0}{Lt} \frac{s(5s+4)(s+6)}{s(s+2)(3s+1)}$$
$$= \frac{4\times 6}{2\times 1} = 12$$

$$x(\infty) = 12$$

Example 4.26 Consider the pulse shown in Fig. 4.17a. Determine the LT.

Solution

Method 1: The given signal $x(t)$ which is shown in Fig. 4.17a could be split up of step signals as shown in Fig. 4.17b and c. Thus, the following equation is written.

$$x(t) = u(t) - u(t-3)$$

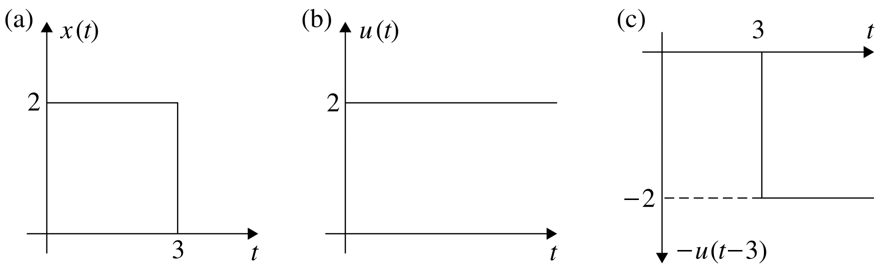


Fig. 4.17 LT of a pulse

Taking LT on both sides, we get

$$X(s) = U(s) - U(s)e^{-3s}$$
$$= [1 - e^{-3s}]U(s)$$

But $U(s) = \dfrac{2}{s}$ (for a step input).

$$\therefore \quad X(s) = \frac{2}{s}[1 - e^{-3s}]$$

Method 2: By definition of LT, the following equation is written for Fig. 4.17a.

$$X(s) = \int_0^3 2e^{-st}dt$$
$$= \frac{2}{(-s)}\left[e^{-st}\right]_0^3$$

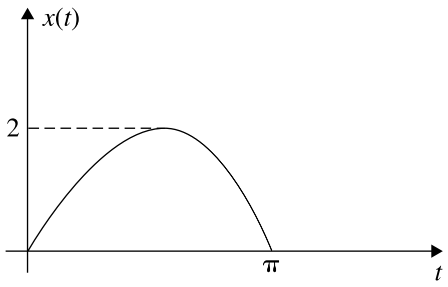
$$X(s) = \frac{2}{s}[1 - e^{-3s}]$$

Example 4.27 For the wave form shown in Fig. 4.18, determine the LT.

Solution For Fig. 4.18, the following equation is written:

$$x(t) = 2\sin t \quad 0 \le t \le \pi$$
$$= \quad 0 \quad\quad t > \pi$$

Fig. 4.18 A sine wave



The LT of the above signal is obtained from the following equation:

$$X(s) = \int_0^{\pi} 2\sin t\, e^{-st} dt$$

Let $u = 2\sin t$ and $du = 2\cos t dt$; $dv = e^{-st} dt$ and $v = -\frac{1}{s}e^{-st}$. Applying $\int u dv = uv - \int v du$, we get

$$X(s) = \left[-\frac{2}{s}\sin t e^{-st}\right]_0^{\pi} + \int_0^{\pi} \frac{2}{s}\cos t e^{-st} dt = 0 + \frac{2}{s}\int_0^{\pi} \cos t e^{-st} dt$$

Let $u = \cos t$ and $du = -\sin t dt$; $dv = e^{-st} dt$ and $v = -\frac{1}{s}e^{-st}$. Substituting the above in equation for $X(s)$ we get

$$\begin{aligned} X(s) &= \frac{2}{s}\left\{\left[-\frac{1}{s}\cos t e^{-st}\right]_0^{\pi} - \int_0^{\pi} \frac{1}{s}\sin t e^{-st} dt\right\} \\ &= \frac{2}{s}\left[\left\{e^{-\pi s} + 1\right\}\frac{1}{s} - \frac{1}{2s}X(s)\right] \qquad \text{since} \int_0^{\pi} \sin t e^{-st} dt = \frac{X(s)}{2} \end{aligned}$$

$$\begin{aligned} \frac{sX(s)}{2} + \frac{1}{2s}X(s) &= \frac{(e^{-\pi s} + 1)}{s} \\ \frac{(s^2 + 1)X(s)}{2s} &= \frac{(e^{-\pi s} + 1)}{s} \end{aligned}$$

$$X(s) = \frac{2(e^{-\pi s} + 1)}{(s^2 + 1)}$$

Example 4.28 Determine the LT of the saw tooth wave form shown in Fig. 4.19.

(*Anna University, April, 2005*)

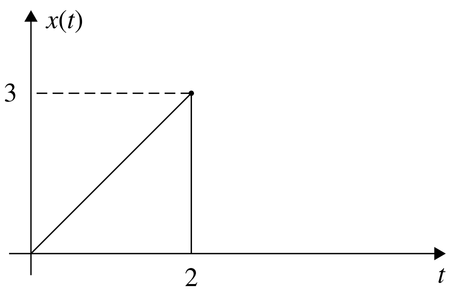


Fig. 4.19 Saw tooth wave form

Solution The saw tooth wave form shown in Fig. 4.19 is expressed as

$$x(t) = \frac{3}{2}t \quad 0 \le t \le 2$$
$$= 0 \quad \text{otherwise}$$

Taking LT for the time function $x(t)$, we get

$$X(s) = \int_0^2 \frac{3}{2} t e^{-st} dt$$

Let

$$u = \frac{3}{2}t \quad \text{and} \quad du = \frac{3}{2}dt$$

$$dv = e^{-st}dt \quad \text{and} \quad v = -\frac{1}{s}e^{-st}$$

Using $\int udv = uv - \int vdu$, we get

$$\begin{aligned} X(s) &= \left[\frac{3}{2}t\left(-\frac{1}{s}\right)e^{-st}\right]_0^2 + \frac{3}{2}\int_0^2 \frac{1}{s}e^{-st}dt \\ &= \frac{-3}{s}e^{-2s} + \frac{3}{2s^2}\left[-1e^{-st}\right]_0^2 \\ &= \frac{-3}{s}e^{-2s} - \frac{3}{2s^2}e^{-2s} + \frac{3}{2s^2} \end{aligned}$$

$$X(s) = \frac{3}{2}\frac{1}{s^2} - \left(\frac{3}{s} + \frac{3}{2s^2}\right)e^{-2s}$$

Example 4.29 Consider the triangular wave form shown in Fig. 4.20. Determine the LT.

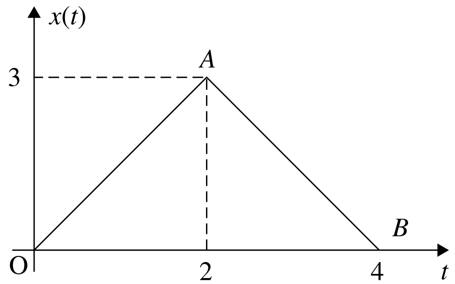


Fig. 4.20 Triangular wave form

Solution For the straight line OA, the slope is $\frac{3}{2}$ and passes through the origin. Hence, the following equation is written:

$$x_1(t) = \frac{3}{2}t \quad 0 \leq t \leq 2$$

For the straight line AB, the slope is negative and it is $-\frac{3}{2}$. The following equation is written

$$x_2(t) = -\frac{3}{2}t + C$$

when $t = 2$, $x_2(t) = 3$. Hence,

$$3 = -\frac{3}{2} \times 2 + C$$

or $C = 6$

$$x_2(t) = -\frac{3}{2}t + 6 \quad 2 \leq t \leq 4$$

From Example 4.29, $X_1(s)$ is written as

$$X_1(s) = \frac{3}{2s^2} - \left(\frac{3}{s} + \frac{3}{2s^2}\right)e^{-2s}$$

Now $X_2(s)$ is written as

$$X_2(s) = \int_2^4 \left(6 - \frac{3}{2}t\right)e^{-st}dt$$

Let

$$u = \left(6 - \frac{3}{2}t\right) \quad \text{and} \quad du = -\frac{3}{2}dt$$

$$dv = \int e^{-st}dt \quad \text{and} \quad v = -\frac{1}{s}e^{-st}$$

Using $\int udv = uv - \int vdu$, we get

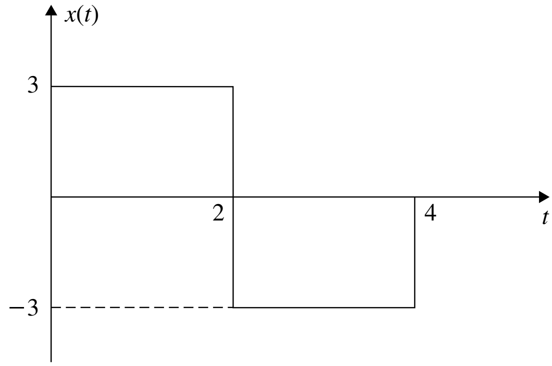


Fig. 4.21 A rectangular wave

$$X_2(s) = \left[\left(6 - \frac{3}{2}t\right)\left(-\frac{1}{s}\right)e^{-st}\right]_2^4 - \frac{3}{2s}\int_2^4 e^{-st}dt$$

$$= \left[\frac{3}{s}e^{-2s}\right] + \frac{3}{2s^2}\left[e^{-st}\right]_2^4$$

$$= \frac{3}{s}e^{-2s} + \frac{3}{2s^2}e^{-4s} - \frac{3}{2s^2}e^{-2s}$$

$$X(s) = X_1(s) + X_2(s)$$

$$= \frac{3}{2s^2} - \left(\frac{3}{s} + \frac{3}{2s^2}\right)e^{-2s} + \frac{3}{s}e^{-2s} + \frac{3}{2s^2}e^{-4s} - \frac{3}{2s^2}e^{-2s}$$

$$X(s) = \frac{3}{2s^2} - \left(\frac{3}{s^2}e^{-2s}\right) + \frac{3}{2s^2}e^{-4s}$$

Example 4.30 Consider the rectangular wave form shown in Fig. 4.21. Determine the LT.

Solution Consider the rectangular wave shown in Fig. 4.21 for the time interval

$$x_1(t) = 3 \quad 0 \le t \le 2.$$

The LT of $x_1(t)$ is found from the equation

$$X_1(s) = \int_0^2 3e^{-st}dt$$
$$= \frac{-3}{s}\left[e^{-st}\right]_0^2$$
$$= \frac{3}{s}[1 - e^{-2s}]$$

Consider rectangular wave

$$x_2(t) = -3 \quad 2 \le t \le 4$$

Using shift theorem $X_2(s)$ is obtained as

$$X_2(s) = -X_1(s)e^{-2s}$$
$$\therefore \qquad X(s) = X_1(s) + X_2(s)$$
$$= \frac{3}{s}(1 - e^{-2s}) - \frac{3}{s}(1 - e^{-2s})e^{-2s}$$
$$= \frac{3}{s}(1 - e^{-2s})(1 - e^{-2s})$$

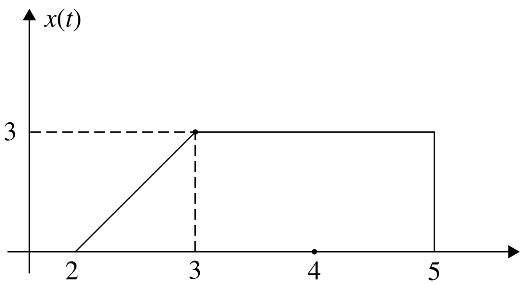
$$X(s) = \frac{3}{s}(1 - e^{-2s})^2$$

Example 4.31 Consider the wave form shown in Fig. 4.22. Determine the LT.

Solution The mathematical description of the wave form shown in Fig. 4.22 is written as follows:

$$x(t) = 3t \quad 2 \le t \le 3$$
$$= 3 \quad 3 \le t \le 5$$

Fig. 4.22 A triangular pulse rectangular wave



The LT of $x(t)$ is written as

$$\begin{aligned} X(s) &= \int_2^3 3te^{-st}dt + \int_3^5 3e^{-st}dt \\ &= X_1(s) + X_2(s) \end{aligned}$$

where

$$\begin{aligned} X_2(s) &= \frac{3}{(-s)}\left[e^{-st}\right]_3^5 \\ &= \frac{3}{s}[e^{-3s} - e^{-5s}] \end{aligned}$$

$X_1(s)$ is determined as follows.

For the triangle $x_1(t)$ is written as follows:

$$x_1(t) = 3t + C$$

When $t = 2, x_1(t) = 0$

$$\begin{aligned} 0 = 3 \times 2 + C \quad &\text{or} \quad C = -6 \\ x_1(t) &= (3t - 6) \\ X_1(s) &= \int_2^3 (3t - 6)e^{-st}dt \end{aligned}$$

Let

$$\begin{aligned} u = (3t - 6) \quad &\text{and} \quad du = 3dt \\ dv = \int e^{-st}dt \quad &\text{and} \quad v = -\tfrac{1}{s}e^{-st} \end{aligned}$$

$$\begin{aligned} X_1(s) &= \left[(3t - 6)\left(-\frac{1}{s}\right)e^{-st}\right]_2^3 + \frac{3}{s^2}\left[e^{-st}\right]_2^3 \\ &= -\frac{3}{s}e^{-3s} + \frac{3}{s^2}(e^{-3s} - e^{-2s}) \\ X(s) &= X_1(s) + X_2(s) \\ &= -\frac{3}{s}e^{-3s} + \frac{3}{s^2}(e^{-3s} - e^{-2s}) + \frac{3}{s}(e^{-3s} - e^{-5s}) \end{aligned}$$

$$X(s) = -\frac{3}{s}e^{-5s} + \frac{3}{s^2}(e^{-3s} - e^{-2s})$$

4.7 Laplace Transform of Periodic Signal

If a signal $x(t)$ is a periodic signal with period T, then the LT of $X(s)$ is given as

$$X(s) = X_1(s)\left[1 + e^{-Ts} + e^{-2Ts} + \ldots\right]$$
$$= \frac{X_1(s)}{(1 - e^{-Ts})}$$

Here $x_1(t)$ is the signal which is repeated for every T.

Example 4.32 Consider the output of a full wave rectifier shown in Fig. 4.23. Determine the LT.

Solution In Example 4.27, $X_1(s)$ is determined as

$$X_1(s) = \frac{2(e^{-\pi s} + 1)}{(s^2 + 1)}$$

If $X(s)$ is the LT of the full wave rectifier

$$X(s) = X_1(s) + X_1(s)e^{-Ts} + X_1(s)e^{-2Ts} + \ldots$$

where $T = \pi$

$$= X_1(s) + X_1(s)e^{-\pi s} + X_1(s)e^{-2\pi s} + \ldots$$
$$= X_1(s)[1 + e^{-\pi s} + e^{-2\pi s} + \ldots]$$
$$= \frac{X_1(s)}{(1 - e^{-\pi s})}$$
$$= \frac{2(e^{-\pi s} + 1)}{(1 - e^{-\pi s})}\frac{1}{(s^2 + 1)}$$

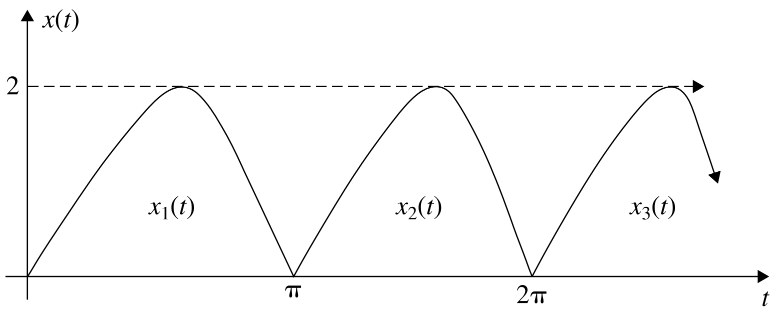


Fig. 4.23 Full wave rectifier

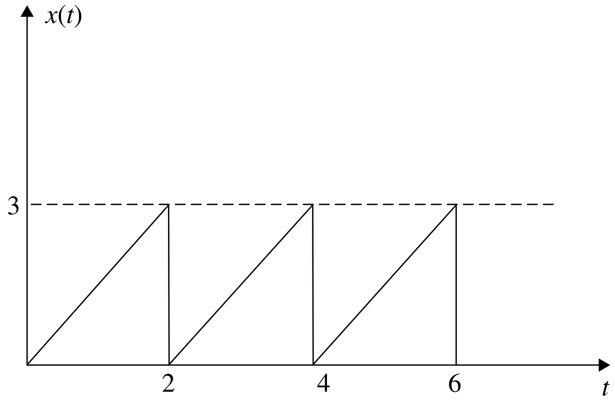


Fig. 4.24 Saw tooth wave

$$X(s) = \frac{2(1 + e^{-\pi s})}{(1 - e^{-\pi s})(1 + s^2)}$$

Example 4.33 Consider the saw tooth wave shown in Fig. 4.24. Determine the LT.

Solution The mathematical description of $x(t)$ for $0 \leq t \leq 2$ is given as $x_1(t)$. In Example 4.28, $X_1(s)$ is determined as

$$X_1(s) = \frac{3}{2s^2} - \left(\frac{3}{s} + \frac{3}{2s^2}\right) e^{-2s}$$

from Fig. 4.24,

$$\begin{aligned} X(s) &= X_1(s)[1 + e^{-2s} + e^{-4s} + \ldots] \\ &= \frac{X_1(s)}{(1 - e^{-2s})} \end{aligned}$$

$$X(s) = \frac{3}{2(1 - e^{-2s})} \left[\frac{1}{s^2} - \left(\frac{2}{s} + \frac{1}{s^2}\right) e^{-2s}\right]$$

Example 4.34 Consider the rectangular periodic wave shown in Fig. 4.25. Determine the LT.

Solution The mathematical description of the periodic wave with period 4 is written as follows:

$$\begin{aligned} x(t) &= 3 \quad 0 \leq t \leq 2 \\ &= -3 \quad 2 \leq t \leq 4 \end{aligned}$$

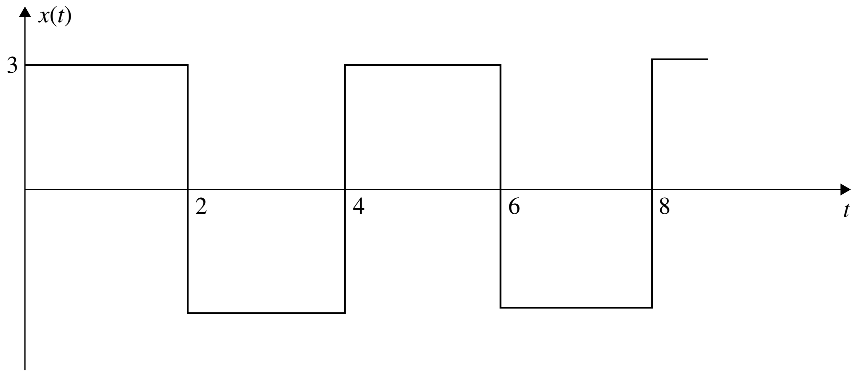


Fig. 4.25 A periodic rectangular wave

Let $X_1(s)$ be the LT of $x(t)$ for the time $0 \leq t \leq 4$. $X_1(s)$ in Example 4.30 has been determined as

$$\begin{aligned} X_1(s) &= \frac{3}{s}(1 - e^{-2s})^2 \\ X(s) &= X_1(s)[1 + e^{-4s} + e^{-8s} + \ldots] \\ &= \frac{X_1(s)}{(1 - e^{-4s})} \end{aligned}$$

$$X(s) = \frac{3(1 - e^{-2s})^2}{s(1 - e^{-4s})}$$

4.8 Inverse Laplace Transform

The time signal $x(t)$ is the Inverse LT of $X(s)$. This is represented by the following mathematical equation:

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds \tag{4.52}$$

Use of Eq. (4.52) to obtain $x(t)$ from $X(s)$ is really a tedious process. The alternative is to express $X(s)$ in polynomial form both in the numerator and the denominator. Both these polynomials are factorized as

$$X(s) = \frac{(s + z_1)(s + z_2) \ldots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)} \tag{4.53}$$

The points in the s-plane at which $X(s) = 0$ are called zeros. Thus, $(s + z_1)$, $(s + z_2), \ldots, (s + z_m)$ are the zeros of $X(s)$ in Eq. (4.53). Similarly, the points in the s-plane at which $X(s) = \infty$ are called poles of $X(s)$ in Eq. (4.53).

The zeros are identified by a small circle O and the poles by a small cross $\times$ in the s-plane. For $m < n$ the degree of the numerator polynomial is less than the degree of the denominator polynomial. Under this condition $X(s)$ in Eq. (4.53) is written in the following partial fraction form:

$$X(s) = \frac{A_1}{s + p_1} + \frac{A_2}{s + p_2} + \frac{A_3}{s + p_3} + \cdots + \frac{A_n}{s + p_n} \tag{4.54}$$

In Eq. (4.54) $A_1, A_2, \ldots, A_n$ are called the residues and are determined by any convenient method. Once the residues are determined, then by using Table 4.2, one can easily obtain $x(t)$ which is the required inverse LT of $X(s)$.

4.8.1 Graphical Method of Determining the Residues

The residues in Eq. (4.54) are determined by analytical as well as graphical method. The graphical method has the following advantages:

- It is less time consuming.
- It does not require any graph to be drawn.
- The results are obtained in compact form very quickly even if the poles and zeros are complex and repeated.

Both analytical and graphical methods are given wherever necessary. The following simple example illustrates both analytical and graphical methods.

Example 4.35

$$X(s) = \frac{10(s + 2)(s - 3)}{s(s + 4)(s - 5)}$$

Find $x(t)$.

Solution The given $X(s)$ is expressed in partial fraction form as follows:

$$X(s) = \frac{A_1}{s} + \frac{A_2}{(s + 4)} + \frac{A_3}{(s - 5)}$$

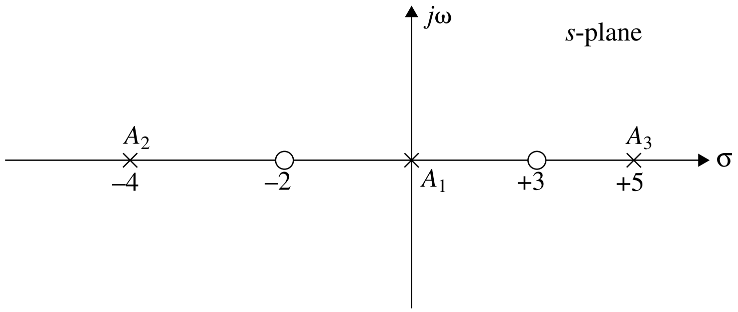


Fig. 4.26 Poles and zeros of $X(s)$ (pole zero diagram)

Method 1. Analytical Method

1. The poles and zeros of $X(s)$ are represented in Fig. 4.26. $X(s)$ is expressed in the following form:

$$X(s) = \frac{A_1(s+4)(s-5) + A_2 s(s-5) + A_3 s(s+4)}{s(s+4)(s-5)}$$

The numerator polynomial of $X(s)$ should be same and therefore the following equation is written.

$$10(s+2)(s-3) = A_1(s+4)(s-5) + A_2 s(s-5) + A_3 s(s+4)$$

2. Substitute $s = 0$ in the above equation which will eliminate A_2 and A_3. Thus,

$$10(2)(-3) = A_1(4)(-5) + 0 + 0$$
$$A_1 = \frac{60}{20} = 3$$

Substitute $s = -4$ in $X(s)$. This eliminates A_1 and A_3. Thus,

$$10(-4+2)(-4-3) = 0 - A_2 4(-4-5) + 0$$
$$10(-2)(-7) = A_2 36$$
$$A_2 = \frac{140}{36} = \frac{35}{9}$$

Substitute $s = 5$ in $X(s)$. This eliminates A_1 and A_2. Thus,

$$10(5+2)(5-3) = 0 + 0 + A_3(5)(5+4)$$
$$A_3 = \frac{140}{45} = \frac{28}{9}$$

3. With the values of residues obtained in step 2, $X(s)$ is expressed as follows:

$$X(s) = \frac{3}{s} + \frac{35}{9}\frac{1}{(s+4)} + \frac{28}{9}\frac{1}{(s-5)}$$

4. From the Table 4.2, the inverse Laplace transform is obtained for $\frac{1}{s}$, $\frac{1}{(s+4)}$ and $\frac{1}{s-5}$.
5. To check whether the residues determined are correct, the following procedure is followed:

$$X(s) = \frac{10(s+2)(s-3)}{s(s+4)(s-5)} = \frac{3}{s} + \frac{35}{9(s+4)} + \frac{28}{9(s-5)}$$

Choose any value of s so that $X(s)$ does not become zero or infinity. Let us choose $s = 1$

$$\frac{10(3)(-2)}{1(5)(-4)} = \frac{3}{1} + \frac{35}{9 \times 5} + \frac{28}{9(-4)}$$
$$3 = 3 + \frac{7}{9} - \frac{7}{9} = 3$$
$$\text{LHS} = \text{RHS}.$$

Hence, A_1, A_2 and A_3 determined are correct.

$$x(t) = \left(3 + \frac{35}{9}e^{-4t} + \frac{28}{9}e^{5t}\right)u(t)$$

Method 2. Graphical Method of Determining the Residues

1. According to the graphical method, the residue A at any pole is obtained from

$$A = \frac{\text{Constant term} \times \text{Directed Vector distances drawn from all zeros to the concerned point}}{\text{Directed vector distances drawn from all poles to the concerned point}}$$

2. For the given problem, refer to the pole zero diagram of Fig. 4.26. From the Figure we obtain, A_1 by drawing vectors from poles and zeros of $X(s)$ to $s = 0$.

$$A_1 = \frac{10(2)(-3)}{4(-5)} = 3$$

A_2 is determined by drawing vectors from poles and zeros of $X(s)$ to $s = -4$.

$$A_2 = \frac{10(-2)(-7)}{(-4)(-9)} = \frac{35}{9}$$

A_3 is obtained by drawing vectors from poles and zeros of $X(s)$ to $s = 5$.

$$A_3 = \frac{10(7)(2)}{(5)(9)} = \frac{28}{9}$$

It is to be noted that the directed distances drawn from any pole or zero drawn towards right, a +ve sign is added and for the directed distance drawn towards left, a −ve sign in each case has to be included.

3. It is seen that the residues determined by analytical method and graphical method are same. Hence, inverse LT of $X(s)$ is written as

$$x(t) = \left(3 + \frac{35}{9}e^{-4t} + \frac{28}{9}e^{5t}\right)u(t)$$

In the expression for $x(t)$ it is necessary to include $u(t)$ in the right-side of the equation. This indicates that the inverse LT is right-sided or unilateral. It is also to be noted that the pole zero diagram of Fig. 4.26 need not be drawn to any scale. Mere location of poles and zeros with the appropriate values is enough.

Example 4.36 Find the inverse LT of

$$X(s) = \frac{10e^{-3s}}{(s-2)(s+2)}.$$

Solution Consider the function

$$X_1(s) = \frac{10}{(s-2)(s+2)}$$

Putting this into partial fraction, we get

$$\begin{aligned} X_1(s) &= \frac{A_1}{(s-2)} + \frac{A_2}{(s+2)} \\ &= \frac{A_1(s+2) + A_2(s-2)}{(s-2)(s+2)} \\ 10 &= A_1(s+2) + A_2(s-2) \end{aligned}$$

Substitute $s = -2$

$$\begin{aligned} 10 &= 0 + A_2(-2-2) \\ A_2 &= -2.5 \end{aligned}$$

Substitute $s = 2$

$$\begin{aligned} 10 &= A_1(2+2) + 0 \\ A_1 &= 2.5 \\ X_1(s) &= 2.5\left[\frac{1}{s-2} - \frac{1}{s+2}\right] \end{aligned}$$

Taking inverse LT, we get

$$x_1(t) = 2.5[e^{+2t} - e^{-2t}]u(t)$$

According to time shifting property of LT

$$X(s) = X_1(s)e^{-3s}$$

$$x(t) = 2.5[e^{2(t-3)} - e^{-2(t-3)}]u(t-3)$$

Example 4.37 Find the inverse LT of

$$X(s) = \frac{(s+1) + 3e^{-4s}}{(s+2)(s+3)}.$$

Solution The given function is written in the following form:

$$\begin{aligned} X(s) &= \frac{(s+1)}{(s+2)(s+3)} + \frac{3e^{-4s}}{(s+2)(s+3)} \\ &= X_1(s) + X_2(s) \\ X_1(s) &= \frac{(s+1)}{(s+2)(s+3)} \\ &= \frac{A_1}{(s+2)} + \frac{A_2}{(s+3)} \\ &= \frac{A_1(s+3) + A_2(s+2)}{(s+2)(s+3)} \\ (s+1) &= A_1(s+3) + A_2(s+2) \end{aligned}$$

Put $s = -3$

$$\begin{aligned} (-3+1) &= 0 + A_2(-3+2) \\ A_2 &= 2 \end{aligned}$$

Put $s = -2$

$$\begin{aligned}(-2+1) &= A_1(-2+3) + 0\\ A_1 &= -1\\ X_1(s) &= -\frac{1}{s+2} + \frac{2}{s+3}\\ x_1(t) &= (-e^{-2t} + 2e^{-3t})u(t).\end{aligned}$$

Now consider $X_2(s)$ without delay as $X_3(s)$

$$\begin{aligned}X_3(s) &= \frac{3}{(s+2)(s+3)}\\ &= \frac{A_1}{(s+2)} + \frac{A_2}{(s+3)}\\ 3 &= A_1(s+3) + A_2(s+2)\end{aligned}$$

Put $s = -2$

$$3 = A_1$$

Put $s = -3$

$$3 = A_2(-3+2)$$

$$\begin{aligned}A_2 &= -3\\ X_3(s) &= 3\left[\frac{1}{s+2} - \frac{1}{s+3}\right]\\ X_2(s) &= X_3(s)e^{-4s}\\ x_3(t) &= 3[e^{-2t} - e^{-3t}]u(t)\\ x_2(t) &= 3[e^{-2(t-4)} - e^{-3(t-4)}]u(t-4)\\ x(t) &= x_1(t) + x_2(t)\end{aligned}$$

$$x(t) = \left[-e^{-2t} + 2e^{-3t}\right]u(t) + 3\left[e^{-2(t-4)} - e^{-3(t-4)}\right]u(t-4)$$

Example 4.38 Find the inverse LT of

$$(1) \qquad X(s) = \frac{(s+1)(s+3)}{(s+2)(s+4)}$$

$$(2) \qquad X(s) = \frac{(s+1)(s+3)e^{-2s}}{(s+2)(s+4)}$$

Solution (1)

$$X(s) = \frac{(s+1)(s+3)}{(s+2)(s+4)}$$

Here both numerator polynomial and denominator polynomial have the same degree and therefore it is an improper function. Now $X(s)$ is written in the polynomial form as given below:

$$X(s) = \frac{(s^2+4s+3)}{(s^2+6s+8)}$$

By synthetic division, we get

$$\begin{array}{r|l} & \quad 1 \\ s^2+6s+8 & \ s^2+4s+3 \\ & \ s^2+6s+8 \\ \hline & \quad -2s-5 \end{array}$$

$$\therefore \qquad X(s) = 1 - \frac{(2s+5)}{(s+2)(s+4)}$$

Now consider

$$\begin{aligned} X_1(s) &= \frac{(2s+5)}{(s+2)(s+4)} \\ &= \frac{A_1}{(s+2)} + \frac{A_2}{(s+4)} \\ (2s+5) &= A_1(s+4) + A_2(s+2) \end{aligned}$$

Put $s = -4$

$$\begin{aligned} (-8+5) &= 0 + A_2(-4+2) \\ A_2 &= \frac{3}{2} \end{aligned}$$

Put $s = -2$

$$\begin{aligned} (-4+5) &= A_1(-2+4) + 0 \\ A_1 &= \frac{1}{2} \\ X_1(s) &= \frac{1}{2}\left[\frac{1}{(s+2)} + \frac{3}{(s+4)}\right] \\ X(s) &= 1 - \frac{1}{2}\left[\frac{1}{(s+2)} + \frac{3}{(s+4)}\right] \end{aligned}$$

Taking inverse LT, we get

$$x(t) = \delta(t) - \left[0.5e^{-2t} + 1.5e^{-4t}\right] u(t)$$

(2) Now consider

$$X(s) = \frac{(s+1)(s+3)e^{-2s}}{(s+2)(s+4)}$$

Using the time shifting property the results obtained in the previous example is modified and written as

$$x(t) = \delta(t-2) - \left[0.5e^{-2(t-2)} + 1.5e^{-4(t-2)}\right] u(t-2)$$

Example 4.39 Find the inverse LT of the following function:

$$X(s) = \frac{10(s+4)}{s^2(s+2)}$$

Solution The given function $X(s)$ is written in the partial fraction form as follows:

$$X(s) = \frac{A_1}{s^2} + \frac{A_2}{s} + \frac{A_3}{s+2}$$
$$10(s+4) = A_1(s+2) + A_2 s(s+2) + A_3 s^2$$

Put $s = 0$

$$40 = 2A_1 \quad \text{or} \quad A_1 = 20$$

Put $s = -2$

$$10(-2+4) = 0 + 0 + A_3 4$$
$$A_3 = \frac{20}{4} = 5$$

Comparing the coefficients of s term, we get

$$10 = (A_1 + 2A_2)$$
$$10 = 20 + 2A_2$$
$$A_2 = -5$$
$$X(s) = \frac{20}{s^2} - \frac{5}{s} + \frac{5}{s+2}$$

$$x(t) = (20t - 5 + e^{-2t})u(t)$$

Example 4.40 Find the inverse LT of the following function:

$$X(s) = \frac{2}{s(s^2 + 2s + 2)}$$

Solution Method 1.

$$(s^2 + 2s + 2) = (s + 1 + j)(s + 1 - j)$$

$$X(s) = \frac{A_1}{s} + \frac{A_2}{s + 1 + j} + \frac{A_3}{s + 1 - j}$$
$$2 = A_1(s^2 + 2s + 2) + A_2 s(s + 1 - j) + A_3 s(s + 1 + j)$$

Put $s = 0$

$$2 = A_1 2 \quad \text{or} \quad A_1 = 1$$

Put $s = -1 + j$

$$2 = 0 + 0 + A_3(-1 + j)(-1 + j + 1 + j)$$
$$= A_3(-1 + j)2j$$
$$A_3 = \frac{1}{(-1 + j)j}$$

But $(-1 + j)$ is expressed in polar form as

$$(-1 + j) = \sqrt{2}\angle 135^\circ$$

$$A_3 = \frac{1}{\sqrt{2}\angle 135^\circ + 90^\circ}$$
$$= 0.707\angle{+135^\circ}$$
$$= 0.707 e^{+j135^\circ}$$

A_2 is the conjugate of A_3

$$A_2 = 0.707\angle{-135^\circ} = 0.707 e^{-j135^\circ}$$

$$X(s) = \frac{1}{s} + 0.707 \left[e^{-j135^\circ} \frac{1}{(s + 1 + j)} + \frac{e^{+j135^\circ}}{s + 1 - j} \right]$$

Taking inverse LT, we get

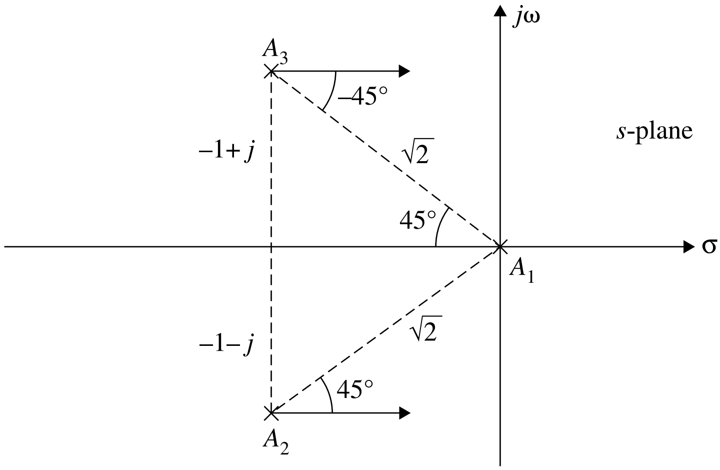


Fig. 4.27 Pole-zero configuration of Example 4.39

$$\begin{aligned}
x(t) &= 1 + 0.707[e^{-j135^\circ} e^{-(+1+j)t} + e^{+j135^\circ} e^{-(1-j)t)}] \\
&= 1 + 1.414e^{-t}\left[\frac{e^{j(135^\circ+t)} + e^{-j(135^\circ+t)}}{2}\right] \\
&= 1 + 1.414e^{-t}\cos(135^\circ + t) \\
&= 1 - 1.414e^{-t}\sin(t + 45^\circ)
\end{aligned}$$

$$x(t) = 1 - 1.414e^{-t}\sin\left(t + \frac{\pi}{4}\,\text{rad}\right)$$

Method 2. Graphical Method

From the pole-zero configuration of $X(s)$ shown in Fig. 4.27, we get

$$\begin{aligned}
A_1 &= \frac{2}{\sqrt{2}\angle 45^\circ \sqrt{2}\angle -45^\circ} = 1 \\
A_2 &= \frac{2}{\sqrt{2}\angle -135^\circ 2\angle -90^\circ} = 0.707\angle -135^\circ \\
A_3 &= \text{conjugate of } A_2 \\
A_3 &= 0.707\angle 135^\circ
\end{aligned}$$

By graphical method, the residues A_1, A_2 and A_3 are obtained with ease. Substituting these values in $X(s)$ and taking inverse LT, the following result is obtained as in Method 1

$$x(t) = 1 - 1.414e^{-t}\sin\left(t + \frac{\pi}{4}\text{ rad}\right)$$

Example 4.41 Find the inverse LT of the following function:

$$X(s) = \frac{(3s^2 + 8s + 23)}{(s+3)(s^2 + 2s + 10)}.$$

(*Anna University, April, 2005*)

Solution

$$s^2 + 2s + 10 = (s + 1 + j3)(s + 1 - j3)$$

The given $X(s)$ is put into partial fraction as follows:

$$\begin{aligned} X(s) &= \frac{A_1}{(s+3)} + \frac{A_2}{(s+1+j3)} + \frac{A_3}{(s+1-j3)} \\ (3s^2 + 8s + 23) &= A_1(s^2 + 2s + 10) + A_2(s+3)(s+1-j3) \\ &\quad + A_3(s+3)(s+1+j3) \end{aligned}$$

Let $s = -3$

$$\begin{aligned} 27 - 24 + 23 &= A_1(9 - 6 + 10) \\ A_1 &= \frac{26}{13} = 2 \end{aligned}$$

Put $s = -1 - j3$

$$\begin{aligned} 3(+1 + j3)^2 - 8(1 + j3) + 23 &= A_2(-1 - j3 + 3)(-j6) \\ 3(-8 + j6) - 8 - 24j + 23 &= A_2(j6 - 18 - j18) \\ (-24 - 8 + 23) + j18 - 24j &= A_2(-18 - j12) \\ -9 - j6 &= -A_2(18 + j12) \\ A_2 &= \frac{(3 + j2)}{(6 + j4)} = \frac{3.6\angle 33.7^\circ}{7.2\angle 33.7^\circ} \\ &= 0.5 \\ A_3 &= \text{conjugate of } A_2 \\ &= 0.5 \\ X(s) &= \left[\frac{2}{s+3} + \frac{0.5}{s+1+j3} + \frac{0.5}{s+1-j3}\right] \end{aligned}$$

Taking inverse LT, we get

$$
\begin{aligned}
x(t) &= 2e^{-3t} + 0.5\left\{e^{-(1+j3)t} + e^{-(1-j3)t}\right\} \\
&= 2e^{-3t} + e^{-t}\,\frac{\{e^{-j3t} + e^{j3t}\}}{2}
\end{aligned}
$$

$$x(t) = 2e^{-3t} + e^{-t}\cos 3t$$

Example 4.42 Find the inverse LT of

$$X(s) = \frac{3s^2 + 8s + 6}{(s+2)(s^2 + 2s + 1)}.$$

(*Anna University, December, 2007*)

Solution

$$
\begin{aligned}
(s^2 + 2s + 1) &= (s+1)^2 \\
X(s) &= \frac{(3s^2 + 8s + 6)}{(s+2)(s+1)^2} \\
&= \frac{A_1}{(s+2)} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)} \\
&= \frac{A_1(s^2 + 2s + 1) + A_2(s+2) + A_3(s+1)(s+2)}{(s+2)(s+1)^2} \\
3s^2 + 8s + 6 &= A_1(s^2 + 2s + 1) + A_2(s+2) + A_3(s+1)(s+2)
\end{aligned}
$$

Put $s = -2$

$$
\begin{aligned}
12 - 16 + 6 &= A_1(4 - 4 + 1) + 0 + 0 \\
A_1 &= 2 \\
3s^2 + 8s + 6 &= (A_1 + A_3)s^2 + (2A_1 + A_2 + 3A_3)s + (A_1 + 2A_2 + 2A_3)
\end{aligned}
$$

Comparing the coefficients of s^2, we get

$$
\begin{aligned}
3 &= A_1 + A_3 \\
A_3 &= 3 - A_1 = 3 - 2 \\
A_3 &= 1
\end{aligned}
$$

Comparing the coefficients of s, we get

$$
\begin{aligned}
8 &= 2A_1 + A_2 + 3A_3 \\
&= 4 + A_2 + 3 \\
A_2 &= 1
\end{aligned}
$$

Substituting the values of $A_1,\ A_2$ and A_3 in $X(s)$ we get

$$X(s) = \frac{2}{(s+2)} + \frac{1}{(s+1)^2} + \frac{1}{(s+1)}$$

Taking inverse LT of $X(s)$, we get

$$x(t) = (2e^{-2t} + te^{-t} + e^{-t})u(t)$$

Example 4.43 Find the inverse LT of the following function:

$$X(s) = \frac{10s^2}{(s+2)(s^2+4s+5)}$$

Solution

Method 1.

$$\begin{aligned}(s^2+4s+5) &= (s+2+j)(s+2-j)\\ X(s) &= \frac{10s^2}{(s+2)(s+2+j)(s+2-j)}\\ &= \frac{A_1}{(s+2)} + \frac{A_2}{(s+2+j)} + \frac{A_3}{(s+2-j)}\\ 10s^2 &= A_1(s^2+4s+5) + A_2(s+2)(s+2-j) + A_3(s+2)(s+2+j)\end{aligned}$$

Put $s = -2$

$$\begin{aligned}40 &= A_1(4-8+5) + 0 + 0\\ A_1 &= 40\end{aligned}$$

Put $s = -2 - j$

$$\begin{aligned}10(-2-j)^2 &= 0 + A_2(-2-j+2)(-2-j+2-j) + 0\\ 10(4-1+4j) &= A_2(-j)(-2j)\\ -10(3+4j) &= 2A_2\\ A_2 &= -5(3+4j)\\ &= 25\angle{-126.88^\circ} = 25e^{-j126.88^\circ}\end{aligned}$$

$A_3 =$ conjugate of A_2

$$\begin{aligned}A_3 &= 25\angle{+126.88^\circ} = e^{j126.88^\circ}\\ X(s) &= \frac{40}{(s+2)} + \frac{25e^{-j126.88^\circ}}{(s+2+j)} + \frac{25e^{j126.88^\circ}}{(s+2-j)}\end{aligned}$$

Taking inverse LT, we get

$$
\begin{aligned}
x(t) &= 40e^{-2t} + 25\left\{e^{-j126.88^\circ}e^{-(2+j)t} + e^{j126.88^\circ}e^{-(2-j)t}\right\} \\
&= 40e^{-2t} + e^{-2t}50\frac{\{e^{-j(t+126.88^\circ)} + e^{+j(126.88^\circ+t)}\}}{2} \\
x(t) &= [40e^{-2t} + 50e^{-2t}\cos(t + 126.88^\circ)]
\end{aligned}
$$

$$
x(t) = [40 - 50\sin(t + 0.644\,\text{rad})]e^{-2t}u(t)
$$

Method 2. Graphical Method

The pole zero configuration of $X(s)$ is shown in Fig. 4.28. From Fig. 4.28 the residues A_1, A_2 and A_3 are obtained as follows:

$$
A_1 = \frac{10(-2)(-2)}{1\angle 90^\circ 1\angle -90^\circ} = 40
$$

$$
\begin{aligned}
A_3 &= \frac{10\sqrt{5}\angle 153.44^\circ \sqrt{5}\angle 153.44^\circ}{1\angle 90^\circ\ 2\angle 90^\circ} \\
&= 25\angle 126.88^\circ = 25e^{j126.88^\circ}
\end{aligned}
$$

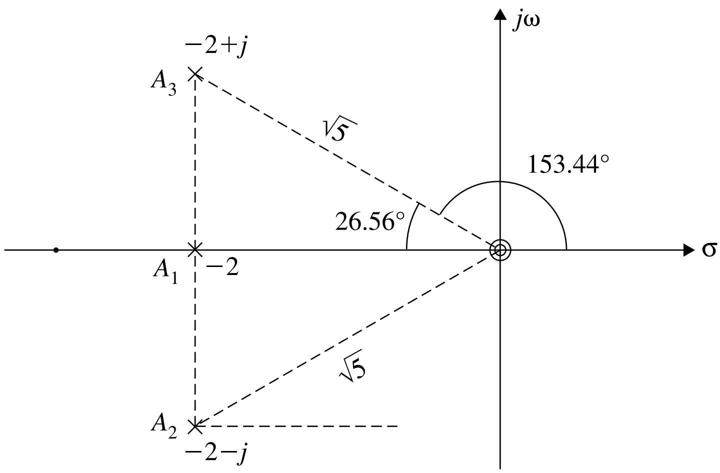


Fig. 4.28 Pole-zero diagram of $X(s)$ of Example 4.43

A_2 = conjugate of A_3

$$A_2 = 25\angle{-126.88^\circ} = 25e^{-j126.88^\circ}$$

The residues A_1, A_2 and A_3 obtained by graphical method are same as obtained by analytical method. Thus by proceeding as in Method 1, the inverse LT is obtained as

$$x(t) = [40 - 50\sin(t + 0.644\,\text{rad})]e^{-2t}u(t)$$

4.9 Solving Differential Equation

Laplace transform is a very powerful tool in the analysis of linear time invariant dynamic system. It provides

- Solutions to LTI dynamic systems described by linear differential equations by converting the differential equation to algebraic equation.
- For test signals of different kind, solutions are obtained for the differential equations with and without initial conditions.
- The dynamic systems are represented in terms of transfer function which is nothing but the ratio of the LT of the output variable to the LT of the input variable.
- The transfer function is made use of to determine the frequency response of the system.
- The transfer function is also made use of to determine the stability of the system using the well-known Routh-Hurwitz criterion and Nyquist stability criterion.
- The structure of the dynamic system is realized using the transfer function.

Now we give below the method of solving differential equation using LT.

4.9.1 Solving Differential Equation Without Initial Conditions

1. If $y(t)$ is the output variable and $x(t)$ is the input variable, convert the differential equation to algebraic equation which is obtained by simple multiplication of Laplace complex variable s.
2. These algebraic equations are obtained using the following LT when the initial conditions are zero.

$$L[y(t)] = Y(s)$$
$$L\left[\frac{dy}{dt}\right] = sY(s)$$
$$L\left[\frac{d^2y}{dt}\right] = s^2Y(s)$$
$$L\left[\frac{d^3y}{dt^3}\right] = s^3Y(s)$$
$$\vdots$$
$$L\left[\frac{d^ny}{dt^n}\right] = s^nY(s)$$

Similarly for the input $x(t)$, we convert

$$L[x(t)] = X(s)$$
$$L\left[\frac{dx}{dt}\right] = sX(s)$$
$$L\left[\frac{d^2x}{dt^2}\right] = s^2X(s)$$
$$\vdots$$
$$L\left[\frac{d^nx}{dt^n}\right] = s^nX(s)$$

The following examples illustrate the method of solving differential equation using LT when the initial conditions are zero for the input as well as the output.

Example 4.44 Consider an LTIC system with the following differential equation with zero initial conditions for the input and output.

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Find the impulse response of the system.

(*Anna University, December, 2006*)

Solution Taking LT on both sides of the given differential equation, we get

$$(s^2 + 4s + 3)Y(s) = (s + 2)X(s)$$

The transfer function is obtained as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{(s+2)}{(s^2+4s+3)} = \frac{(s+2)}{(s+1)(s+3)}$$
$$Y(s) = \frac{(s+2)X(s)}{(s+1)(s+3)}$$

From Table 4.1, for an impulse input $x(t) = \delta(t)$, $X(s) = 1$. Substituting this in the above equation, we get

$$\begin{aligned} Y(s) &= \frac{(s+2)}{(s+1)(s+3)} \\ &= \frac{A_1}{(s+1)} + \frac{A_2}{(s+3)} \\ (s+2) &= A_1(s+3) + A_2(s+1) \end{aligned}$$

Put $s = -1$

$$\begin{aligned} (-1+2) &= A_1(-1+3) + 0 \\ A_1 &= 0.5 \end{aligned}$$

Put $s = -3$

$$\begin{aligned} (-3+2) &= 0 + A_2(-3+1) \\ A_2 &= 0.5 \\ \therefore \qquad Y(s) &= \frac{0.5}{(s+1)} + \frac{0.5}{(s+3)} \end{aligned}$$

Taking inverse LT, we get

$$y(t) = 0.5\left[e^{-t} + e^{-3t}\right]u(t)$$

Example 4.45 Using LT, find the impulse response of an LTI system described by the following differential equation.

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

Assume zero initial conditions.

(*Anna University, April, 2004*)

Solution Taking LT on both sides of the given differential equation, we get

$$
\begin{aligned}
(s^2 - s - 2)Y(s) &= X(s) \\
Y(s) &= \frac{X(s)}{(s^2 - s - 2)} \\
&= \frac{X(s)}{(s+1)(s-2)}
\end{aligned}
$$

For an impulse $X(s) = 1$

$$
\begin{aligned}
Y(s) &= \frac{1}{(s+1)(s-2)} \\
&= \frac{A_1}{(s+1)} + \frac{A_2}{(s-2)} \\
1 &= A_1(s-2) + A_2(s+1)
\end{aligned}
$$

Put $s = -1$

$$A_1 = -\frac{1}{3}$$

Put $s = 2$

$$
\begin{aligned}
A_2 &= \frac{1}{3} \\
Y(s) &= \frac{1}{3}\left[\frac{1}{s-2} - \frac{1}{s+1}\right]
\end{aligned}
$$

$$y(t) = \frac{1}{3}\left[e^{2t} - e^{-t}\right]u(t)$$

Example 4.46 Consider the LTI system with the following differential equation with zero initial conditions.

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$$

where $x(t) = e^{-4t}u(t)$. Find an expression for $y(t)$ using LT method.

Solution The given differential equation is written as follows:

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{-4t}u(t)$$

Taking LT on both sides, we get

$$(s^2+5s+6)Y(s) = \frac{1}{(s+4)}$$
$$Y(s) = \frac{1}{(s+4)(s^2+5s+6)}$$
$$= \frac{1}{(s+3)(s+2)(s+4)}$$
$$= \frac{A_1}{(s+3)} + \frac{A_2}{(s+2)} + \frac{A_3}{(s+4)}$$
$$1 = A_1(s+2)(s+4) + A_2(s+3)(s+4) + A_3(s+3)(s+2)$$

Put $s = -3$

$$1 = A_1(-3+2)(-3+4)$$
$$A_1 = -1$$

Put $s = -2$

$$1 = A_2(-2+3)(-2+4)$$
$$A_2 = \frac{1}{2} = 0.5$$

Put $s = -4$

$$1 = A_3(-4+3)(-4+2)$$
$$A_3 = \frac{1}{2} = 0.5$$
$$Y(s) = \frac{-1}{(s+3)} + \frac{0.5}{(s+2)} + \frac{0.5}{(s+4)}$$

$$y(t) = \left(-e^{-3t} + 0.5e^{-2t} + 0.5e^{-4t}\right)u(t)$$

Example 4.47 Consider the following differential equation with zero initial conditions.

$$\frac{d^2y(t)}{dt} + 2\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + x(t)$$

For $x(t) = u(t)$, a unit step input find the response $y(t)$ of the system.

Solution

Method 1

Taking LT on both sides of the differential equation, we get

$$(s^2 + 2s + 2)Y(s) = (s + 1)X(s)$$
$$Y(s) = \frac{(s + 1)}{(s^2 + 2s + 2)}X(s)$$

For unit step $X(s) = \frac{1}{s}$. Substituting this in the above equation, we get

$$Y(s) = \frac{(s + 1)}{s(s^2 + 2s + 2)}$$
$$(s^2 + 2s + 2) = (s + 1 + j)(s + 1 - j)$$
$$\therefore \quad Y(s) = \frac{(s + 1)}{s(s + 1 + j)(s + 1 - j)}$$
$$= \frac{A_1}{s} + \frac{A_2}{(s + 1 + j)} + \frac{A_3}{(s + 1 - j)}$$
$$(s + 1) = A_1(s^2 + 2s + 2) + A_2 s(s + 1 - j) + A_3 s(s + 1 + j)$$

Put $s = 0$

$$1 = 2A_1 \quad \text{or} \quad A_1 = 0.5$$

Put $s = -(1 + j)$

$$(-1 - j + 1) = 0 + A_2(-1 - j)(-1 - j + 1 - j) + 0$$
$$-j = A_2(-1 - j)(-2j)$$
$$= A_2(2j - 2) = 2A_2(j - 1)$$
$$A_2 = \frac{0.5j}{1 - j} = \frac{0.5}{\sqrt{2}}\frac{\angle 90^\circ}{\angle -45^\circ}$$
$$= 0.354\angle 135^\circ = 0.354e^{j135^\circ}$$
$$A_3 = \text{conjugate of } A_2$$
$$= 0.354e^{-j135^\circ}$$
$$Y(s) = \frac{0.5}{s} + \frac{0.354e^{j135^\circ}}{s + 1 + j} + \frac{0.354e^{-j135^\circ}}{(s + 1 - j)}$$

$$\begin{aligned}
y(t) &= 0.5 + 0.354e^{j135^\circ}e^{-(1+j)t} + 0.354e^{-j135^\circ}e^{-(1-j)t} \\
&= 0.5 + 0.708e^{-t}\frac{[e^{j(135^\circ - t)} + e^{-j(135^\circ - t)}]}{2} \\
&= 0.5 + 0.708e^{-t}\cos(135^\circ - t) \\
&= 0.5 - 0.708e^{-t}\sin(45^\circ - t)
\end{aligned}$$

$$y(t) = \left[0.5 + 0.708e^{-t}\sin\left(t - \frac{\pi}{4}\text{rad}\right)\right]u(t)$$

Method 2

The pole-zero diagram of $Y(s)$ is shown in Fig. 4.29. The residues A_1, A_2 and A_3 are determined as follows:

$$\begin{aligned}
A_1 &= \frac{1\angle 0^\circ}{\sqrt{2}\angle 45^\circ \sqrt{2}\angle -45^\circ} = 0.5 \\
A_2 &= \frac{1\angle -90^\circ}{\sqrt{2}\angle 225^\circ 2\angle -90^\circ} = 0.354\angle 135^\circ \\
A_3 &= \text{conjugate of } A_2 = 0.354\angle -135^\circ
\end{aligned}$$

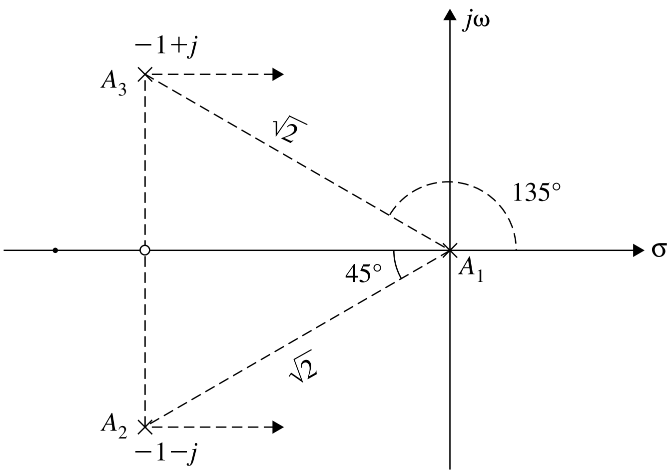


Fig. 4.29 Pole-zero diagram of Example 4.47

The residues determined by graphical method is same as determined by analytical method. Therefore $y(t)$ is written as

$$y(t) = \left[0.5 + 0.708e^{-t} \sin\left(t - \frac{\pi}{4}\,\text{rad}\right)\right] u(t)$$

4.9.2 Solving Differential Equation with the Initial Conditions

1. When the initial conditions are specified for the given differential equation, they have to be accounted for when LT is taken to convert the differential equation to algebraic equation. Thus,

$$L\left[\frac{dy}{dt}\right] = sY(s) - y(0^-)$$
$$L\left[\frac{d^2y}{dt^2}\right] = s^2Y(s) - sy(0^-) - \dot{y}(0^-)$$
$$L\left[\frac{d^3y}{dt^3}\right] = s^3Y(s) - s^2y(0^-) - s\dot{y}(0^-) - \ddot{y}(0^-)$$

   The initial conditions $y(0^-)$, $\dot{y}(0^-)$ and $\ddot{y}(0^-)$, are meant that the system initial conditions are given just before the input is applied to the system.
   The initial condition $y(0^+)$ indicates that the initial condition is given to the system after the input is applied which is not realistic. Unless otherwise mentioned, $y(0^-)$ means $y(0)$ and $y(0)$ is not $y(0^+)$.
2. The zero initial conditions explained in step 1 is applicable to the input also. Thus,

$$\frac{dx}{dt} = sX(s) - x(0^-)$$

3. The initial conditions for an input multiplied by $u(t)$ implies that the signals are zero prior to $t = 0$.
4. The solution of the differential equation contains two components. The first component is the response due to the initial conditions only where the input is assumed to be absent. The response is called the zero input response. The second component is the response due to the input alone and the initial conditions here are assumed to be zero. Such response is called zero state response.
5. The total response = zero state response + zero input response.
6. If one is interested to find out the zero initial conditions for verification of the results, only the zero input response has to be considered and not the total response. The total response satisfies the initial conditions at $t = 0^+$.

The following examples illustrate the method of obtaining total response which is due to initial conditions and the input.

Example 4.48 A certain system is described by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 12y(t) = x(t)$$

Use LT to determine the response of the system to unit step input applied at $t = 0$. Assume the initial conditions are $y(0^-) = -2$ and $\frac{dy(0^-)}{dt} = 0$.

(*Anna University, May, 2007*)

Solution Taking LT on both sides of the given equation, we get

$$s^2Y(s) - sy(0^-) - \dot{y}(0^-) + 7Y(s) - 7y(0^-) + 12Y(s) = X(s)$$

$$\begin{aligned}(s^2 + 7s + 12)Y(s) + 2s + 14 &= \frac{1}{s}\\ (s^2 + 7s + 12)Y(s) &= -2s - 14 + \frac{1}{s}\\ &= \frac{(-2s^2 - 14s + 1)}{s}\end{aligned}$$

$$\begin{aligned}Y(s) &= \frac{(-2s^2 - 14s + 1)}{s(s^2 + 7s + 12)}\\ &= \frac{(-2s^2 - 14s + 1)}{s(s+3)(s+4)}\\ &= \frac{A_1}{s} + \frac{A_2}{(s+3)} + \frac{A_3}{(s+4)}\\ -2s^2 - 14s + 1 &= A_1(s+3)(s+4) + A_2s(s+4) + A_3s(s+3)\end{aligned}$$

Put $s = 0$

$$\begin{aligned}1 &= A_1(12)\\ A_1 &= \frac{1}{12}\end{aligned}$$

Put $s = -3$

$$\begin{aligned}-18 + 42 + 1 &= A_2(-3)(-3+4)\\ A_2 &= \frac{-25}{3}\end{aligned}$$

Put $s = -4$

$$\begin{aligned}-32 + 56 + 1 &= A_3(-4)(-4+3)\\ A_3 &= \frac{25}{4}\\ Y(s) &= \frac{1}{12}\frac{1}{s} - \frac{25}{3}\frac{1}{(s+3)} + \frac{25}{4}\frac{1}{(s+4)}\end{aligned}$$

The total response is obtained by taking inverse LT

$$y(t) = \left[\frac{1}{12} - \frac{25}{3}e^{-3t} + \frac{25}{4}e^{-4t}\right]u(t)$$

Example 4.49 Solve

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + x(t)$$

if the initial conditions are $y(0^+) = \frac{9}{4}$; $\dot{y}(0^+) = 5$, if the input is $e^{-3t}u(t)$.

(*Anna University, December, 2007*)

Solution Taking LT on both sides of the equation, we get

$$s^2Y(s) - sy(0^+) - \dot{y}(0^+) + 4sY(s) - 4y(0^+) + 4Y(s) = sX(s) + X(s) - x(0^+)$$

If the initial conditions are given at $t = 0^+$ for the output, then the initial conditions must be applied to the input also

$$\begin{aligned}L\left[\frac{d}{dt}x(t) + x(t)\right] &= sX(s) - x(0^+) + X(s)\\ x(t) &= e^{-3t}\\ X(s) &= \frac{1}{(s+3)}\end{aligned}$$

Since $x(0^+) = \underset{t\to 0}{Lt}\, e^{-3t} = 1$

$$\begin{aligned}(s+1)X(s) - x(0^+) &= \frac{(s+1)}{s+3} - 1\\ &= \frac{(s+1-s-3)}{(s+3)}\\ &= \frac{-2}{(s+3)}\end{aligned}$$

Alternatively,

$$x(t) = e^{-3t}$$
$$\frac{dx(t)}{dt} = -3e^{-3t}$$
$$L\left[\frac{dx(t)}{dt}\right] = \frac{-3}{s+3}$$
$$L[x(t)] = \frac{1}{(s+3)}$$
$$\therefore \quad L\left[\frac{dx(t)}{dt} + x(t)\right] = \frac{1}{s+3}[-3+1]$$
$$= \frac{-2}{(s+3)}$$

Substituting $y(0^+)$ and $\dot{y}(0^+)$ in the given equation we get

$$(s^2 + 4s + 4)Y(s) - \frac{9}{4}s - 5 - 9 = \frac{-2}{(s+3)}$$

$$(s^2 + 4s + 4)Y(s) = \frac{9}{4}s + 14 - \frac{2}{(s+3)}$$

$$Y(s) = \frac{(9s^2 + 83s + 160)}{4(s+3)(s^2 + 4s + 4)}$$

$$= \frac{(9s^2 + 83s + 160)}{4(s+3)(s+2)^2} \quad \text{(a)}$$

$$= \frac{A_1}{(s+3)} + \frac{A_2}{(s+2)^2} + \frac{A_3}{(s+2)} \quad \text{(b)}$$

$$\frac{1}{4}[9s^2 + 83s + 160] = A_1(s+2)^2 + A_2(s+3) + A_3(s+2)(s+3)$$

Put $s = -3$

$$\frac{1}{4}[81 - 249 + 160] = A_1 \quad \text{or} \quad A_1 = -2$$

Put $s = -2$

$$\frac{1}{4}[36 - 166 + 160] = A_2 \quad \text{or} \quad A_2 = 7.5$$

Compare the coefficients of s^2 on both sides

$$\frac{9}{4} = A_1 + A_3 = -2 + A_3 \quad \text{or} \quad A_3 = 4.25$$

$$Y(s) = \frac{-2}{s+3} + \frac{7.5}{(s+2)^2} + \frac{4.25}{(s+2)}$$

Taking inverse LT, we get

$$y(t) = -2e^{-3t} + 7.5te^{-2t} + 4.25e^{-2t} \quad t \geq 0$$

To check whether the residues are correctly determined
Choose any value of s such that when substituted in $X(s)$ it does not become zero or infinitive. Find the value of $Y(s)$ in (a) and (b). If both are same, the residues determined are correct.

For $s = 0$;

$$\frac{160}{4 \times 3 \times 4} = \frac{-2}{3} + \frac{7.5}{4} + \frac{4.25}{2}$$
$$\frac{40}{12} = \frac{-8 + 22.5 + 25.5}{12} = \frac{40}{12}$$
$$\text{LHS} = \text{RHS}$$

Hence, the values of A_1, A_2 and A_3 determined are correct.

4.9.3 Zero Input and Zero State Response

As described earlier, the response of the system due to the input $x(t)$ with all initial conditions are zero is called zero state response. The response of the system due to the initial conditions with zero input is called zero input response. The total response of the system is the sum of the zero state response and zero input response. This is illustrated in the following example.

Example 4.50 Consider the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = \frac{dx(t)}{dt} + 3x(t)$$

$$x(t) = u(t)$$

$$y(0^-) = 1 \quad \text{and} \quad \dot{y}(0^-) = 2$$

Find the zero state, zero input and total response. Verify, from the expression for the response, the initial conditions given.

Solution Zero State Response

For zero state response the initial conditions are assumed to be zero. Under this condition, the output is denoted as $y_i(t)$.

Taking LT on both sides of the given differential equation, we get

$$(s^2 + 6s + 8)Y_i(s) = (s + 3)X(s)$$

Substituting $X(s) = \frac{1}{s}$ and $(s^2 + 6s + 8) = (s + 2)(s + 4)$, we get

$$\begin{aligned} Y_i(s) &= \frac{(s+3)}{s(s+2)(s+4)} \\ &= \frac{A_1}{s} + \frac{A_2}{(s+2)} + \frac{A_3}{(s+4)} \\ (s+3) &= A_1(s+2)(s+4) + A_2 s(s+4) + A_3 s(s+2) \end{aligned}$$

Put $s = 0$

$$3 = A_1 8 \quad \text{or} \quad A_1 = \frac{3}{8}$$

Put $s = -2$

$$\begin{aligned} (-2+3) &= A_2(-2)(-2+4) \\ A_2 &= -\frac{1}{4} \end{aligned}$$

Put $s = -4$

$$\begin{aligned} (-4+3) &= A_3(-4)(-4+2) \\ A_3 &= -\frac{1}{8} \\ Y_i(s) &= \frac{3}{8}\frac{1}{s} - \frac{1}{4}\frac{1}{(s+2)} - \frac{1}{8}\frac{1}{(s+4)} \end{aligned}$$

$$y_i(t) = \left(\frac{3}{8} - \frac{1}{4}e^{-2t} - \frac{1}{8}e^{-4t}\right)u(t)$$

Zero Input Response

Under zero input condition the output is denoted as $y_s(t)$. The given differential equation becomes,

$$\frac{d^2y_s(t)}{dt^2} + 6\frac{dy_s(t)}{dt} + 8y_s(t) = 0$$

Taking LT with initial conditions, we get

$$s^2Y_s(s) - sy_s(0^-) - \dot{y}_s(0^-) + 6Y_s(s) - 6y_s(0) + 8Y_s(s) = 0$$

$$\begin{aligned}
(s^2 + 6s + 8)Y_s(s) &= (s + 2 + 6) = (s + 8) \\
Y_s(s) &= \frac{(s+8)}{(s+2)(s+4)} \\
&= \frac{A_1}{(s+2)} + \frac{A_2}{(s+4)} \\
(s+8) &= A_1(s+4) + A_2(s+2)
\end{aligned}$$

Put $s = -2$

$$\begin{aligned}
(-2+8) &= A_1(-2+4) \\
A_1 &= 3
\end{aligned}$$

Put $s = -4$

$$(-4+8) = A_2(-4+2)$$

$$\begin{aligned}
A_2 &= -2 \\
Y_s(s) &= \frac{3}{s+2} - \frac{2}{s+4}
\end{aligned}$$

$$y_s(t) = \left(3e^{-2t} - 2e^{-4t}\right)u(t)$$

Total Response

The total response is denoted by the letter $y(t)$.

$$\begin{aligned} y(t) &= y_i(t) + y_s(t) \\ &= \left(\frac{3}{8} - \frac{1}{4}e^{-2t} - \frac{1}{8}e^{-4t}\right)u(t) + (3e^{-2t} - 2e^{-4t})u(t) \end{aligned}$$

$$y(t) = \left[\frac{3}{8} + \frac{11}{4}e^{-2t} - \frac{17}{8}e^{-4t}\right]u(t)$$

To verify the initial condition, consider the zero input response $y_s(t)$

$$\begin{aligned} y_s(t) &= 3e^{-2t} - 2e^{-4t} \\ y_s(0) &= y(0) = 3 - 2 = 1 \\ \dot{y}(0) &= \left.\frac{dy_s(t)}{dt}\right|_{t=0} = -6e^{-2t} + 8e^{-4t}\Big|_{t=0} = -6 + 8 \\ \dot{y}(0) &= 2 \end{aligned}$$

The given initial conditions are satisfied. On the other hand, consider the expression for the total response

$$\begin{aligned} y(t) &= \frac{3}{8} + \frac{11}{4}e^{-2t} - \frac{17}{8}e^{-4t} \\ y(0) &= \frac{3}{8} + \frac{11}{4} - \frac{17}{8} \\ &= 1 \\ \dot{y}(t) = \frac{dy(t)}{dt} &= -\frac{22}{4}e^{-2t} + \frac{68}{8}e^{-4t} \\ \dot{y}(0) &= \frac{-22}{4} + \frac{68}{8} \\ \dot{y}(0) &= 3 \end{aligned}$$

The result obtained is erroneous. **Therefore, the initial conditions are verified from zero input response and not the total response**.

4.9.4 Natural and Forced Response Using LT

Consider the differential equation of Example 4.50 which is given below:

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = \frac{d}{dx}x(t) + 3x(t)$$

Taking LT on both sides of the above equation with zero conditions, we get

$$(s^2 + 6s + 8)Y(s) = (s + 3)X(s)$$
$$(s + 2)(s + 4)Y(s) = (s + 3)X(s)$$

The transfer function is the ratio of $Y(s)$ to $X(s)$ and is written as

$$\frac{Y(s)}{X(s)} = \frac{(s + 3)}{(s + 2)(s + 4)}$$

In the above equation, $s^2 + 6s + 8 = 0$ is called characteristic equation and $s = -2$ and $s = -4$ are called characteristic roots or eigen values. In the total response of $y(t)$, corresponding to these eigen values, the characteristic modes are found. In the above example the characteristic modes are e^{-2t} and e^{-4t}. In the total response of the system which is composed of zero input response and zero state response, if we can lump together all the terms corresponding to the characteristic mode, it is called natural response $y_n(t)$. The remaining non-characteristic mode terms are lumped together and the response is called forced response and is denoted by $y_f(t)$. Thus, in Example 4.49, the eigen values are $s = -2$ and $s = -4$. The characteristic modes are e^{-2t} and e^{-4t}. Thus,

$$y(t) = \frac{3}{8} + \frac{11}{4}e^{-2t} - \frac{17}{8}e^{-4t}$$

The natural response

$$y_n(t) = \frac{11}{4}e^{-2t} - \frac{17}{8}e^{-4t}$$

The forced response

$$y_f(t) = \frac{3}{8}$$

Example 4.51 Find the forced response of the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = \frac{dx}{dt} + x(t)$$

where $x(t) = t^2$.

Solution Taking LT of the given differential equation, we get

$$\begin{aligned}(s^2 + 6s + 8)Y(s) &= (s + 1)X(s)\\ (s^2 + 6s + 8) &= (s + 2)(s + 4)\end{aligned}$$

The eigen values are $s = -2$ and $s = -4$. The characteristic modes are e^{-2t} and e^{-4t}. The terms involving these characteristic mode will correspond to the natural response of the system. The remaining terms will correspond to forced response of the system. Substituting $X(s) = \frac{2}{s^3}$, we get

$$\begin{aligned}Y(s) &= \frac{2(s+1)}{(s+2)(s+4)s^3}\\ &= \frac{A_1}{(s+2)} + \frac{A_2}{(s+4)} + \frac{A_3}{s^3} + \frac{A_4}{s^2} + \frac{A_5}{s}\\ 2(s+1) &= A_1s^3(s+4) + A_2s^3(s+2) + A_3(s+2)(s+4)\\ &\quad + A_4s(s+2)(s+4) + A_5s^2(s+2)(s+4)\end{aligned}$$

Put $s = 0$

$$2 = 8A_3 \quad \text{or} \quad A_3 = \frac{1}{4}$$

Compare the coefficients of s

$$\begin{aligned}2 &= 6A_3 + 8A_4\\ A_4 &= \frac{1}{16}\end{aligned}$$

Compare the coefficients of s^2

$$\begin{aligned}0 &= A_3 + 6A_4 + 8A_5\\ &= \frac{1}{4} + \frac{6}{16} + 8A_5\\ A_5 &= -\frac{10}{128} = -\frac{5}{64}\end{aligned}$$

The residues A_1 and A_2 are determined as follows. Put $s = -2$

$$2(-2+1) = A_1(-8)(-2+4) + 0 + 0 + 0 + 0$$
$$A_1 = \frac{1}{8}$$

Put $s = -4$

$$2(-4+1) = A_2 64(-4+2)$$
$$A_2 = \frac{3}{64}$$

$$Y(s) = \frac{1}{8}\frac{1}{(s+2)} + \frac{3}{64}\frac{1}{s+4} + \frac{1}{4}\frac{1}{s^3} + \frac{1}{16}\frac{1}{s^2} - \frac{5}{64}\frac{1}{s}$$

Taking inverse LT, we get

$$y(t) = \underbrace{\frac{1}{8}e^{-2t} + \frac{3}{64}e^{-4t}}_{\text{Natural response}} + \underbrace{\frac{1}{8}t^2 + \frac{1}{16}t - \frac{5}{64}}_{\text{Forced response}}$$

The natural response which is due to the characteristic modes e^{-2t} and e^{-4t} is given by

$$y_n(t) = \frac{1}{8}e^{-2t} + \frac{3}{64}e^{-4t} \qquad t \geq 0$$

The forced response is the response which does not contain the characteristic mode. Thus,

$$y_f(t) = \frac{1}{8}t^2 + \frac{1}{16}t - \frac{5}{64} \qquad t \geq 0$$

4.10 Time Convolution Property of the Laplace Transform

If

$$x_1(t) \overset{L}{\longleftrightarrow} X_1(s)$$

and

$$x_2(t) \overset{L}{\longleftrightarrow} X_2(s)$$

then

$$x_1(t) * x_2(t) \overset{L}{\longleftrightarrow} X_1(s)X_2(s)$$

This property of LT is used to determine

$$y(t) = x_1(t) * x_2(t)$$

The following examples illustrate this.

Example 4.52 Using the convolution property of the LT determine $y(t) = x_1(t) * x_2(t)$ where $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = e^{-3t}u(t)$.

Solution

$$\begin{aligned}
X_1(s) &= L[e^{-2t}u(t)] = \frac{1}{(s+2)} \\
X_2(s) &= L[e^{-3t}u(t)] = \frac{1}{(s+3)} \\
Y(s) &= X_1(s)X_2(s) \\
&= \frac{1}{(s+2)}\frac{1}{(s+3)} \\
&= \frac{1}{(s+2)} - \frac{1}{(s+3)}
\end{aligned}$$

$$y(t) = \left[e^{-2t} - e^{-3t}\right]u(t)$$

Example 4.53 Given

$$\begin{aligned}
x_1(t) &= e^{-2t}u(t) \\
x_2(t) &= (1 + e^{-3t})u(t)
\end{aligned}$$

Determine $y(t) = x_1(t) * x_2(t)$.

Solution

$$\begin{aligned}
X_1(s) &= L[x_1(t)] = \frac{1}{(s+2)} \\
X_2(s) &= L[x_2(t)] = \left[\frac{1}{s} + \frac{1}{s+3}\right]
\end{aligned}$$

$$\begin{aligned}
Y(s) &= X_1(s)X_2(s)\\
&= \frac{1}{(s+2)}\left[\frac{1}{s}+\frac{1}{s+3}\right]\\
&= \frac{(2s+3)}{s(s+2)(s+3)}\\
&= \frac{A_1}{s}+\frac{A_2}{s+2}+\frac{A_3}{s+3}\\
(2s+3) &= A_1(s+3)(s+2)+A_2 s(s+3)+A_3 s(s+2)
\end{aligned}$$

Put $s = 0$

$$\begin{aligned}
3 &= A_1(2)(3)\\
A_1 &= \frac{1}{2}
\end{aligned}$$

Put $s = -2$

$$\begin{aligned}
(-4+3) &= A_2(-2)(-2+3)\\
A_2 &= \frac{1}{2}
\end{aligned}$$

Put $s = -3$

$$\begin{aligned}
(-6+3) &= A_3(-3)(-3+2)\\
A_3 &= -1\\
Y(s) &= \frac{1}{2s}+\frac{1}{2(s+2)}-\frac{1}{(s+3)}
\end{aligned}$$

$$y(t) = \left(\frac{1}{2}+\frac{1}{2}e^{-2t}-e^{-3t}\right)u(t)$$

Example 4.54

$$\begin{aligned}
x_1(t) &= e^{a_1 t}u(t)\\
x_2(t) &= e^{a_2 t}u(-t)\\
y(t) &= x_1(t) * x_2(t)
\end{aligned}$$

Find $y(t)$ by Convolution method.

Solution

$$X_1(s) = \frac{1}{(s - a_1)}$$
$$X_2(s) = \frac{-1}{(s - a_2)}$$
$$Y(s) = X_1(s)X_2(s)$$
$$= \frac{-1}{(s - a_1)(s - a_2)}$$
$$= \frac{A_1}{(s - a_1)} + \frac{A_2}{(s - a_2)}$$
$$-1 = A_1(s - a_2) + A_2(s - a_1)$$

Put $s = a_1$

$$-1 = A_1(a_1 - a_2)$$
$$A_1 = \frac{1}{a_2 - a_1}$$

Put $s = a_2$

$$-1 = A_2(a_2 - a_1)$$
$$A_2 = \frac{-1}{(a_2 - a_1)}$$
$$Y(s) = \frac{1}{(a_2 - a_1)}\left[\frac{1}{(s - a_1)} - \frac{1}{(s - a_2)}\right]$$

$$y(t) = \frac{1}{(a_2 - a_1)}\left[e^{a_1 t}u(t) + e^{a_2 t}u(-t)\right]$$

Example 4.55 Given

$$x_1(t) = e^{3t}u(-t)$$
$$x_2(t) = u(t - 2)$$

Determine $y(t) = x_1(t) * x_2(t)$.

Solution

$$X_1(s) = L[x_1(t)] = \frac{-1}{(s - 3)}$$
$$X_2(s) = L[x_2(t)] = \frac{e^{-2s}}{s}$$

$$\begin{aligned} Y(s) &= X_1(s)X_2(s) \\ &= \frac{-e^{-2s}}{s(s-3)} \\ &= \frac{1}{3}\left[\frac{1}{s} - \frac{1}{s-3}\right]e^{-2s} \end{aligned}$$

Let

$$\begin{aligned} Y_1(s) &= \frac{1}{3}\left[\frac{1}{s} - \frac{1}{s-3}\right] \\ y_1(t) &= L^{-1}[Y_1(s)] \\ &= \frac{1}{3}[u(t) + e^{3t}u(-t)] \end{aligned}$$

By using the shifting property,

$$y(t) = y_1(t-2)$$

$$y(t) = \frac{1}{3}\left[u(t-2) + e^{3(t-2)}u(-t+2)\right]$$

4.11 Network Analysis Using Laplace Transform

An electrical network consists of passive elements like resistors, capacitors and inductors. They are connected in series, parallel and series parallel combinations. The currents through and voltages across these elements are obtained by solving integro differential equations using LT technique. Alternatively, the elements in the network are transformed from time domain and an algebraic equation is obtained which is expressed in terms of input and output. The commonly used inputs are impulse, step, ramp, sinusoids, exponentials *etc.* The desired response is expressed as a function of time for the given input. When writing the integro differential equation for a given network, the initial conditions must be taken into account. The energy storing elements such as inductor and a capacitor have initial conditions. At time $t = 0$ the capacitor is initially charged and has the initial voltage $v_c(0)$. Similarly, at $t = 0$, the current through the inductor is denoted as $i_L(0)$. These initial conditions are expressed $v_c(0^-)$ $v_c(0^+)$ and $i_L(0^-)$ and $i_L(0^+)$. The input is assumed to start at $t = 0$ which is considered as the reference point. The condition just before the input is applied $(t = 0^-)$ is denoted as $v_c(0^-)$ and the condition just after the input is applied $(t = 0^+)$ is denoted as $v_c(0^+)$. In many cases $v_c(0^-)$ and $v_c(0^+)$ are same but not always. Unless otherwise it is specified, $v_c(0)$ or $i_L(0)$ has to be taken as $v_c(0^-)$ or $i_L(0^-)$ which is more practical.

4.11.1 Mathematical Description of R-L-C-Elements

(a) Resistor

Consider the resistor connected across the voltage source $v_i(t)$. The loop equation for the above circuit is written as follows:

$$\begin{aligned} v_i(t) &= i(t)R \\ v_R(t) &= i(t)R \end{aligned} \tag{4.57a}$$

(b) Inductor

Consider the inductor connected across the voltage source $v_i(t)$ as shown in Fig. 4.30b. The loop equation for the above circuit is written as follows:

$$v_i(t) = L\frac{di(t)}{dt}$$

$$v_L(t) = L\frac{di(t)}{dt}$$

Taking LT on both sides of the above equation, with the initial current $i(t) = i(0^-)$, we get

$$V_i(s) = LsI(s) - Li(0^-) \tag{4.57b}$$

(c) Capacitor

For the capacitor circuit shown in Fig. 4.30c the following equation is written.

$$v_i(t) = \frac{1}{C}\int i(t)dt$$

Taking LT on both sides of the above equation with the capacitor initially charged with $v_c(0^-)$, the following equation is obtained

$$V_i(s) = \frac{1}{Cs}I(s) + v_c(0^-) \tag{4.57c}$$

Equation 4.57a, b and c are called integro differential equations. If the initial conditions are zero, these equations can respectively be written as

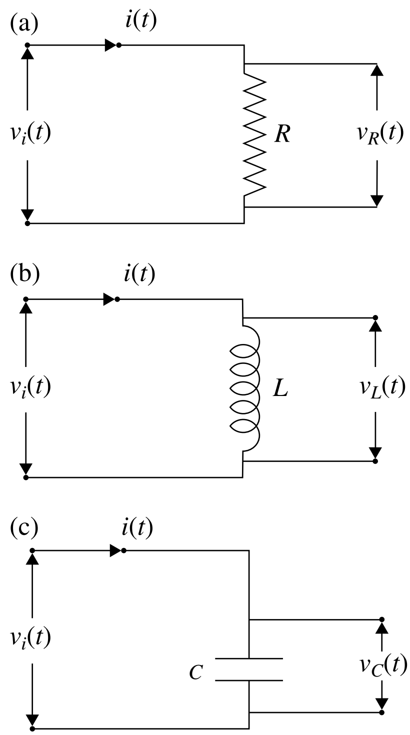


Fig. 4.30 **a** Circuit with a resistor. **b** Circuit with an inductor. **c** Circuit with a capacitor

$$V_i(s) = I(s)R$$
$$V_i(s) = I(s)Ls$$
$$V_i(s) = \frac{1}{Cs} I(s) \tag{4.58}$$

Equation (4.58) is called algebraic equation. These equations can be written in the frequency domain with the impedance function for the resistor, inductor and capacitor respectively as R, Ls and $\frac{1}{Cs}$.

4.11.2 Transfer Function and Pole-Zero Location

Consider the R-L-C series circuit shown in Fig. 4.31. $v_i(t)$ is the input, $v_0(t)$ is the output and $i(t)$ is the current flowing through the series circuit. For Fig. 4.31 the following integro differential equation is written.

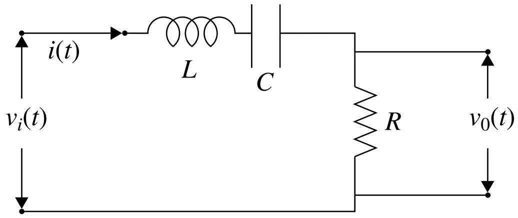


Fig. 4.31 R-L-C series circuit

$$v_i(t) = L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt + Ri(t) \tag{4.59}$$
$$v_0(t) = i(t)R$$

Taking LT on both sides of the above equations, we get the following algebraic equation.

$$V_i(s) = LsI(s) - Li(0^-) + \frac{1}{Cs}I(s) + v_c(0^-) + RI(s)$$
$$V_0(s) = RI(s) \tag{4.60}$$

In Eq. (4.60) if the initial conditions $i(0^-)$ and $v_c(0^-)$ are zero, the following equations could be written.

$$V_i(s) = \left(Ls + \frac{1}{Cs} + R\right) I(s)$$
$$V_0(s) = RI(s)$$

Dividing one by the other, we get

$$\frac{V_0(s)}{V_i(s)} = \frac{R}{Ls + \frac{1}{Cs} + R} \tag{4.61}$$
$$= \frac{RCs}{LCs^2 + RCs + 1}$$

Denoting $\frac{V_0(s)}{V_i(s)} = G(s)$, the above equation can be written in the following form:

$$G(s) = \frac{RCs}{LCs^2 + RCs + 1} \tag{4.62}$$

Equation (4.62) is called the transfer function of the given electric circuit.

Transfer function: Transfer function is therefore defined as the ratio of the LT of the output variable to the LT of the input variable with all the initial conditions being assumed to be zero.

In Eq. (4.62) if we put $L = 1, C = 1$ and $R = 2.5$, the transfer function is obtained as

$$G(s) = \frac{2.5s}{(s^2 + 2.5s + 1)} = \frac{2.5s}{(s + 2)(s + 0.5)} \tag{4.63}$$

The transfer function $G(s)$ becomes zero at $s = 0$.

The points at which the transfer function becomes zero in the s-plane are called zeros and are marked with a circle 0 in the s-plane.

The transfer function $G(s)$ becomes infinity at points $s = -2$ and $s = -0.5$ in the s-plane. These points are called poles of the transfer function and are marked with a small cross $\times$ in the s-plane.

The poles of the transfer function are defined as the points in the s-plane at which the transfer function becomes infinity.

The zeros of the transfer function are obtained by factorizing the numerator polynomial and putting each factor to zero. The poles of a transfer function are obtained by factorizing the denominator polynomial and putting each factor to zero. It is to be noted that the transfer function is not defined if the initial conditions are not zero. The poles and zeros of equation (4.63) are shown in Fig. 4.32. The s-plane is a complex plane whose real axis is represented by σ and the imaginary axis by $j\omega$.

The following examples illustrate electric circuit analysis using LT method.

Example 4.64 Consider the R.L.C. series circuit shown in Fig. 4.31 with $L = 1H$, $C = 1F$ and $R = 2.5$ ohms. Derive an expression for the output voltage $v_0(t)$ if the input is an (a) impulse (b) unit step. Assume zero initial conditions.

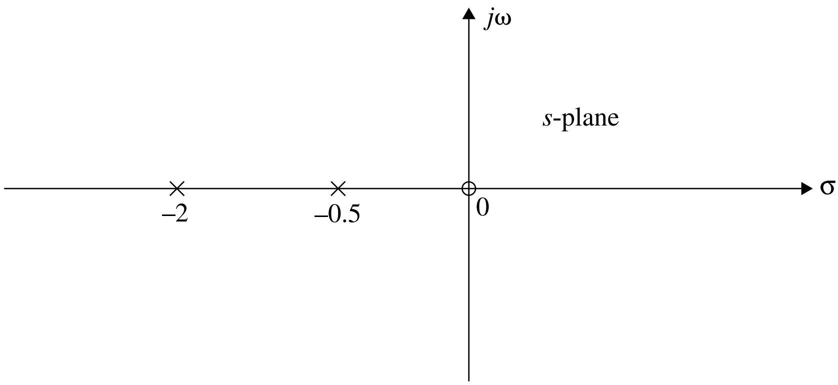


Fig. 4.32 Pole-zero locations of $G(s) = \frac{2.5s}{(s + 2)(s + 0.5)}$

Solution With zero initial conditions, for the circuit shown in Fig. 4.31, the following equation is obtained.

$$L\frac{di(t)}{dt} + \frac{1}{C}\int dt + Ri(t) = v_i(t)$$
$$v_0(t) = i(t)R.$$

Taking LT on both sides and substituting the numerical values for R, L and C we get

$$\frac{V_0}{V_i}(s) = \frac{2.5s}{(s+2)(s+0.5)}$$

(a) Impulse Response of the System

For an impulse input $V_i(s) = 1$

$$\therefore \quad V_0(s) = \frac{2.5s}{(s+2)(s+0.5)}$$
$$= \frac{A_1}{(s+2)} + \frac{A_2}{(s+0.5)}$$
$$2.5s = A_1(s+0.5) + A_2(s+2)$$

Put $s = -2$

$$(2.5)(-2) = A_1(-2+0.5)$$
$$A_1 = \frac{5}{1.5} = \frac{10}{3}$$

Put $s = -0.5$

$$(2.5)(-0.5) = A_2(-0.5+2)$$
$$A_2 = \frac{1.25}{1.5} = \frac{-5}{6}$$
$$\therefore \quad V_0(s) = \frac{10}{3}\frac{1}{(s+2)} - \frac{5}{6}\frac{1}{(s+0.5)}$$

Taking inverse LT, we get

$$v_0(t) = \left(\frac{10}{3}e^{-2t} - \frac{5}{6}e^{-0.5t}\right)u(t)$$

(b) Step Response of the System

$$\frac{V_0}{V_i}(s) = \frac{2.5s}{(s+2)(s+0.5)}$$

For unit step input, $V_i(s) = \frac{1}{s}$

$$\therefore \qquad V_0(s) = \frac{2.5s}{s(s+2)(s+0.5)}$$
$$= \frac{2.5}{(s+2)(s+0.5)}$$
$$= \frac{A_1}{(s+2)} + \frac{A_2}{(s+0.5)}$$
$$2.5 = A_1(s+0.5) + A_2(s+2)$$

Put $s = -2$

$$2.5 = A_1(-2+0.5)$$
$$A_1 = -\frac{2.5}{1.5} = \frac{-5}{3}$$

Put $s = -0.5$

$$2.5 = A_2(-0.5+2)$$
$$A_2 = \frac{2.5}{1.5} = \frac{5}{3}$$
$$\therefore \qquad V_0(s) = \frac{5}{3}\left(-\frac{1}{(s+2)} + \frac{1}{(s+0.5)}\right)$$

$$v_0(t) = \frac{5}{3}(-e^{-2t} + e^{-0.5t})u(t)$$

Note: For an impulse input $V_i(s) = 1$ and for a step input $V_i(s) = \frac{1}{s}$. By integrating the impulse response one can get the step response. Similarly by differentiating the step response, the impulse response can be obtained.

In the above example, consider the step response.

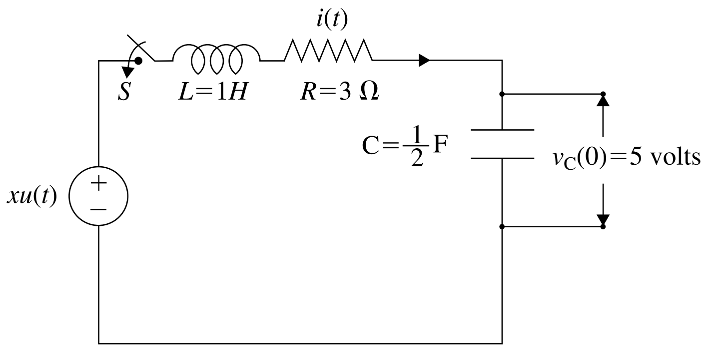


Fig. 4.33 R.L.C. series circuit with initial conditions

$$v_0(t) = \frac{5}{3}(-e^{-2t} + e^{-0.5t})u(t)$$

$$\frac{dv_0(t)}{dt} = \frac{5}{3}(2e^{-2t} - 0.5e^{-0.5t})u(t)$$

$$= \left(\frac{10}{3}e^{-2t} - \frac{5}{6}e^{-0.5t}\right)u(t)$$

The above response is nothing but the impulse response.

Example 4.65 Consider the R.L.C series circuit shown in Fig. 4.33. The circuit parameters are $R = 3$ ohm; $L = 1H$ and $C = \frac{1}{2}F$. The capacitor C is initially charged with a voltage of $v_c(0^-) = 5$ Volts. The initial current $i(0^-)$ before the input is applied is 2 amps. Find the current in the R-L-C circuit if the input is unit step. Also find the voltages across these elements for the above case.

Solution For the Circuit shown in Fig. 4.33, the loop equations is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i(t)dt = x(t)$$

Taking LT on both sides of the equation, we get the following transformation term by term

$$L\left[L\frac{di}{dt}\right] = LsI(s) - Li(0) = (sI(s) - 2)$$

$$L[Ri] = RI(s) = 3I(s)$$

$$L\left[\frac{1}{C}\int i(t)dt\right] = \frac{1}{Cs}I(s) + \frac{v_c(0^-)}{s}$$

$$= \frac{2I(s)}{s} + \frac{5}{s}$$

Thus, the differential equation after taking LT is written as

$$sI(s) - 2 + 3I(s) + \frac{2I(s)}{s} + \frac{5}{s} = X(s)$$

$$\left[s + 3 + \frac{2}{s}\right]I(s) = 2 - \frac{5}{s} + X(s)$$

$$\frac{(s^2 + 3s + 2)}{s}I(s) = \frac{2s - 5 + sX(s)}{s}$$

Step Response

For step input $X(s) = \frac{1}{s}$

$$I(s) = \frac{(2s - 5) + s\frac{1}{s}}{(s + 1)(s + 2)}$$

$$= \frac{(2s - 4)}{(s + 1)(s + 2)}$$

$$= \frac{A_1}{(s + 1)} + \frac{A_2}{(s + 2)}$$

$$(2s - 4) = A_1(s + 2) + A_2(s + 1)$$

Put $s = -1$

$$(-2 - 4) = A_1(-1 + 2)$$

$$A_1 = -6$$

Put $s = -2$

$$(-4 - 4) = A_2(-2 + 1)$$

$$A_2 = 8$$

$$I(s) = \frac{-6}{(s + 1)} + \frac{8}{s + 2}$$

Taking inverse LT, we get

$$i(t) = (-6e^{-t} + 8e^{-2t})u(t)$$

The voltage across the resistor is given by

$$\begin{aligned} v_R(t) &= i(t)R \\ &= 3i(t) \end{aligned}$$

$$v_R(t) = (-18e^{-t} + 24e^{-2t})u(t)$$

The voltage across the inductor is given by

$$\begin{aligned} v_L(t) &= L\frac{di(t)}{dt} \\ &= \frac{di(t)}{dt} \end{aligned}$$

$$v_L(t) = (6e^{-t} - 16e^{-2t})u(t)$$

The voltage across the capacitor is given by

$$\begin{aligned} v_c(t) &= \frac{1}{C}\int i(t)dt \\ &= 2\int(-6e^{-t} + 8e^{-2t})dt \\ &= 12e^{-t} - 8e^{-2t} + C. \end{aligned}$$

At $t = 0$, $v_c(0) = 5$

$$5 = 12 - 8 + C \qquad \text{or} \quad C = 1$$

$$v_c(t) = (12e^{-t} - 8e^{-2t} + 1)u(t)$$

Example 4.66 Consider the R-L-C circuit shown in Fig. 4.34 with the numerical values given. The initial current through the inductor and the initial voltage across the capacitor at $t = 0^+$ is zero. Derive an expression for the source current as a function of time for $t \geq 0$ when the switch S is closed.

Solution The expression for $i(t)$ is obtained by writing the algebraic equation rather than the integro differential equation when the initial conditions are zero.

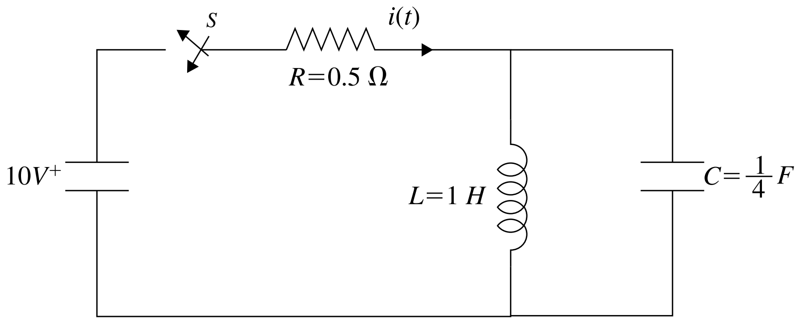


Fig. 4.34 R-L-C- circuit of Example 4.66

1. The impedance function for the inductor L is taken as $Z_1(s)$.

$$Z_1(s) = Ls$$
$$= s$$

2. The impedance function for the capacitor C is taken as $Z_2(s)$.

$$Z_2(s) = \frac{1}{Cs} = \frac{4}{s}$$

3. $Z_1(s)$ and $Z_2(s)$ are in parallel. Let $Z_3(s)$ be impedance of the parallel combination of $Z_1(s)$ and $Z_2(s)$. Thus,

$$Z_3(s) = \frac{Z_1(s)Z_2(s)}{Z_1(s) + Z_2(s)}$$
$$= \frac{\frac{4}{s}s}{\frac{4}{s} + s}$$
$$= \frac{4s}{s^2 + 4}$$

4. R and $Z_3(s)$ are in series. Let $Z(s)$ be the impedance of the series combination of R and $Z_3(s)$. Thus,

$$Z(s) = R + Z_3(s)$$
$$Z(s) = 0.5 + \frac{4s}{s^2 + 4}$$
$$Z(s) = \frac{(0.5s^2 + 4s + 2)}{s^2 + 4}$$

5.
$$I(s) = \frac{V(s)}{Z(s)}$$

For a step input $V(s) = \frac{V}{S} = \frac{10}{s}$

$$\begin{aligned} I(s) &= \frac{10}{s}\frac{(s^2+4)}{(0.5s^2+4s+2)} \\ &= \frac{20(s^2+4)}{s(s^2+8s+4)} \end{aligned}$$

But $(s^2 + 8s + 4) = (s + 7.464)(s + 0.536)$.

6. Putting $I(s)$ into partial fraction, we get

$$\begin{aligned} I(s) &= \frac{A_1}{s} + \frac{A_2}{(s+7.464)} + \frac{A_3}{(s+0.536)} \\ 20(s^2+4) &= A_1(s^2+8s+4) + A_2 s(s+0.536) + A_3 s(s+7.464) \end{aligned}$$

Put $s = 0$

$$\begin{aligned} 80 &= 4A_1 \\ A_1 &= 20 \end{aligned}$$

Put $s = -7.464$

$$\begin{aligned} (1114.23 + 80) &= A_2(-7.464)(-7.464 + 0.536) \\ A_2 &= 23.1 \end{aligned}$$

Put $s = -0.536$

$$\begin{aligned} (5.746 + 80) &= A_3(-0.536)(-0.536 + 7.464) \\ A_3 &= -23.1 \\ I(s) &= \frac{20}{s} + \frac{23.1}{s+7.464} - \frac{23.1}{s+0.536} \end{aligned}$$

7. Taking inverse LT, we get

$$i(t) = \left(20 + 23.1e^{-7.464t} - 23.1e^{-0.536t}\right) u(t)$$

Example 4.67 Find the unit step response of the circuit shown in Fig. 4.35.

(*Anna University, December, 2007*)

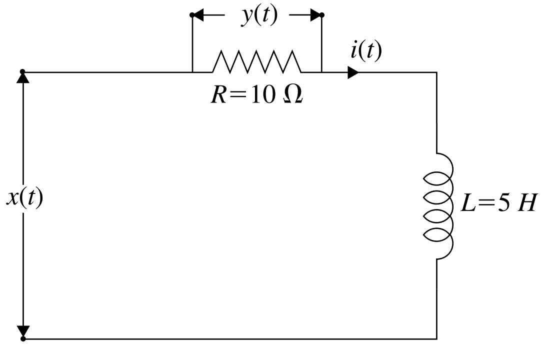


Fig. 4.35 R.L. series circuit

Solution

1. Since the initial condition is zero, the total impedance of the circuit is written as

$$Z(s) = R + Ls$$
$$= 10 + 5s$$

2. The current through the series circuit is

$$I(s) = \frac{X(s)}{Z(s)}$$

Since $X(s) = \frac{1}{s}$ for unit step

$$I(s) = \frac{1}{sZ(s)} = \frac{1}{s(10+5s)}$$

3. The output $Y(s) = I(s)R$

$$= \frac{10}{s(10+5s)} = \frac{2}{s(s+2)}$$
$$= \frac{1}{s} - \frac{1}{s+2}$$

4. Taking inverse LT, we get

$$i(t) = L^{-1}I(s) = (1 - e^{-2t})u(t)$$

Example 4.68 Consider the R-C-Circuit shown in Fig. 4.36. The input $x(t) = u(t) - u(t-2)$. Derive an expression for the voltage output across the capacitor C as a function of time when the switch S is closed at $t = 0$. Assume zero initial condition.

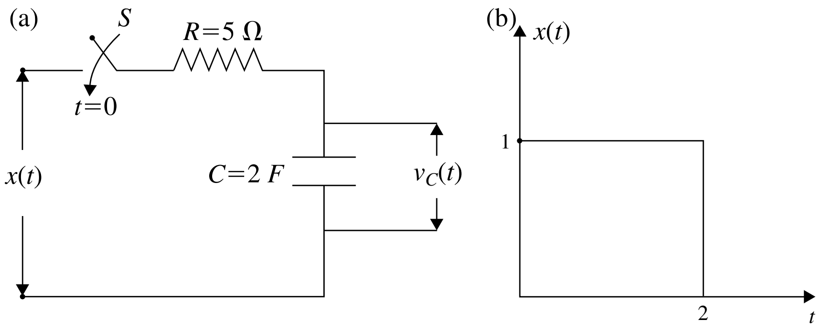


Fig. 4.36 **a** R.C. circuit and **b** $x(t) = u(t) - u(t-2)$

Solution

1.

$$x(t) = u(t) - u(t-2)$$
$$X(s) = \left[\frac{1}{s} - \frac{1}{s}e^{-2s}\right]$$

2. Since the initial condition is zero, the impedance of the circuit is written as

$$\begin{aligned} Z(s) &= R + \frac{1}{Cs} \\ &= 5 + \frac{1}{2s} \\ &= \frac{(10s+1)}{2s} \end{aligned}$$

3. The current in the circuit is

$$\begin{aligned} I(s) &= \frac{X(s)}{Z(s)} \\ &= \left[\frac{1}{s} - \frac{1}{s}e^{-2s}\right]\frac{2s}{(10s+1)} \\ &= (1 - e^{-2s})\frac{2}{(10s+1)} \end{aligned}$$

4. The impedance of the capacitor C is

$$\begin{aligned} Z_c(s) &= \frac{1}{Cs} \\ &= \frac{1}{2s} \end{aligned}$$

5. The output voltage across the capacitor C is given by

$$\begin{aligned} V_c(s) &= I(s)Z_c(s) \\ &= (1-e^{-2s})\frac{2}{(10s+1)}\frac{1}{2s} \\ &= \frac{(1-e^{-2s})0.1}{s(s+0.1)} \\ &= \frac{0.1}{s(s+0.1)} - \frac{0.1e^{-2s}}{s(s+0.1)} \end{aligned}$$

6.

$$\begin{aligned} v_c(t) &= L^{-1}V_c(s) \\ v_c(t) &= L^{-1}\left[\frac{0.1}{s(s+0.1}\right] - L^{-1}\left[\frac{0.1e^{-2s}}{s(s+0.1)}\right] \end{aligned}$$

Now consider $\frac{0.1}{s(s+0.1)}$ which can be expressed as $\frac{1}{s} - \frac{1}{s+0.1}$. Thus,

$$L^{-1}\left[\frac{1}{s} - \frac{1}{s+0.1}\right] = (1-e^{-0.1t})u(t)$$

Using shift theorem, $L^{-1}\left[\frac{0.1e^{-2s}}{s(s+0.1)}\right] = (1-e^{-0.1(t-2)})u(t-2)$. Thus,

$$v_c(t) = (1-e^{-0.1t})u(t) - (1-e^{-0.1(t-2)})u(t-2)$$

Example 4.69 Consider the R.L. series circuit shown in Fig. 4.37a. At $t = 0$, the switch S is closed. Derive an expression for the current in the series circuit as a function of time. The mathematical description of the input is given by

$$\begin{aligned} x(t) &= \frac{3}{2}t & 0 \le t \le 2 \\ &= \left(6 - \frac{3}{2}t\right) & 2 \le t \le 4 \end{aligned}$$

Solution

1. The mathematical description of $x(t)$ is represented as a triangular wave and is shown in Fig. 4.37b.
2. For Fig. 4.37b, the LT is determined (*see* Example 4.28) as

$$X(s) = \left[\frac{3}{2}\frac{1}{s^2} - \frac{3}{s^2}e^{-2s} + \frac{3}{2s^2}e^{-4s}\right]$$

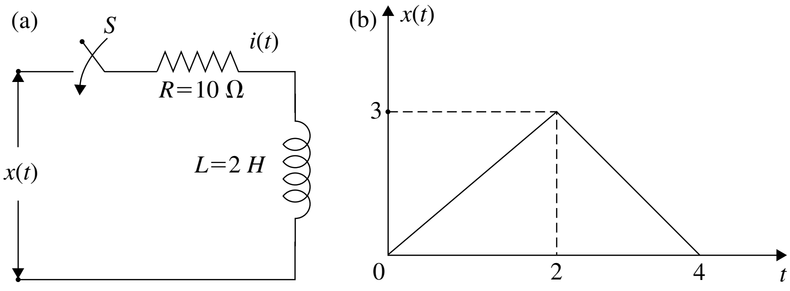


Fig. 4.37 **a** R-L-series circuit; **b** $x(t)$

3. For the circuit shown in Fig. 4.37a, the impedance function is

$$\begin{aligned} Z(s) &= (R + Ls) \\ &= (10 + 2s) \\ &= 2(s + 5) \end{aligned}$$

4. The current through the series circuit is

$$\begin{aligned} I(s) &= \frac{X(s)}{Z(s)} \\ &= \left[\frac{3}{2}\frac{1}{s^2} - \frac{3}{s^2}e^{-2s} + \frac{3}{2}\frac{1}{s^2}e^{-4s}\right]\frac{1}{2(s+5)} \end{aligned}$$

5.

$$\begin{aligned} i(t) &= L^{-1}I(s) \\ &= L^{-1}\left[\frac{3}{4}\frac{1}{s^2(s+5)} - \frac{3}{2s^2}\frac{e^{-2s}}{(s+5)} + \frac{3}{4s^2}\frac{e^{-4s}}{(s+5)}\right] \\ \frac{1}{s^2(s+5)} &= \frac{A_1}{s^2} + \frac{A_2}{s} + \frac{A_3}{(s+5)} \\ 1 &= A_1(s+5) + A_2 s(s+5) + A_3 s^2 \end{aligned}$$

Put $s = 0$

$$A_1 = \frac{1}{5}$$

Put $s = -5$

$$\begin{aligned} 1 &= A_3 25 \\ A_3 &= \frac{1}{25} \end{aligned}$$

Compare the coefficients of s^2.

$$A_2 + A_3 = 0$$
$$A_2 = -A_3$$
$$= -\frac{1}{25}$$

6.

$$L^{-1}\left[\frac{3}{4}\frac{1}{s^2(s+5)}\right] = \frac{3}{4}L^{-1}\left[\frac{1}{5s^2} - \frac{1}{25s} + \frac{1}{25(s+5)}\right] = \frac{3}{4}\left[\frac{1}{5}t - \frac{1}{25} + \frac{1}{25}e^{-5t}\right]u(t) \quad \text{(a)}$$

$$L^{-1}\left[-\frac{3}{2}\frac{e^{-2s}}{s^2(s+5)}\right] = -\frac{3}{2}\left[\frac{1}{5}(t-2) - \frac{1}{25} + \frac{1}{25}e^{-5(t-2)}\right]u(t-2) \quad \text{(b)}$$

$$L^{-1}\left[\frac{3}{4}\frac{e^{-4s}}{s^2(s+5)}\right] = \frac{3}{4}\left[\frac{1}{5}(t-4) - \frac{1}{25} + \frac{1}{25}e^{-5(t-4)}\right]u(t-4) \quad \text{(c)}$$

7.

$$i(t) = (a) + (b) + (c)$$
$$i = \left[\frac{3}{4}\left\{\frac{1}{5}t - \frac{1}{25} + \frac{1}{25}e^{-5t}\right\}u(t) - \frac{3}{2}\left\{\frac{1}{5}(t-2) - \frac{1}{25} + \frac{1}{25}e^{-5(t-2)}\right\}u(t-2) + \frac{3}{4}\left\{\frac{1}{5}(t-4) - \frac{1}{25} + \frac{1}{25}e^{-5(t-4)}\right\}u(t-4)\right]$$

Example 4.70 Consider the circuit shown in Fig. 4.38. Initially the switch is in position 1. At $t = 0$, the switch is moved to position 2. Find the expression for the current in the inductor L as a function of time t.

Solution

1. When the switch S is in position 1, the current i passing through the inductor at steady state is decided by the source voltage and resistance R_2.

$$i = \frac{40}{10} = 4 \text{ amp.}$$

2. When the switch S is in position 2, the following dynamic equation for the $R_1 - R_2 - L$ series circuit is written

$$L\frac{di}{dt} + (R_1 + R_2)i = 0$$

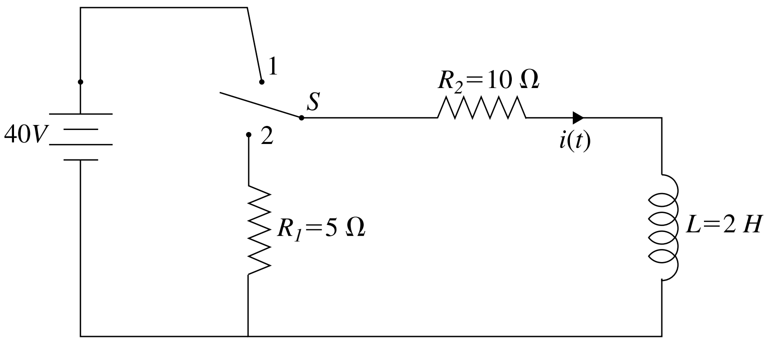


Fig. 4.38 Circuit of Example 4.70

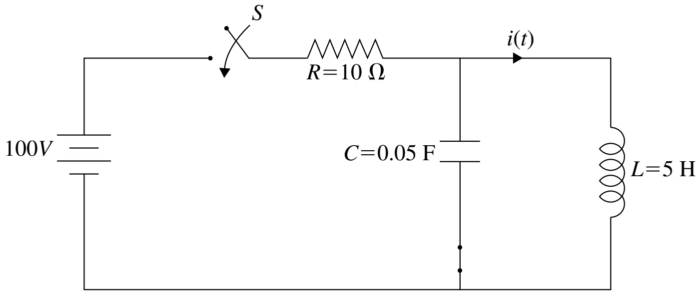


Fig. 4.39 Circuit for Example 4.71

3. Taking LT on both sides of the above equation, we get

$$sLI(s) - Li(0^+) + (R_1 + R_2)I(s) = 0$$

   Substituting the numerical values, we get

$$(2s + 15)I(s) = 2 \times 4 = 8$$
$$I(s) = \frac{8}{2s + 15}$$
$$= \frac{4}{s + 7.5}$$

4. Taking inverse LT, we get

$$i(t) = 4e^{-7.5t}u(t)$$

Example 4.71 Consider the circuit shown in Fig. 4.39. The switch S is initially closed. Derive an expression for the current through the inductor as a function of time when the switch S is suddenly opened at $t = 0$.

Solution

1. When the switch S is closed, the current is passing through R, and L and C is open circuited. Under this condition the initial current is limited by R only. Thus,

$$i(0) = \frac{100}{10} = 10 \text{ Amps.}$$

2. The initial charge across the capacitor is zero because the entire voltage is applied across R only. Therefore, the following Loop equation for L.C. circuit is written when the switch S is open.

$$L\frac{di}{dt} - Li(0) + \frac{1}{C}\int i(t)dt + v_c(0) = 0$$

3. Taking LT and substituting $i(0) = 10$ and $v_c(0) = 0$, we get

$$\begin{aligned}
\left[Ls + \frac{1}{Cs}\right]I(s) &= Li(0)\\
\left(5s + \frac{1}{0.05s}\right)I(s) &= 50\\
(5s^2 + 20)I(s) &= 50s\\
(s^2 + 4)I(s) &= 10s\\
(s + j2)(s - j2)I(s) &= 10s\\
I(s) &= \frac{A_1}{(s + j2)} + \frac{A_2}{(s - j2)}\\
10s &= A_1(s - j2) + A_2(s + j2)
\end{aligned}$$

Put $s = -j2$

$$\begin{aligned}
10(-j2) &= 4A_1(-j)\\
A_1 &= 5\\
A_2 &= A_1^* = 5\\
I(s) &= 5\left[\frac{1}{s + j2} + \frac{1}{s - j2}\right]
\end{aligned}$$

4. Taking inverse LT, we get

$$i(t) = 5[e^{-j2t} + e^{j2t}]$$

$$i(t) = 10\cos 2t$$

Example 4.72 Find the transfer function of LTI system described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 2\frac{dx(t)}{dt} - 3x(t)$$

(*Anna University, May, 2008*)

Solution Taking LT on both sides assuming zero initial conditions, we get

$$(s^2 + 3s + 2)Y(s) = (2s - 3)X(s)$$

The transfer function is $\frac{Y(s)}{X(s)}$

$$\frac{Y(s)}{X(s)} = \frac{(2s - 3)}{(s^2 + 3s + 2)}$$

Example 4.73 Consider an LTI system with input $x(t) = e^{-t}u(t)$ and impulse response
$h(t) = e^{-2t}u(t)$

- Determine the LT of $x(t)$ and $h(t)$.
- Using the convolution property, determine the LT $Y(s)$ of the output $y(t)$.
- From the LT of $y(t)$ as obtained in part (2) determine $y(t)$.
- Verify your result in part (2) by explicitly convolving $x(t)$ and $h(t)$.

(*Anna University, May, 2008*)

Solution

1.

$$x(t) = e^{-t}u(t)$$

From LT table

$$X(s) = \frac{1}{(s+1)} \qquad \text{ROC: Re}(s) > -1$$

$$h(t) = e^{-3t}u(t)$$

From LT table

$$H(s) = \frac{1}{(s+3)} \qquad \text{ROC: Re}(s) > -3$$

2.

$$Y(s) = X(s)H(s)$$

$$Y(s) = \frac{1}{(s+1)(s+3)}$$

3.

$$Y(s) = \frac{1}{(s+1)(s+3)}$$
$$= \frac{A_1}{(s+1)} + \frac{A_2}{(s+3)}$$
$$1 = A_1(s+3) + A_2(s+1)$$

Put $s = -1$

$$1 = A_1(-1+3)$$
$$A_1 = \frac{1}{2}$$

Put $s = -3$

$$1 = A_2(-3+2)$$
$$A_2 = -\frac{1}{2}$$
$$Y(s) = \frac{1}{2}\left[\frac{1}{s+1} - \frac{1}{s+3}\right]$$
$$y(t) = L^{-1}Y(s) = \frac{1}{2}[e^{-t} - e^{-3t}]u(t)$$

$$y(t) = \frac{1}{2}[e^{-t} - e^{-3t}]u(t)$$

4.

$$x(t) = e^{-t}$$
$$x(t-\tau) = e^{-(t-\tau)}$$
$$h(\tau) = e^{-3\tau}$$

Since $x(t)$ and $h(t)$ are casual, the limit of integration varies from 0 to t. Thus,

$$
\begin{aligned}
y(t) &= \int_0^t e^{-(t-\tau)} e^{-3\tau} d\tau \\
&= e^{-t} \int_0^t e^{-2\tau} d\tau \\
&= \frac{e^{-t}}{(-2)} \left[e^{-2\tau}\right]_0^t
\end{aligned}
$$

$$
y(t) = \frac{1}{2}\left[e^{-t} - e^{-3t}\right] u(t)
$$

Example 4.74 Determine the impulse response $h(t)$ of the system whose input-output is related by the differential equation where $x(t)$ is the input, $y(t)$ is the output

$$
\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)
$$

with all initial conditions to be zeros.

(*Anna University, April, 2004*)

Solution

1. Taking LT on both sides of the given differential equation, we get

$$
\begin{aligned}
(s^2 + 3s + 2)Y(s) &= X(s) \\
s^2 + 3s + 2 &= (s+1)(s+2)
\end{aligned}
$$

For an impulse,

$$
\begin{aligned}
X(s) &= 1 \\
Y(s) &= \frac{1}{(s+1)(s+2)}
\end{aligned}
$$

2. Putting into partial fraction, we get

$$
\begin{aligned}
Y(s) &= \frac{A_1}{(s+1)} + \frac{A_2}{(s+2)} \\
1 &= A_1(s+2) + A_2(s+1)
\end{aligned}
$$

Put $s = -1$

$$
\begin{aligned}
1 &= A_1(-1+2) \\
A_1 &= 1
\end{aligned}
$$

Put $s = -2$

$$1 = A_2(-2+1)$$
$$A_2 = -1$$
$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

3. Taking inverse LT, we get

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

For impulse input $y(t) = h(t)$

$$h(t) = (e^{-t} - e^{-2t})u(t)$$

Example 4.75 Determine the output response of the system whose impulse response

$$h(t) = e^{-at}u(t)$$

for the step input.

(*Anna University, April, 2004*)

Solution

1.
$$H(s) = L[h(t)] = L[e^{-at}u(t)]$$
$$= \frac{1}{s+a}$$
$$H(s) = \frac{Y(s)}{X(s)}$$

For step input $X(s) = \frac{1}{s}$.

2. Substituting in $H(s)$, we get

$$Y(s) = \frac{1}{s(s+a)}$$

The residues are obtained by intuition

$$Y(s) = \left[\frac{1}{s} - \frac{1}{s+a}\right]\frac{1}{a}$$

3. Taking inverse LT, we get

$$y(t) = \frac{1}{a}[1 - e^{-at}]u(t)$$

Example 4.76 Consider an LTI system whose response to the input $x(t) = (e^{-t} + e^{-3t})u(t)$ is $y(t) = (2e^{-t} - 2e^{-4t})u(t)$. Find the system's impulse response.

(*Anna University, December, 2007*)

Solution

1. The LT of $x(t)$ is $X(s)$

$$X(s) = L[e^{-t} + e^{-3t}] = \frac{1}{(s+1)} + \frac{1}{(s+3)} = \frac{2(s+2)}{(s+1)(s+3)}$$

The LT of $y(t)$ is $Y(s)$

$$Y(s) = L[2e^{-t} - 2e^{-4t}] = 2\left[\frac{1}{s+1} - \frac{1}{s+4}\right] = \frac{6}{(s+1)(s+4)}$$

2. The transfer function is

$$\begin{aligned} H(s) = \frac{Y(s)}{X(s)} &= \frac{6}{(s+1)(s+4)} \frac{(s+1)(s+3)}{2(s+2)} \\ &= \frac{3(s+3)}{(s+2)(s+4)} \end{aligned}$$

3. For an impulse $X(s) = 1$. Now $Y(s)$ can be put into partial fraction as given below.

$$\begin{aligned} Y(s) &= \frac{3(s+3)}{(s+2)(s+4)} \\ &= \frac{A_1}{(s+2)} + \frac{A_2}{(s+4)} \\ 3(s+3) &= A_1(s+4) + A_2(s+2) \end{aligned}$$

Put $s = -2$

$$\begin{aligned} 3(-2+3) &= A_1(-2+4) \\ A_1 &= \frac{3}{2} \end{aligned}$$

Put $s = -4$

$$\begin{aligned} 3(-4+3) &= A_2(-4+2) \\ A_2 &= \frac{3}{2} \\ Y(s) &= \frac{3}{2}\left(\frac{1}{s+2} + \frac{1}{s+4}\right) \end{aligned}$$

4. Taking inverse LT of $Y(s)$, we get

$$y(t) = L^{-1}Y(s)$$
$$= \frac{3}{2}L\left(\frac{1}{s+2} + \frac{1}{s+4}\right)$$

$$y(t) = \frac{3}{2}(e^{-2t} + e^{-4t})u(t)$$

Example 4.77 Determine the response of the system with impulse response $h(t) = u(t)$ for the input $x(t) = e^{-2t}u(t)$.

(*Anna University, April, 2004*)

Solution **Method 1:**

1. Taking LT for $h(t)$ and $x(t)$, we get

$$H(s) = L(u(t)) = \frac{1}{s}$$
$$X(s) = L[e^{-2t}u(t)] = \frac{1}{(s+2)}$$

2.

$$y(t) = x(t) * h(t)$$
$$Y(s) = X(s)H(s)$$
$$= \frac{1}{s(s+2)}$$

3. Putting into partial fraction and by intuition the residues are obtained. Thus, $Y(s)$ is written as

$$Y(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)$$

4. Taking Laplace inverse for $Y(s)$, we get $y(t)$

$$y(t) = L^{-1}Y(s)$$
$$= L^{-1}\frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)$$

$$y(t) = \frac{1}{2}(1 - e^{-2t})$$

Method 2: $y(t)$ can be derived by using Convolution Integral

1. Both $h(t)$ and $x(t)$ are casual. Hence, the following convolution integral is written for $y(t)$

$$\begin{aligned} y(t) &= \int_0^t h(\tau)x(t-\tau)d\tau \\ &= \int_0^t e^{-2(t-\tau)}d\tau \\ &= e^{-2t}\int_0^t e^{2\tau}d\tau \\ &= \frac{e^{-2t}}{2}\left[e^{2\tau}\right]_0^t \\ &= \frac{e^{-2t}}{2}[e^{2t}-1] \end{aligned}$$

$$y(t) = \frac{1}{2}[1-e^{-2t}]$$

Example 4.78 Find the output of an LTI system with impulse response $h(t) = \delta(t-3)$ for the input $x(t) = \cos 4t + \cos 7t$.

(*Anna University, April, 2004*)

Solution

$$\begin{aligned} h(t) &= \delta(t-3) \\ H(s) &= e^{-3s} \\ X(s) &= L[\cos 4t + \cos 7t] \\ Y(s) &= H(s)X(s) = L[\cos 4t + \cos 7t]e^{-3s} \end{aligned}$$

$$y(t) = \cos 4(t-3) + \cos 7(t-3)$$

Example 4.79 Find the initial and final values for

$$X(s) = \frac{(s+5)}{(s^2+3s+2)}$$

(*Anna University, June, 2007*)

Solution

1. Initial value of $x(0)$. According to initial value theorem

$$x(0) = \underset{s\to\infty}{Lt}\ sX(s)$$
$$= \underset{s\to\infty}{Lt}\ \frac{s^2+5s}{s^2+3s+2}$$
$$= \underset{s\to\infty}{Lt}\ \frac{1+\frac{5}{s}}{1+\frac{3}{s}+\frac{2}{s^2}}$$

$$x(0) = 1$$

2.
$$(s^2+3s+2) = (s+1)(s+2)$$

Here the poles are at $s = -1$ and $s = -2$ and are in LHP. No pole of $X(s)$ is in RHP. Hence, the application of initial value theorem is correct.

3. Final value of $x(\infty)$. According to final value theorem,

$$x(\infty) = \underset{s\to 0}{Lt}\ sX(s)$$
$$= \underset{s\to 0}{Lt}\ \frac{s^2+5s}{s^2+3s+2}$$

$$x(\infty) = 0$$

Example 4.80 Find the step response of the system whose impulse response is given as

$$h(t) = u(t+1) - u(t-1)$$

(*Anna University, June, 2007*)

Solution

1. By taking LT for $h(t)$, using time shifting property, we get

$$H(s) = L[h(t)]$$
$$= \frac{1}{s}e^{s} - \frac{1}{s}e^{-s}$$
$$= \frac{1}{s}[e^{s} - e^{-s}]$$

2.
$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s}[e^{s} - e^{-s}]$$

For step input $X(s) = \frac{1}{s}$

$$\begin{aligned} Y(s) &= \frac{1}{s^2}[e^{s} - e^{-s}] \\ &= Y_1(s)[e^{s} - e^{-s}] \end{aligned}$$

where
$$Y_1(s) = \frac{1}{s^2}$$

3.
$$\begin{aligned} y_1(t) &= L^{-1}Y_1(s) \\ &= L^{-1}\frac{1}{s^2} \\ &= t \end{aligned}$$

4.
$$y(t) = y_1(t)[u(t+1) - u(t-1)]$$

$$y(t) = (t+1)u(t+1) - (t-1)u(t-1)$$

Example 4.81 Find the response of the system whose impulse response is

$$\begin{aligned} h(t) &= e^{-3t}u(t) \\ x(t) &= u(t-3) - u(t-5) \end{aligned}$$

(*Anna University, June, 2007*)

Solution

1. The LT of $h(t)$ is

$$\begin{aligned} H(s) &= L[e^{-3t}u(t)] \\ &= \frac{1}{(s+3)} \end{aligned}$$

2. The LT of the input $x(t)$ is

$$\begin{aligned} X(s) &= L[u(t-3) - u(t-5)] \\ &= \frac{1}{s}[e^{-3s} - e^{-5s}] \end{aligned}$$

3.

$$H(s) = \frac{Y(s)}{X(s)}$$
$$Y(s) = H(s)X(s)$$
$$= \frac{1}{s(s+3)}[e^{-3s} - e^{-5s}]$$
$$= Y_1(s)[e^{-3s} - e^{-5s}]$$

where $Y_1(s) = \dfrac{1}{s(s+3)}$.

4. Now $Y_1(s)$ can be put into partial fraction as

$$Y_1(s) = \frac{1}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right]$$
$$y_1(t) = L^{-1}Y_1(s)$$
$$= \frac{1}{3}[1 - e^{-3t}]$$

5. The response $y(t)$ is obtained from $y_1(t)$ and applying time shifting property

$$y(t) = \frac{1}{3}[1 - e^{-3(t-3)}]u(t-3) - \frac{1}{3}[1 - e^{-3(t-5)}]u(t-5)$$

$$y(t) = \frac{1}{3}[1 - e^{-3(t-3)}]u(t-3) - \frac{1}{3}[1 - e^{-3(t-5)}]u(t-5)$$

Example 4.82 Draw the wave forms $\delta(t-2)$ and $u(t+2)$.

Solution

1. The unit sample is shown in Fig. 4.40a. The time delayed signal (right-shifted by $t = 2$) is shown by its side.
2. The unit step signal is shown in Fig. 4.40b. The unit step signal is left shifted by $t = -2$ and is shown in the figure shown by its side.

Example 4.83 A system has the transfer function

$$H(s) = \frac{(3s-1)}{(s+3)(s-2)}$$

Find the impulse response assuming the system is stable and causal.

(*Anna University, December, 2007*)

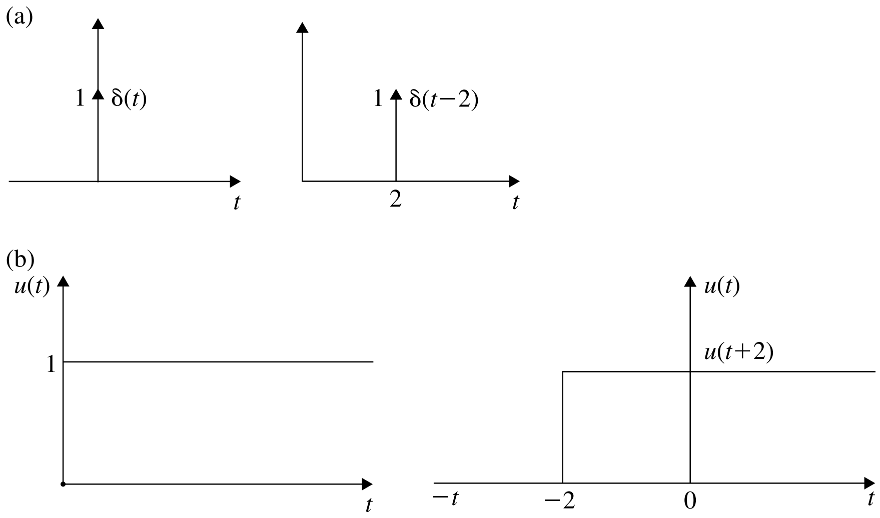


Fig. 4.40 Time shifted unit sample and unit step

Solution

1.

$$
\begin{aligned}
H(s) &= \frac{(3s-1)}{(s+3)(s-2)} \\
&= \frac{A_1}{(s+3)} + \frac{A_2}{(s-2)} \\
(3s-1) &= A_1(s-2) + A_2(s+3)
\end{aligned}
$$

Put $s = -3$

$$
\begin{aligned}
(-9-1) &= A_1(-3-2) \\
A_1 &= 2
\end{aligned}
$$

Put $s = 2$

$$
\begin{aligned}
(6-1) &= A_2(2+3) \\
A_2 &= 1 \\
H(s) &= \frac{2}{(s+3)} + \frac{1}{(s-2)}
\end{aligned}
$$

2. The poles of $H(s)$ are at $s = 2$ and $s = -3$. If the system is stable, the pole at $s = 2$ contributes to the left-sided term to the impulse response and the pole at $s = -3$ contributes right-sided term. Thus, we have

$$h(t) = 2e^{-3t}u(t) - e^{2t}u(-t)$$

3. If the system is causal, then both the poles should contribute right-sided term to the impulse response which is obtained as

$$h(t) = [2e^{-3t} + e^{2t}]u(t)$$

Due to $e^{2t}u(t)$ the system is not stable.
4. Hence the given system cannot be both stable and causal due to the pole at $s = 2$.

4.12 Connection between Laplace Transform and Fourier Transform

The bilateral LT of a signal $x(t)$ as defined earlier is written as follows:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{4.64}$$

Substituting $s = j\omega$ in the above equation, we get

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{4.65}$$

Thus, the FT is a special case of LT which is obtained by putting $X(s)|_{s=j\omega}$ with the following constraints:

- $x(t)$ is absolutely integrable.
- ROC of $X(s)$ includes the $j\omega$ axis.

Many commonly used signals have $x(t) = 0$ for $t \leq 0$ and ROC of the LT includes the $j\omega$ axis. Under this condition,

$$X(j\omega) = X(s)|_{s=j\omega}$$

Consider the following signals

$$x(t) = e^{-2t}u(t)$$
$$X(s) = \frac{1}{(s+2)} \qquad \text{ROC: Re}(s) > -2$$

Put $s = j\omega$

$$X(j\omega) = \frac{1}{j\omega + 2}.$$

Now by FT method, we get

$$X(j\omega) = \int_0^{\infty} e^{-2t} e^{-j\omega t} dt$$
$$= \frac{1}{(j\omega + 2)}$$

In the above case ROC includes the $j\omega$ axis.

Now consider the step function $u(t)$. The LT of a step function is

$$L[u(t)] = \frac{1}{s}.$$

But the FT of $u(t)$ is obtained as

$$F[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

Thus, the FT of $u(t)$ cannot be obtained from its LT as it is not absolutely integrable.

4.13 Causality of Continuous-Time Invariant System

A linear time invariant continuous time system is said to be causal iff the impulse response $h(t)$ of the system is zero for $t < 0$. Thus, the system which possesses right-sided impulse response is said to be causal. For this, the ROC of the system transfer function $H(s)$ which is rational, should be in the right half plane and to the right of the right most pole.

Consider the following impulse response function

$$h(t) = e^{-2t} u(t)$$
$$H(s) = \frac{1}{(s+2)} \qquad \text{ROC: Re}(s) > -2$$

The above transfer function is rational because the degree of the denominator polynomial is greater than the degree of the numerator polynomial. The ROC is to the right of the right most pole $s = -2$. Hence, the system is causal. The ROC is shown in Fig. 4.41a. Now consider the following impulse response function

$$h(t) = e^{-|t|}$$

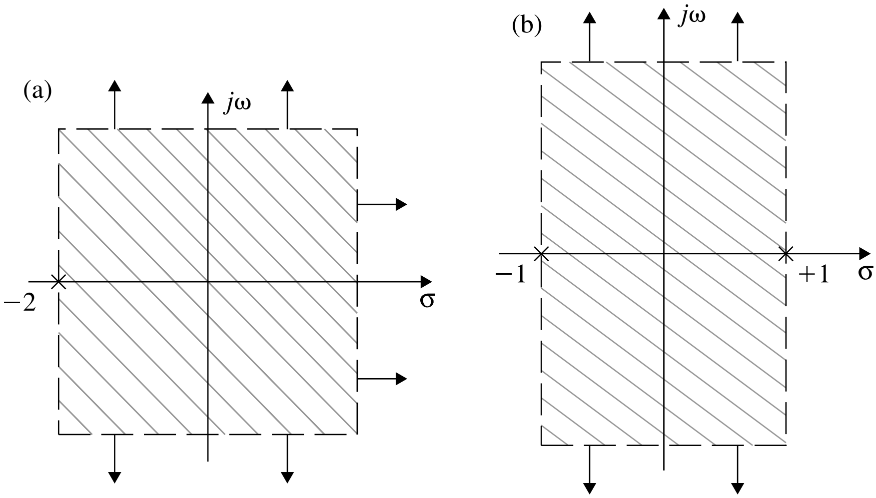


Fig. 4.41 **a** ROC of $h(t) = e^{-2t}$ (causal); **b** ROC of $h(t) = e^{-|t|}$ (non-causal)

The above function can be written as

$$\begin{aligned} h(t) &= e^{-t} \qquad t \geq 0 \\ &= e^{t} \qquad t \leq 0 \\ H(s) &= \int_{-\infty}^{0} e^{t} e^{-st} dt + \int_{0}^{\infty} e^{-t} e^{-st} dt \\ &= -\frac{1}{(s-1)} + \frac{1}{(s+1)} = \frac{-2}{(s-1)(s+1)} \end{aligned}$$

The transfer function is rational. The ROC is shown in Fig. 4.41b. The right most pole is at $s = 1$. The ROC is not to the right of the right most pole. Hence, the system is not causal.

4.14 Stability of Linear Time Invariant Continuous System

As already derived a linear time invariant system is said to be stable if the area under the impulse response $h(t)$ curve is finite (absolutely integrable). The impulse response of a causal system is absolutely integrable if the response curve decays exponentially as time increases. Consider the transfer function of an LTIC system.

$$H(s) = \frac{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_0}{a_n s^m + a_{m-1}s^{m-1} + \cdots + a_0}$$

For a rational function $H(s)$, $m > n$. The above transfer function can be written in terms of factors.

$$H(s) = \frac{A_1}{(s+p_1)} + \frac{A_2}{(s+p_2)} + \cdots + \frac{A_n}{(s+p_m)}$$

The impulse response of $H(s)$ is obtained by taking inverse LT.

$$h(t) = L^{-1}H(s) = A_1 e^{-p_1 t} + A_2 e^{-p_2 t} + \cdots + A_m e^{-p_m t}$$

For $h(t)$ to be absolutely integrable, the following conditions are to be satisfied.

- All the poles of $H(s)$ should lie in the left half of the s-plane.
- No repeated pole should be in the imaginary axis. Under these conditions, the system is said to be stable.
- The stability is also assessed by ROC. The ROC of $H(s)$ should include $j\omega$ axis.

Example 4.84 A Certain causal linear time invariant system has the following transfer function. Test whether the system is stable.

$$(a) \qquad H(s) = \frac{(s-4)}{(s+2)(s-1)}$$

$$(b) \qquad H(s) = \frac{(s-4)}{s^2(s+1)}$$

$$(c) \qquad H(s) = \frac{(s-4)}{s(s+1)(s+4)}$$

$$(d) \qquad H(s) = \frac{(s-4)}{(s-3)(s+4)} \qquad \text{ROC: } -4 < \text{Re}(s) < 3$$

Solution (a) Since the system is causal, the pole $s = 1$ which lies in RHP makes the system unstable.

(b) There are two poles repeated at the origin. The system is unstable.

(c) All the poles are in LHP. The system is stable. It is to be noted that the locations of zeros do not have any influence on the system stability.

(d) This is a non-causal system. ROC strip is enclosing the $j\omega$ axis. Hence, the system is stable.

4.15 The Bilateral Laplace Transform

The unilateral LT is applicable for causal signals and/or systems. However, for non-causal signals and systems, the LT pair is defined as follows:

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{4.66}$$

$$L^{-1}[X(s)] = x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+\infty} X(s)e^{st}ds \tag{4.67}$$

It is to be noted here that the unilateral LT pair defined earlier is the special case of bilateral LT.

4.15.1 Representation of Causal and Anti-causal Signals

The signal $x(t)$ shown in Fig. 4.42a is a non-causal signal which has two components. $x(t)$ can be split up into two components as $x(t) = x_1(t) + x_2(t)$. The signal $x_1(t)$ is a causal signal (positive time) and is also called as right-sided signal. This is shown in Fig. 4.42b. The signal $x_2(t)$ is called non-causal or anti-causal (negative time) signal. It is also called left-sided signal. $x_2(t)$ is shown in Fig. 4.42c. These signals are given the following mathematical description.

$$x_1(t) = x(t)u(t) \qquad -0 < t < \infty \tag{4.68}$$

$$x_2(t) = x(t)u(-t) \qquad -\infty < t < -0 \tag{4.69}$$

The LT of $x_1(t)$, the causal component is

$$X_1(s) = L[x_1(t)] = \int_{0^-}^{\infty} x_1(t)e^{-st}dt \tag{4.70}$$

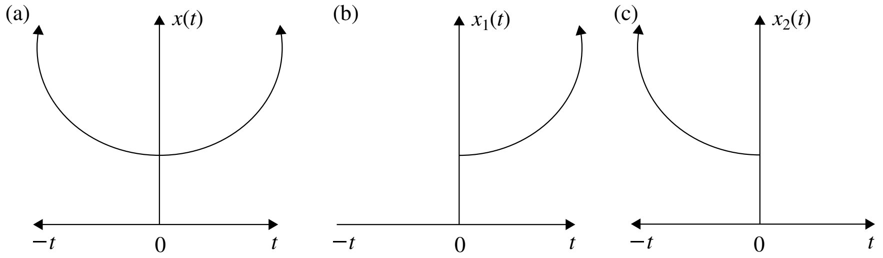


Fig. 4.42 **a** Signal $x(t)$; **b** Causal signal; **c** Anti-causal signal

The LT of $x_2(t)$, the non-causal component is

$$X_2(s) = L[x_2(t)] = \int_{-\infty}^{0^-} x_2(t)e^{-st}dt \tag{4.71}$$

It is to be noted that if $x(t)$ has any impulse or its derivatives at the origin they should be included in the causal signal $x_1(t)$and $x_2(t) = 0$ at the origin.

4.15.2 ROC of Bilateral Laplace Transform

Consider the following signal

$$\begin{aligned} x(t) &= e^{-2t}u(t) + e^{3t}u(-t) \\ x_1(t) &= e^{-2t}u(t) \\ X_1(s) &= \frac{1}{(s+2)} \qquad \text{ROC: Re } s > -2 \\ x_2(t) &= e^{3t}u(-t) \end{aligned}$$

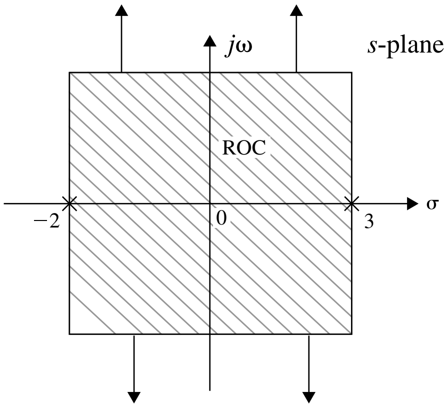
$$\begin{aligned} X_2(s) &= \int_{-\infty}^{-0} e^{3t}e^{-st}dt \\ &= \int_{-\infty}^{-0} e^{-(s-3)t}dt \\ &= -\frac{1}{(s-3)}\left[e^{-(s-3)t}\right]_{-\infty}^{0^-} \\ &= -\frac{1}{(s-3)}[-e^{-(s-3)(-\infty)} + 1] \end{aligned}$$

$e^{-(s-3)(-\infty)}$ converges iff $(s-3) < 0$ or $s < 3$. Hence, the ROC of the left-sided (anti-causal signal) is to the left of the pole at $s = 3$

$$\begin{aligned} X_2(s) &= -\frac{1}{(s-3)} \qquad \text{ROC: Re } s < 3 \\ \therefore \quad X(s) &= X_1(s) + X_2(s) \\ X(s) &= \frac{1}{(s+2)} - \frac{1}{s-3} \qquad \text{ROC: } -2 < \text{Re } s < 3 \end{aligned}$$

Unless the ROC is mentioned, the inverse LT is not unique. In the above case the ROC is a strip between $-2 < \text{Re } s < 3$ and is shown in Fig. 4.43.

Fig. 4.43 ROC of $x(t)$



Example 4.85 Consider the following function:

$$X(s) = \frac{10}{(s+4)(s-2)}$$

Find $x(t)$ if the ROC is (a) Re $s > 2$; (b) Re $s < -4$; (c) $-4 <$ Re $s > 2$.

Solution

$$\begin{aligned} X(s) &= \frac{10}{(s+4)(s-2)} \\ &= \frac{A_1}{(s+4)} + \frac{A_2}{s-2} \\ 10 &= A_1(s-2) + A_2(s+4) \end{aligned}$$

Put $s = -4$

$$\begin{aligned} 10 &= A_1(-4-2) \\ A_1 &= -\frac{5}{3} \end{aligned}$$

Put $s = 2$

$$\begin{aligned} 10 &= A_2(2+4) \\ A_2 &= \frac{5}{3} \\ X(s) &= \frac{5}{3}\left[\frac{-1}{s+4} + \frac{1}{s-2}\right] \end{aligned}$$

(a) ROC > 2.

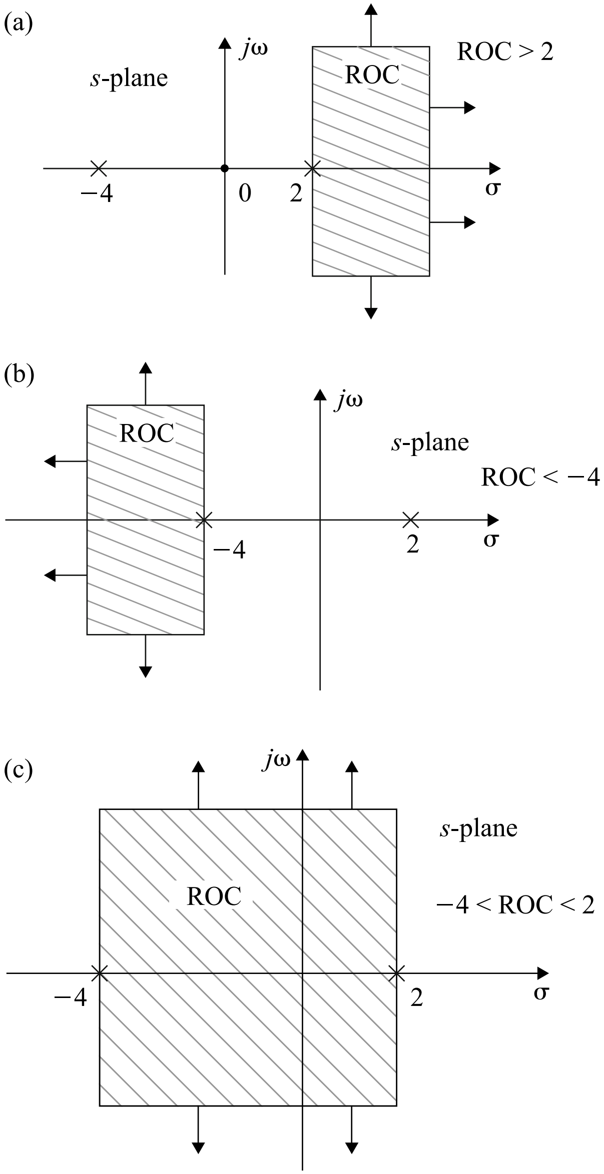


Fig. 4.44 ROCs of $X(s) = \dfrac{10}{(s+4)(s-2)}$. Example 4.85

Figure 4.44a represents pole-zero locations for ROC $>$ 2. Figure 4.44b represents pole-zero locations for ROC < -4. Figure 4.44c represents pole-zero locations for $-4 <$ ROC < 2.

Here the ROC is right-sided for both the poles at $s = -4$ and $s = 2$. Hence, the system is causal (Fig. 4.44).

$$x(t) = \frac{5}{3}[-e^{-4t} + e^{2t}]u(t)$$

(b) ROC Re $s < -4$.
Here the system poles $s = -4$ and $s = 2$ are both left-sided since they lie left to the ROC. Both are non-causal.

$$X(s) = \frac{5}{3}\left[\frac{-1}{(s+4)} + \frac{1}{s-2}\right]$$

$$x(t) = \frac{5}{3}(e^{-4t} - e^{2t})u(-t)$$

(c) ROC $-4 <$ Re $s < 2$.

Here the pole $s = -4$ is to the left of the ROC and it is a right-sided signal. It is therefore causal. The pole $s = 2$ is to the right of the ROC and hence it is a left-sided signal. It is non-causal. Hence

$$x(t) = \frac{5}{3}\left[-e^{-4t}u(t) - e^{2t}u(-t)\right]$$

Example 4.86 The impulse response function of a certain system is

$$H(s) = \frac{10}{s-5} \qquad \text{ROC: Re } s < 5$$

The system is excited by $x(t) = e^{-3t}u(t)$. Derive an expression for the output $y(t)$ as a function of time.

Solution

$$H(s) = \frac{10}{(s-5)} \qquad \text{ROC: Re } s < 5$$

$$X(s) = L^{-1}[e^{-3t}u(t)] = \frac{1}{(s+3)} \qquad \text{ROC: Re } s > -3$$

$$Y(s) = H(s)X(s) = \frac{10}{(s-5)(s+3)} \qquad \text{ROC: } -3 < \text{Re } s < 5$$

Putting into partial fraction, we get

$$Y(s) = \frac{A_1}{s-5} + \frac{A_2}{s+3}$$

$$10 = A_1(s+3) + A_2(s-5)$$

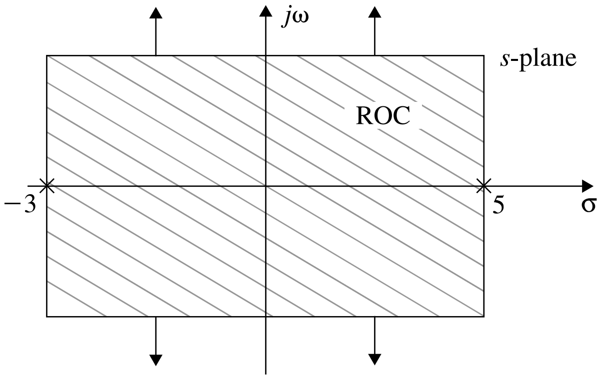


Fig. 4.45 ROC of $Y(s) = \dfrac{10}{(s-5)(s+3)}$

Put $s = 5$

$$10 = A_1(5+3)$$
$$A_1 = \frac{5}{4}$$

Put $s = -3$

$$10 = A_2(-3-5)$$
$$A_2 = -\frac{5}{4}$$

Hence, $Y(s) = \frac{5}{4}\left(\frac{1}{s-5} - \frac{1}{s+3}\right)$.

The ROC is shown in Fig. 4.45. From the ROC, the pole $\frac{1}{(s-5)}$ is left-sided (right to the ROC) and the pole $\frac{1}{(s+3)}$ is right-sided (left to the ROC). Hence, $\frac{1}{(s-5)}$ is non-causal and $\frac{1}{(s+3)}$ is causal. $y(t)$ is obtained by taking inverse LT.

$$y(t) = \frac{5}{4}\left(-e^{5t}u(-t) - e^{-3t}u(t)\right)$$

Example 4.87 The impulse response function of a certain system is given by

$$H(s) = \frac{1}{(s+10)} \qquad \text{ROC: Re } s > -10$$

The system is excited by the following input.

$$x(t) = -2e^{-2t}u(-t) - 3e^{-3t}u(t)$$

Derive an expression for the output $y(t)$ as a function of time.

Solution By taking LT for $x(t)$, we get

$$\begin{aligned}
X(s) &= L[-2e^{-2t}u(-t) - 3e^{-3t}u(t)] \\
&= \frac{2}{(s+2)} - \frac{3}{(s+3)} \qquad \text{ROC: } -3 < \text{Re } s < -2 \\
&= \frac{2s+6-3s-6}{(s+2)(s+3)} \\
&= \frac{-s}{(s+2)(s+3)}
\end{aligned}$$

$$\begin{aligned}
H(s) &= \frac{1}{(s+10)} \qquad \text{ROC: Re } s > -10 \\
Y(s) &= \frac{-s}{(s+2)(s+3)(s+10)} \qquad \text{ROC: } -3 < \text{Re } s < -2
\end{aligned}$$

The ROC of $Y(s)$ is shown in Fig. 4.46. The ROC of $H(s)$ is automatically satisfied if ROC Re $s > -3$. Putting $Y(s)$ into partial fraction, we get

$$\begin{aligned}
Y(s) &= \frac{A_1}{(s+2)} + \frac{A_2}{(s+3)} + \frac{A_3}{(s+10)} \\
-s &= A_1(s+3)(s+10) + A_2(s+2)(s+10) + A_3(s+2)(s+3)
\end{aligned}$$

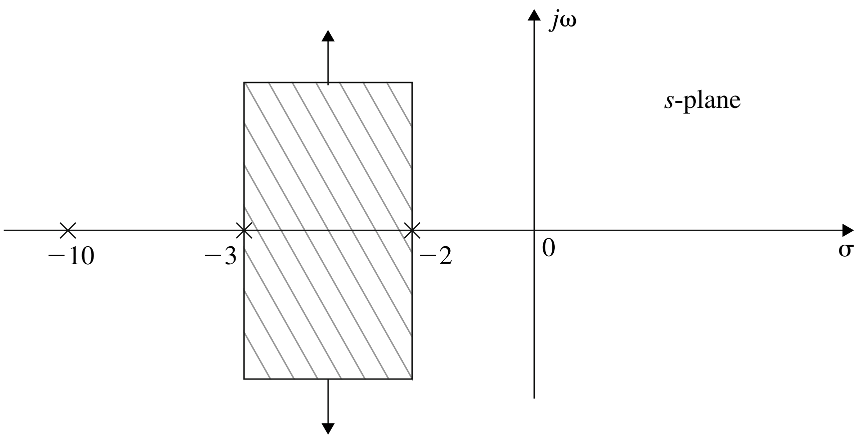


Fig. 4.46 ROC of Example 4.87

Put $s = -2$

$$\begin{aligned} 2 &= A_1(-2+3)(-2+10) \\ A_1 &= \frac{1}{4} \end{aligned}$$

Put $s = -3$

$$\begin{aligned} 3 &= A_2(-3+2)(-3+10) \\ A_2 &= -\frac{3}{7} \end{aligned}$$

Put $s = -10$

$$\begin{aligned} 10 &= A_3(-10+2)(-10+3) \\ A_3 &= \frac{5}{28} \end{aligned}$$

Hence

$$Y(s) = \frac{1}{4}\frac{1}{(s+2)} - \frac{3}{7}\frac{1}{(s+3)} + \frac{5}{28}\frac{1}{(s+10)}$$

From Fig. 4.46 it is evident that the pole $\frac{1}{(s+10)}$ of the system and the pole $\frac{1}{(s+3)}$ of the input are right-sided (to the left of ROC) and hence causal. On the other hand, the pole $\frac{1}{(s+2)}$ is left-sided (right to the ROC) and hence non-causal. Thus, $y(t)$ is obtained by taking inverse LT.

$$y(t) = -\frac{1}{4}e^{-2t}u(-t) - \frac{3}{7}e^{-3t}u(t) + \frac{5}{28}e^{-10t}u(t)$$

Example 4.88 The impulse response of a certain system is given by $h(t) = \delta(t) + e^{-3|t|}$. The system is excited by the following signal $x(t) = e^{-4t}u(t) + e^{-2t}u(-t)$. Find the response of the system $y(t)$.

Solution

$$\begin{aligned} H(s) &= L[h(t)] \\ &= L(\delta(t)) + L(e^{-3|t|}) \\ &= 1 + L(e^{-3|t|}) \\ L(e^{-3|t|}) &= \int_{-\infty}^{0^-} e^{+3t}e^{-st}dt + \int_{0^-}^{\infty} e^{-3t}e^{-st}dt \\ &= -\frac{1}{(s-3)} + \frac{1}{s+3} \end{aligned}$$

$$H(s) = 1 - \frac{1}{(s-3)} + \frac{1}{(s+3)} \qquad \text{ROC: } -3 < \text{Re}\, s < 3$$
$$= \frac{(s^2-15)}{(s-3)(s+3)}$$
$$X(s) = L[e^{-4t}u(t) + e^{-2t}u(-t)] = \frac{1}{(s+4)} - \frac{1}{s+2}$$
$$= \frac{-2}{(s+2)(s+4)} \qquad \text{ROC: } -4 < \text{Re}\, s < -2$$
$$H(s) = \frac{Y(s)}{X(s)}$$
$$Y(s) = H(s)X(s)$$
$$= \frac{(s^2-15)(-2)}{(s-3)(s+3)(s+2)(s+4)}$$
$$= \frac{A_1}{s-3} + \frac{A_2}{s+3} + \frac{A_3}{s+2} + \frac{A_4}{(s+4)}$$

$$(15-s^2)2 = A_1(s+3)(s+2)(s+4) + A_2(s-3)(s+2)(s+4) + A_3(s-3)(s+3)(s+4) + A_4(s-3)(s+3)(s+2)$$

Put $s = 3$

$$2(15-9) = A_1(6)(5)(7)$$
$$A_1 = \frac{2}{35}$$

Put $s = -3$

$$2(15-9) = A_2(-6)(-1)(1)$$
$$A_2 = 2$$

Put $s = -2$

$$2(15-4) = A_3(-5)(1)(2)$$
$$A_3 = -\frac{11}{5}$$

Put $s = -4$

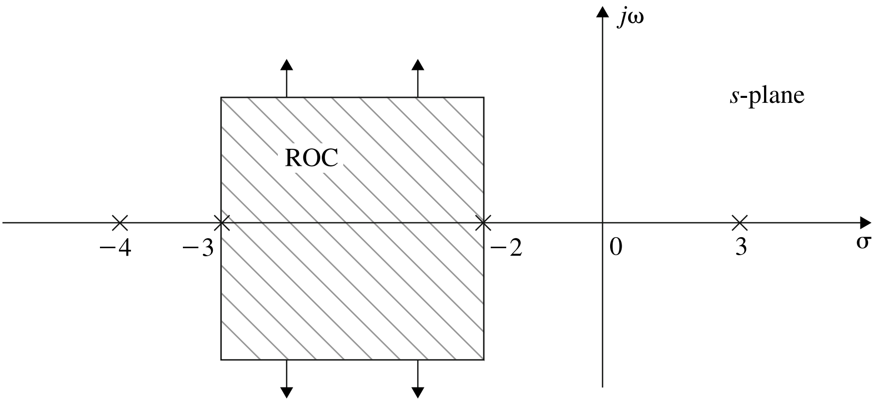


Fig. 4.47 ROC of Example 4.88

$$
\begin{aligned}
2(15-16) &= A_4(-7)(-1)(-2) \\
A_4 &= \frac{1}{7} \\
Y(s) &= \frac{2}{35}\frac{1}{s-3} + \frac{2}{s+3} - \frac{11}{5}\frac{1}{(s+2)} + \frac{1}{7}\frac{1}{(s+4)}
\end{aligned}
$$

The ROC for $Y(s)$ is shown in Fig. 4.47. From Fig. 4.47 the poles $\frac{1}{(s+4)}$ and $\frac{1}{(s+3)}$ are right-sided and hence causal. However, the poles $\frac{1}{(s+2)}$ and $\frac{1}{(s-3)}$ are left-sided and hence non-causal. Taking the ROC into account $y(t)$ is obtained as given below:

$$
y(t) = \left[\left(2e^{-3t} + \frac{1}{7}e^{-4t}\right)u(t) + \left(-\frac{2}{35}e^{3t} + \frac{11}{5}e^{-2t}\right)u(-t)\right]
$$

Example 4.89 Consider the R.L.C. series circuit shown in Fig. 4.48a. The excitation voltage $x(t) = e^{-3t}u(t) + e^{4t}u(-t)$. Derive the expression for the current in the series circuit. Assume zero initial conditions.

Solution

1. The impedance of the R.L.C. circuit is,

$$
\begin{aligned}
Z(s) &= R + Ls + \frac{1}{Cs} \\
&= 3 + s + \frac{2}{s} \\
&= \frac{(s^2+3s+2)}{s} = \frac{(s+1)(s+2)}{s}
\end{aligned}
$$

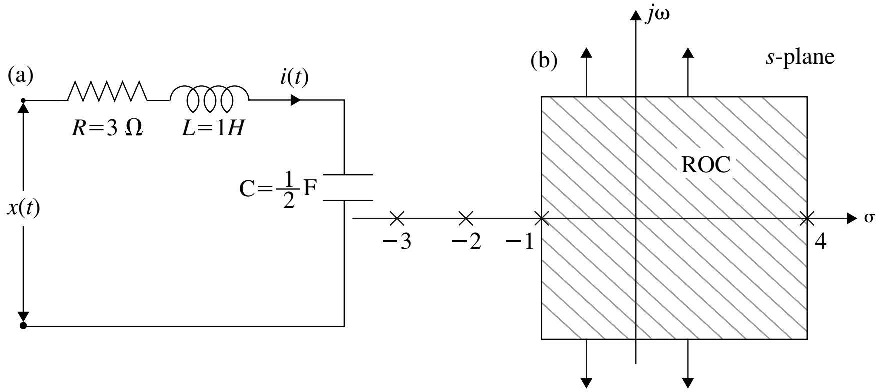


Fig. 4.48 **a** R.L.C circuit; **b** ROC of Example 4.89

2. The excitation voltage

$$x(t) = e^{-3t}u(t) + e^{4t}u(-t)$$
$$X(s) = \frac{1}{(s+3)} - \frac{1}{(s-4)}$$
$$= \frac{-7}{(s+3)(s-4)} \qquad \text{ROC:} -3 < \text{Re } s < 4$$

3. The current flowing in the circuit is

$$I(s) = \frac{X(s)}{Z(s)}$$
$$I(s) = \frac{-7s}{(s+1)(s+2)(s+3)(s-4)}$$

The corresponding ROC: $-1 < \text{Re } s < 4$. The above ROC satisfies the previous ROC also.

$$I(s) = \frac{A_1}{(s+1)} + \frac{A_2}{(s+2)} + \frac{A_3}{(s+3)} + \frac{A_4}{(s-4)}$$
$$-7s = A_1(s+2)(s+3)(s-4) + A_2(s+1)(s+3)(s-4)$$
$$+A_3(s+1)(s+2)(s-4) + A_4(s+1)(s+2)(s+3)$$

Put $s = -1$

$$7 = A_1(-1+2)(-1+3)(-1-4)$$
$$A_1 = -\frac{7}{10}$$

Put $s = -2$

$$14 = A_2(-1)(1)(-6)$$
$$A_2 = \frac{7}{3}$$

Put $s = -3$

$$21 = A_3(-2)(-1)(-7)$$
$$A_3 = -\frac{3}{2}$$

Put $s = 4$

$$-28 = A_4(5)(6)(7)$$
$$A_4 = -\frac{2}{15}$$
$$I(s) = \frac{-7}{10}\frac{1}{(s+1)} + \frac{7}{3}\frac{1}{s+2} - \frac{3}{2}\frac{1}{(s+3)} - \frac{2}{15}\frac{1}{(s-4)}$$

The poles $\frac{1}{s+1}$, $\frac{1}{s+2}$ and $\frac{1}{s+3}$ are right-sided as seen in ROC of Fig. 4.48b. The pole $\frac{1}{s-4}$ is left-sided and hence non-causal. Taking inverse LT for $I(s)$, we get

$$i(t) = \left(\frac{-7}{10}e^{-t} + \frac{7}{3}e^{-2t} - \frac{3}{2}e^{-3t}\right)u(t) + \frac{2}{15}e^{4t}u(-t)$$

Summary

1. The LT is a tool to represent any arbitrary signal $x(t)$ in terms of exponential components.
2. The LT is defined as follows:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

The Laplace inverse transform which converts $X(s)$ into $x(t)$ is expressed as

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

The above two equations are called LT pair.

3. Fourier transform is a special case of LT. Fourier transform is obtained by substituting $s = j\omega$ in LT in many practical cases even though it is not true always.

4. The LT of a causal signal and system is called unilateral LT. The LT of non-causal signal and system is called bilateral LT.
5. The region in the complex s-plane where the LT converges is called the region of convergence which is written in abbreviated form as ROC. For a causal signal the ROC exists to the right of the right most pole of the transfer function. For a non-causal signal the ROC exists to the left of the left most pole of the transfer function. The ROC will not enclose any pole.
6. The unilateral LT is a special case of bilateral LT. Their properties are discussed in details.
7. The inverse LT is conveniently obtained using partial fraction method. Analytical as well as graphical methods are used to determine the residues in the partial fraction.
8. The integro differential equation of LTIC system can be converted into algebraic equations using LT and the solution is obtained with case.
9. By knowing the transfer function using LT one can easily obtain impulse response and step response. Using LT, it is also possible to get zero state response, zero input response, natural response, forced response and total response of the system.
10. The solutions of differential and integro-differential equations are obtained using LT. The initial conditions are applied for zero input. The differential equation can also be solved using classical method. However, in classical method, the zero initial conditions are applied for the total response. The classical method is restricted to a certain class of input and not applicable to any input. In classical method, the total response is expressed in terms of natural response and forced response.
11. Using LT, the electrical network which consists of passive elements can be analyzed.
12. Using time convolution property of LT, it is possible to get the system response $y(t)$.
13. Using LT it's possible to obtain the causality and stability of LTIC system.
14. Non-causal signals and/or systems can be analyzed by the bilateral (two-sided) Laplace transform. Here, the ROC is mostly in the form of a strip. Bilateral Laplace transform can also be used for linear system analysis.
15. The transfer function of an nth order system can be realized using integrators, summers, and multipliers. The following form of realization which is a synthesis problem have been discussed and illustrated with examples.

    (a) Direct Form-I
    (b) Direct Form-II
    (c) Cascade Form
    (d) Parallel Form
    (e) Transposed Form.

Exercise

I. Short Answer Type Questions

1. **What is Laplace Transform?**
The representation of a continuous-time signal $x(t)$ in terms of complex exponential e^{st} is termed as Laplace transform. Mathematically, it is expressed as

$$L[x(t)] = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

where s is a complex variable expressed as $s = \sigma + j\omega$. Thus, by LT the time function $x(t)$ is expressed as a frequency function.
2. **What do you understand by LT pair?**
The LT and inverse LT are called Laplace transform pair. Mathematically, they are expressed as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$
$$x(t) = \frac{1}{2\pi j}\int_{\sigma-\infty}^{\sigma+\infty} X(s)e^{st}ds$$

3. **What is bilateral Laplace transform?**
The LT to handle non-causal signals and systems is called bilateral LT. It is mathematically expressed as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

4. **What is unilateral Laplace transform?**
The LT to handle causal signals and systems is called unilateral LT. Mathematically it is expressed as
$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

5. **What do you understand by LT of right-sided and left-sided signals?**
The LT of a causal signal is called the right-sided LT and is mathematically described as
$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

The LT of a non-causal signal is called the left-sided LT and is mathematically expressed as

$$X(s) = \int_{-\infty}^{0^-} x(t)e^{-st}\,dt$$

6. **What is the connection between LT and FT?**
The FT is a special case of LT which is obtained by putting $X(s)|_{s=j\omega}$ with the constraints that $x(t)$ is absolutely integrable and ROC of $X(s)$ includes the $j\omega$ axis of the s-plane. Thus, the FT $X(j\omega)$ is obtained from LT of $X(s)$ by substituting $s = j\omega$. It is evaluated on the $j\omega$ axis in the s-plane.

7. **What do you understand by Region of convergence?**
The region in the s-plane for which the LT integral

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

converges is called the region of convergence which is written in the abbreviated form as ROC.

8. **How do you identify the ROC of a causal signal?**
The ROC of a causal (or right-sided) signal is identified in the s-plane in the region to the right of the right most pole of the T.F. $H(s)$.

9. **How do you identify the ROC of a non-causal (left-sided) signal?**
The ROC of a non-causal signal is identified in the s-plane in the region to the left of the left most pole of the T.F. $H(s)$.

10. **How do you identify the ROC of a bilateral Laplace transform?**
The region to the right of the right most pole of the causal signal and the region to the left of the left most pole of the non-causal signal are identified as the ROC of bilateral LT. ROC should not include any pole. The ROC is a strip. If ROC does not overlap, LT does not exist.

11. **State any three properties of ROC.**
The three properties of ROC are

    (a) The ROC of LT does not include any pole of $X(s)$.
    (b) For the right-sided (causal) signal ROC exists to the right of right most pole of $X(s)$.
    (c) For the left-sided (noncausal) signal ROC exists to the left of left most pole of $X(s)$.

12. **Identify the ROCs for the following signals and sketch them in the s-plane?**

    (a) $x(t) = e^{-2t}u(t)$
    (b) $x(t) = e^{-3t}u(-t)$
    (c) $x(t) = e^{-2t}u(t) + e^{3t}u(-t)$
    (d) $x(t) = e^{-2|t|}$
    (e) $x(t) = e^{2|t|}$

13. **Sketch the ROC of the following T.F. of a certain causal system and mark the poles and zeros.**
14. **Sketch the ROC of a non-causal system whose T.F. is given as**

$$\boldsymbol{H(s) = \frac{(s+2)(s-2)}{s(s+1)(s-3)}}$$

    Mark the poles and zeros of $H(s)$.
15. **What are initial and final value theorems?**
    Initial value theorem is used to determine the initial value of $x(t)$ (as $t \to 0$) from the LT $X(s)$ which is given below.

$$x(0^+) = \underset{s\to\infty}{Lt}\ sX(s)$$

    provided $x(t)$ and $\frac{dx(t)}{dt}$ are both Laplace transformable and $X(s)$ is proper.

    The final value theorem is used to determine $x(t)$ as t tends to infinity. This can be determined from $X(s)$ using final value theorem as given below.

$$x(\infty) = \underset{s\to 0}{Lt}\ sX(s)$$

    provided that $x(t)$ and $\frac{dx(t)}{dt}$ are both Laplace transformable and $sX(s)$ has no poles in the RHP or on the imaginary axis.
16. **Find the initial and final values of $x(t)$ whose LT is given by**

$$\boldsymbol{X(s) = \frac{(s+5)}{(s^2+3s+2)}}$$

(*Anna University, June, 2007*)

    Initial Value,

$$x(0^+) = \underset{s\to\infty}{Lt}\ \frac{s(s+5)}{(s^2+3s+2)}$$
$$= \underset{s\to\infty}{Lt}\ \frac{s^2(1+5/s)}{s^2(1+\frac{3}{s}+\frac{2}{s^2})}$$

$$x(0^+) = 1$$

Final value

$$x(\infty) = \underset{s\to\infty}{Lt} \frac{s(s+5)}{(s^2+3s+2)}$$

$$x(\infty) = 0$$

17. **Define transfer function.**
The transfer function of a linear time invariant continuous system is defined as the ratio of the LT of the output variable to the LT of the input variable with all initial conditions being assumed to be zero. Thus,

$$\text{T.F.} \quad H(s) = \frac{\text{Laplace transform of zero state response}}{\text{Laplace transform of input signal}}$$

Transfer function does not exist for non-linear and time-varying systems.

18. **Define poles and zeros of the transfer function.**
The pole of a transfer function is defined as the value of s in the s-plane at which the T.F. becomes infinity. The poles are represented by a small cross $\times$. The poles are the roots of the denominator polynomial of the T.F.
The zero of a transfer function is defined as the value of s in the s-plane at which the T.F. becomes zero. They are represented by a small circle 'O' in the s-plane. The zeros are the roots of the numerator polynomial of T.F.

19. **What do you understand by eigenfunction of a system?**
The input for which the system response is also of the same form is called eigenfunction or characteristic function.

20. **What do you understand by causality of an LTIC system?**
An LTIC system with rational T.F. is said to be causal if the impulse response is right-sided. For such a system the ROC is in RHP and to the right of right most pole. An ROC to the right of the right most pole does not simply guarantee causality of the system. The ROC should be in RHP also.

21. **What do you understand by stability of an LTIC system?**
The LTIC system is said to be stable iff the area under the impulse response $h(t)$ curve is finite. In other words the impulse response $h(t)$ should be absolutely integrable. In terms ROC, the T.F. of a stable LTIC system includes the $j\omega$ axis of the s-plane.
An LTIC system which is causal is said to be stable iff all the poles of the transfer function $H(s)$ lie in the LHP and no repeated poles are at the origin of the s-plane.

22. **What do you understand by impulse response and step response of a system?**
The response of the system for the impulse input which is defined as

$$\begin{aligned} \delta(t) &= 1 \qquad t = 0 \\ &= 0 \qquad \text{elsewhere} \end{aligned}$$

is called impulse response of the system. The response of the system for the step input which is defined as

$$\begin{aligned} x(t) &= u(t) \qquad t \geq 0 \\ &= 0 \qquad t < 0 \end{aligned}$$

is called step response of the system.

23. **What do you understand by zero state response and zero input response?**
The system response when the system is in zero state (all the initial conditions are zero) is called zero state response. Here, the response is made up of characteristic mode or the eigen values of the system.
The zero input response of the system is the response due to the initial conditions only. Here the input is made zero. For an LTIC system, the total response is

$$\text{Total response} = \text{zero state response} + \text{zero input response}$$

24. **What do you understand by natural response and forced response of a system?**
The total response of an LTIC system can be expressed in terms of zero input component and zero state component. If we lump together all the characteristic mode terms in the total response, such a response is called natural response. The remaining part of the total response which consists of non-characteristic mode terms is called the forced response of the system.
25. **Are zero input response and natural response and zero state response and forced response same?**
Zero input response is not the same as the natural response and zero state response is also not the same as forced response. However, the total response which is the sum of natural response and forced response and also expressed as the sum of zero state response and zero input response will be the same. In a few cases, the natural response will be same as the zero input response and the zero state response is same as forced response.
26. **Comment on the solutions of the differential equations obtained by the application of LT and by classical method?**

    (a) In the LT method the initial conditions are applied to zero input response. In the classical method, the total response cannot be represented into zero state response and zero input response. Hence, in the classical method, the zero initial conditions are applied to the total response which begins at $t = 0^+$.
    (b) The classical method is restricted to a certain class of inputs, whereas the LT method is applicable to many commonly used signals.

27. **What do you understand by asymptotic stability of an LTIC system?**
An LTIC system is said to be asymptotically stable iff all the roots of the T.F. which may be simple or repeated lie in LHP. Further, there are no repeated roots on the imaginary axis. Under such conditions the system remains in a particular equilibrium state indefinitely in the absence of an external input.
28. **What do you understand by marginal stability of the system?**
An LTIC system is said to be marginally stable iff there are no roots in the RHP and some un-repeated roots are on the imaginary axis.
29. **What do you understand by zero input stability and zero state stability?**
The zero state stability or external stability of the system is obtained when the input is applied with zero initial conditions. The zero input stability or internal stability of the system is obtained by applying initial conditions with no external input.
30. **What do you understand by bounded input and bounded output (BIBO) stability?**
An LTIC system is bounded input bounded output stable iff the area under the impulse response curve is finite. Here all the poles of the T.F. lie in LHP. No repeated poles are on the imaginary axis. An asymptotically stable system is BIBO unstable.
31. **Find the transfer function of LTI system described by the differential equation**

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 2\frac{dx(t)}{dt} - 3x(t)$$

(*Anna University, May, 2008*)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{(2s-3)}{(s^2+3s+2)}$$

32. **Find the LT of $x(t) = e^{-at}u(t)$.** (*Anna University, December, 2006*)

$$X(s) = \int_0^{\infty} e^{-(s+a)t}dt$$

$$X(s) = \frac{1}{(s+a)} \qquad \text{ROC: Re } s > -a$$

33. **Given $\frac{dy(t)}{dt} + 6y(t) = x(t)$. Find the T.F.**
(*Anna University, December, 2006*)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{(s+6)}$$

34. **Find the LT of $u(t) - u(t-a)$ where $a > 0$.**
(*Anna University, December, 2006*)
The LT of $u(t)$ is $\frac{1}{s}$. By using the time shifting property of LT,

$$L[-u(t-a)] = X(s) = -\frac{1}{s}e^{-as} \qquad \text{ROC: Re } s > 0$$

$$L[u(t) - u(t-a)] = \frac{1}{s}[1 - e^{-as}]$$

35. **Find the LT of $x(t) = +e^{-3t}u(t-10)$**?

$$\begin{aligned} X(s) &= \int_{10}^{\infty} e^{-3t} e^{-st} dt \\ &= \frac{1}{(s+3)} e^{-10(s+3)} \qquad \text{ROC: Re } s > -3 \end{aligned}$$

36. **Find the LT of $x(t) = \delta(t-5)$**?

$$X(s) = e^{-5s} \qquad \text{ROC: all } s$$

37. **What is the output of a system whose impulse response $h(t) = e^{-at}$ for a delta input?** (*Anna University, December, 2005*)

$$\begin{aligned} \frac{Y(s)}{X(s)} &= H(s) = \frac{1}{(s+a)} \qquad [X(s) = 1] \\ Y(s) &= \frac{1}{(s+a)} \\ y(t) &= e^{-at}u(t) \qquad \text{ROC: } s > -a \end{aligned}$$

38. **Find the LT of $x(t) = te^{-at}u(t)$ where $a > 0$**? (*Anna University, May, 2005*)

$$\begin{aligned} L[e^{-at}u(t)] &= \frac{1}{(s+a)} \\ L[te^{-at}u(t)] &= \frac{1}{(s+a)^2} \end{aligned}$$

(using frequency differentiation property).

39. **Determine the LT of**

$$\begin{aligned} \boldsymbol{x(t)} &= \mathbf{2}\boldsymbol{t} \qquad \mathbf{0} \leq \boldsymbol{t} \leq \mathbf{1} \\ &= \mathbf{0} \qquad \textbf{otherwise.} \end{aligned}$$

(*Anna University, May, 2005*)

$$X(s) = \int_0^1 2te^{-st}\,dt$$

Integrating by parts, we get

$$\begin{aligned} X(s) &= \left[\frac{-2t}{s}e^{-st}\right]_0^1 - \frac{2}{s^2}[e^{-st}]_0^1 \\ &= \frac{2}{s^2}[1 - e^{-s}(s+1)] \end{aligned}$$

40. **Determine the output response of the system whose impulse response $h(t) = e^{-at}u(t)$ for the step input?** (*Anna University, April, 2004*)

$$\begin{aligned} h(t) &= e^{-at}u(t) \\ H(s) &= \frac{1}{(s+a)} \\ \frac{Y(s)}{X(s)} &= \frac{1}{(s+a)} \qquad X(s) = \frac{1}{s} \\ Y(s) &= \frac{1}{a}\left[\frac{1}{s} - \frac{1}{s+a}\right] \\ y(t) &= \frac{1}{a}[1 - e^{-at}] \qquad \text{ROC: Re } s > 0 \end{aligned}$$

41. **Find the LT and sketch the pole-zero plot with ROC for $x(t) = (e^{-2t} + e^{-3t})u(t)$.** (*Anna University, June 2007*)

$$\begin{aligned} X(s) &= \frac{1}{(s+2)} + \frac{1}{(s+3)} \\ &= \frac{2(s+2.5)}{(s+2)(s+3)} \end{aligned}$$

42. **Find the LT of $x(t) = \delta(t+1) + \delta(t-1)$ and its ROC.**

$$X(s) = e^{s} + e^{-s} \qquad \text{ROC : all } s.$$

43. **Find the LT of $x(t) = u(t+1) + u(t-1)$ and its ROC.**

$$X(s) = \frac{1}{s}[e^{s} + e^{-s}] \qquad \text{ROC: Re } s > 0$$

44. **Using convolution property determine $y(t) = x_1(t) * x_2(t)$ where $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = e^{-3t}u(t)$?**

$$\begin{aligned} X_1(s) &= \frac{1}{(s+2)}; \\ X_2(s) &= \frac{1}{(s+3)} \\ Y(s) &= X_1(s)X_2(s) \\ &= \frac{1}{(s+2)(s+3)} \\ &= \frac{1}{(s+2)} - \frac{1}{(s+3)} \\ y(t) &= (e^{-2t} - e^{-3t})u(t) \qquad \text{ROC: Re } s > -2 \end{aligned}$$

45. **Find the zero input response for the following differential equation.**

$$\begin{aligned} \frac{dy(t)}{dt} + 5y(t) &= u(t); \\ y(0^-) &= 5 \\ Y(s) &= \frac{5}{s+5} \\ y(t) &= 5e^{-5t}u(t) \end{aligned}$$

46. **Find the LT $\frac{d}{dt}[\delta(t)]$.**

$$L\frac{d}{dt}[\delta(t)] = s \qquad \text{ROC: all } s.$$

47. **Find the LT of $x(t) = \delta(2t)$.**

$$X(s) = \frac{1}{2} \qquad \text{ROC: all } s$$

48. **Find the LT of integrated value of $\delta(t)$.**

$$X(s) = \frac{1}{s}$$

49. **Why integrators are preferred to differentiators in structure realization?** Use of differentiators in structure realization enhances noise. That is why differentiators are not preferred.

50. **What are the components required in structure realization?**
The components required in structure realization are (Figs. 4.49, 4.50, 4.51 and 4.52):

(a) Integrators,
(b) Summers, and
(c) Multipliers.

51. **Mention the steps to be followed to realize a transposed structure from canonic form structure.**

(a) Interchange $X(s)$ and $Y(s)$.
(b) Change the directions of arrows.
(c) Replace take off points by summers and *vice versa*.

II. Long Answer Type Questions

1. **Find the LT of $x(t) = e^{-2|t|}$ and ROC.**

$$X(s) = \frac{1}{(s+2)} + \frac{1}{s-2} \qquad \text{ROC: } -2 < \text{Re } s < 2$$

2. **Find the LT of $x(t) = e^{2|t|}$ and ROC.**
ROC do not overlap and $x(t)$ has no LT $X(s)$.

3. **Find the LT of $x(t) = (e^{2t} + e^{-2t})u(t)$ and the ROC.**

$$X(s) = \frac{1}{(s-2)} + \frac{1}{s+2} \qquad \text{ROC: Re } s > 2$$

4. **Find the LT of $x(t) = (e^{2t} + e^{-2t})u(-t)$ and the ROC.**

$$X(s) = -\left(\frac{1}{(s+2)} + \frac{1}{s-2}\right) \qquad \text{ROC: Re } s < -2$$

5. **Find the LT of $x(t) = (e^{-6t} + e^{-4t})u(t) + (e^{-3t} + e^{-2t})u(-t)$**

$$X(s) = \frac{1}{(s+6)} + \frac{1}{(s+4)} - \left(\frac{1}{(s+3)} + \frac{1}{s+2}\right) \quad \text{ROC: } -4 < \text{Re } s < -3$$

6. **Find the LT of**

$$x(t) = (e^{-6t} + e^{-3t})u(t) + (e^{-4t} + e^{-2t})u(-t)$$

ROC does not overlap and hence $x(t)$ has no LT $X(s)$.

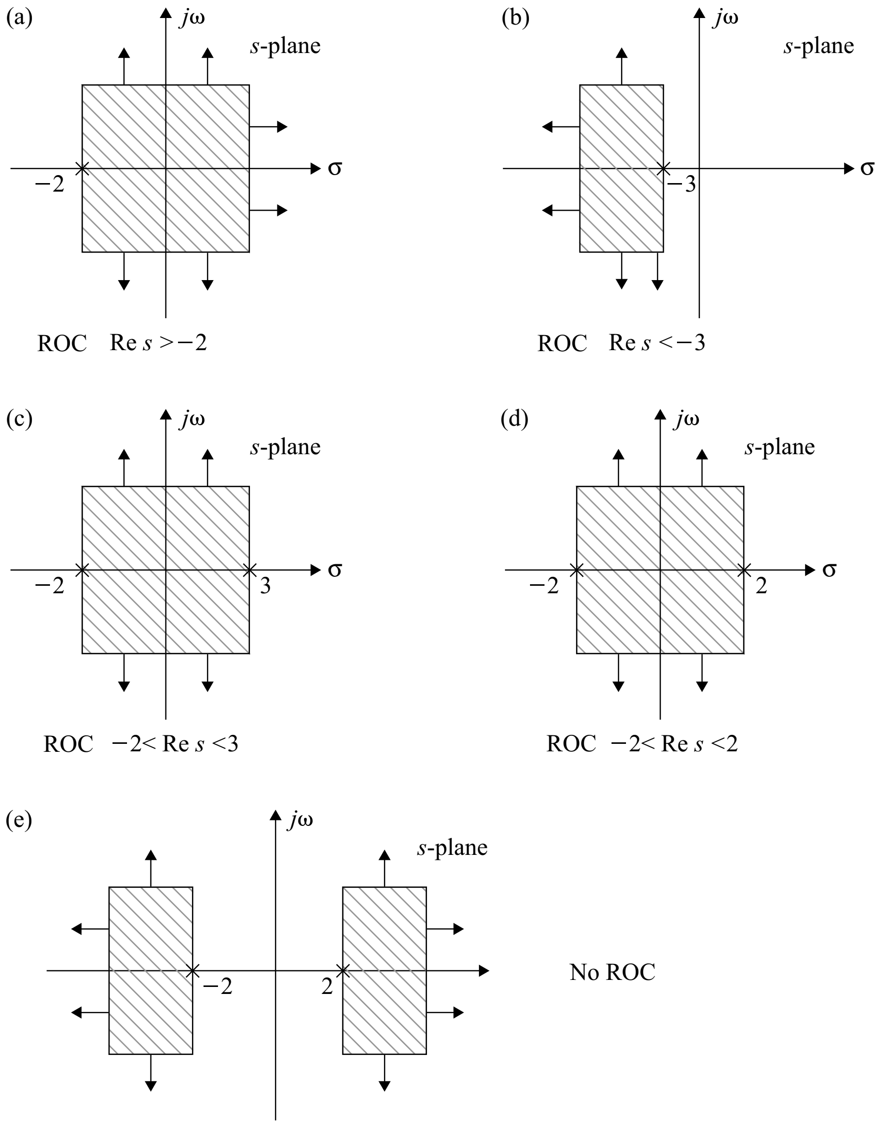


Fig. 4.49 Region of the convergence of different time functions for question 12

Figure 4.49a represents ROC for $x(t) = e^{-2t}u(t)$. Figure 4.49b represents ROC for $x(t) = e^{-3t}u(-t)$. Figure 4.49c represents ROC for $x(t) = e^{-2t}u(t) + e^{-3t}u(-t)$. Figure 4.49d represents ROC for $x(t) = e^{-2|t|}$. Figure 4.49e represents no ROC for $x(t) = e^{2|t|}$.

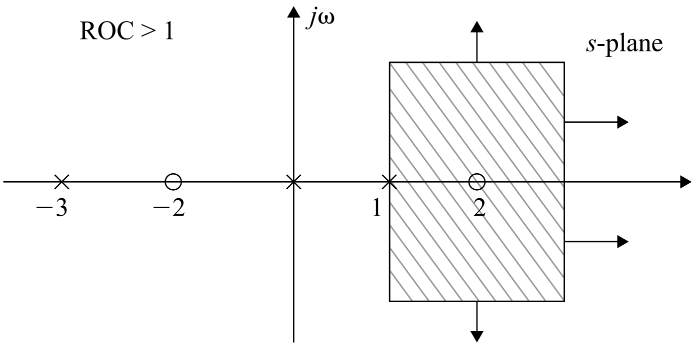


Fig. 4.50 ROC of a causal systematic T.F. $H(s) = \dfrac{10(s-2)(s+2)}{s(s+3)(s-1)}$. Question 13

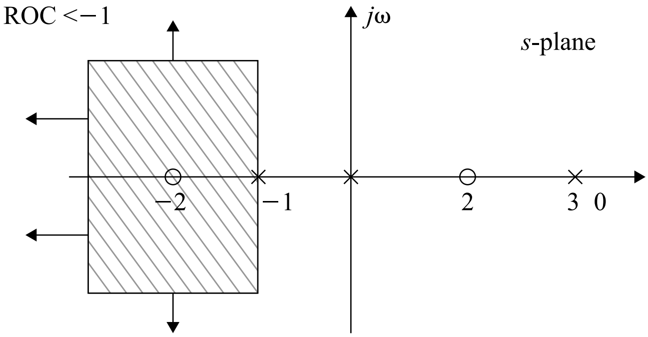


Fig. 4.51 ROC of a non-causal system with the T.F. $H(s) = \dfrac{(s+2)(s-2)}{s(s+1)(s-3)}$. Question 14

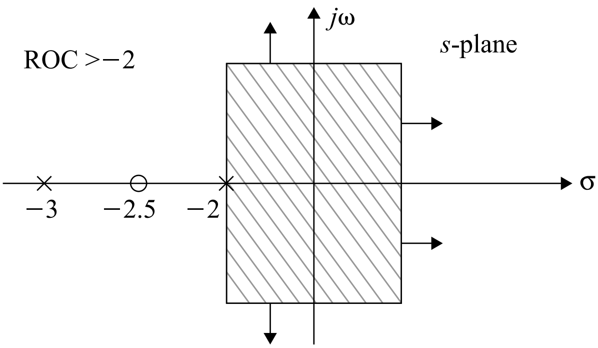


Fig. 4.52 Pole zero plot and ROC of $X(s) = \dfrac{2(s+2.5)}{(s+2)(s+3)}$. Question 41

7. **Find the LT and ROC of**

$$\boldsymbol{x(t) = e^{-3t}[u(t) - u(t-4)]}$$

$X(s) = \left[\frac{1}{(s+3)} - \frac{e^{-4(s+3)}}{(s+3)}\right]$ ROC: Re $s > -3$

8. **Find the inverse LT of the following $X(s)$ for all possible combinations of ROC.**

$$\boldsymbol{X(s) = \frac{4}{(s+1)(s-3)}}$$

(a) $x(t) = (e^{3t} - e^{-t})u(t)$ ROC: Re $s > 3$
(b) $x(t) = (e^{-t} - e^{3t})u(-t)$ ROC: Re $s < -1$
(c) $x(t) = (e^{-t}u(t) - e^{3t}u(-t)$ $-1 < \text{Re } s < 3$

9. **Find the inverse LT of $X(s)$**

$$\boldsymbol{X(s) = \frac{8(s+2)}{s(s^2+4s+8)} \qquad \text{ROC: Re } s > -2}$$

$x(t) = 2\left[1 + \sqrt{2}\sin\left(2t - \frac{\pi}{4}\right)\right]u(t)$

10. **Find the inverse LT of**

$$\boldsymbol{X(s) = \frac{s^2+2s+4)}{(s+2)(s+4)} \qquad \text{ROC: Re } s > -2}$$

$x(t) = \delta(t) + \frac{1}{2}[e^{-2t} - e^{-4t}]u(t)$

11. **Find the inverse LT of**

$$\boldsymbol{X(s) = \frac{(s^2+3s+1)}{(s^2+5s+6)} \qquad \text{ROC: Re } s > -2}$$

$x(t) = \delta(t) - (9e^{-2t} - 11e^{-3t})u(t)$

12. **Find the inverse LT of**

$$\boldsymbol{X(s) = \frac{s^3+8s^2+21s+16}{(s^2+7s+12)} \qquad \text{ROC: Re } s > -3}$$

$x(t) = [\frac{d}{dt}\delta(t) + \delta(t) + 4e^{-4t} - 2e^{-3t}]u(t)$

13. **Find the inverse LT of**

$$X(s) = \frac{10se^{-2s} + 5e^{-4s} + 6}{(s^2 + 13s + 40)} \qquad \text{ROC: Re } s > -5$$

$$x(t) = \left[\frac{80}{3}e^{-8(t-2)} - \frac{50}{3}e^{-5(t-2)}\right]u(t-2) + \frac{5}{3}(e^{-5(t-4)} - e^{-8(t-4)})u(t-4) + 2[e^{-5t} - e^{-8t}]u(t)$$

14. **Find the initial and final value of $y(t)$ if its LT $Y(s)$ is given by**

$$Y(s) = \frac{(s^2 + 2s + 5)}{s(s^2 + 4s + 6)}$$

Initial value $y(0) = 1$. Final value $y(\infty) = \frac{5}{6}$

15.

$$x_1(t) = u(t)$$
$$x_2(t) = e^{-2t}u(t)$$

Using convolution property of LT find $y(t) = x_1(t) * x_2(t)$

$$y(t) = \frac{1}{2}[1 - e^{-2t}]u(t)$$

16. **Consider an LTIC system described by the following differential equation**

$$\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} - 6y(t) = X(s)$$

Determine

(a) the system transfer function.
(b) impulse response of the system if it is causal.
(c) Impulse response of the system if the system is stable.
(d) Impulse response of the system if it is neither causal nor stable.

(a)

$$H(s) = \frac{1}{(s^2 + s - 6)}$$

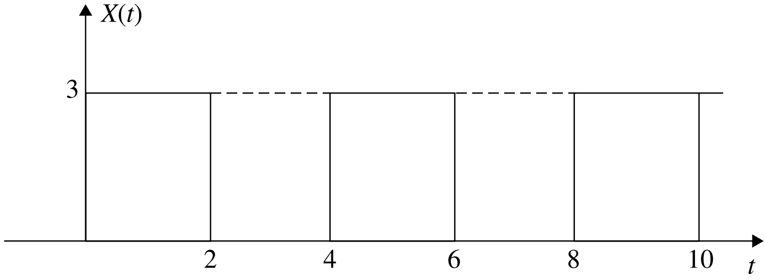


Fig. 4.53 A periodic pulse signal

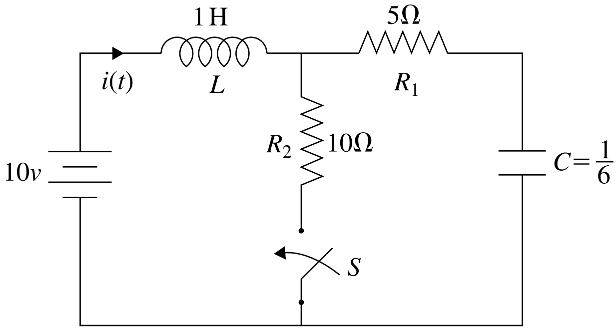


Fig. 4.54 Electrical circuit

(b)

$$y(t) = -\frac{1}{5}[e^{-3t} - e^{2t}]u(t) \qquad \text{ROC: Re } s > 2$$

(c)

$$y(t) = \frac{1}{5}[-e^{2t}u(-t) - e^{-3t}u(t)] \qquad \text{ROC: } -3 < \text{Re } s < 2$$

(d)

$$y(t) = \frac{1}{5}[-e^{2t} + e^{-3t}]u(-t) \qquad \text{ROC: Re } s < -3$$

17. **Determine the LT of the periodic signal shown in Fig. 4.53.**

$$X(s) = \frac{3}{s}\frac{1}{[1 + e^{-2s}]}$$

18. **Consider the electrical circuit shown in Fig. 4.54. Initially the switch S is closed. Derive an expression for the current through the inductor as soon as the switch is open.** $i(t) = [3e^{-3t} - 2e^{-2t}]u(t)$

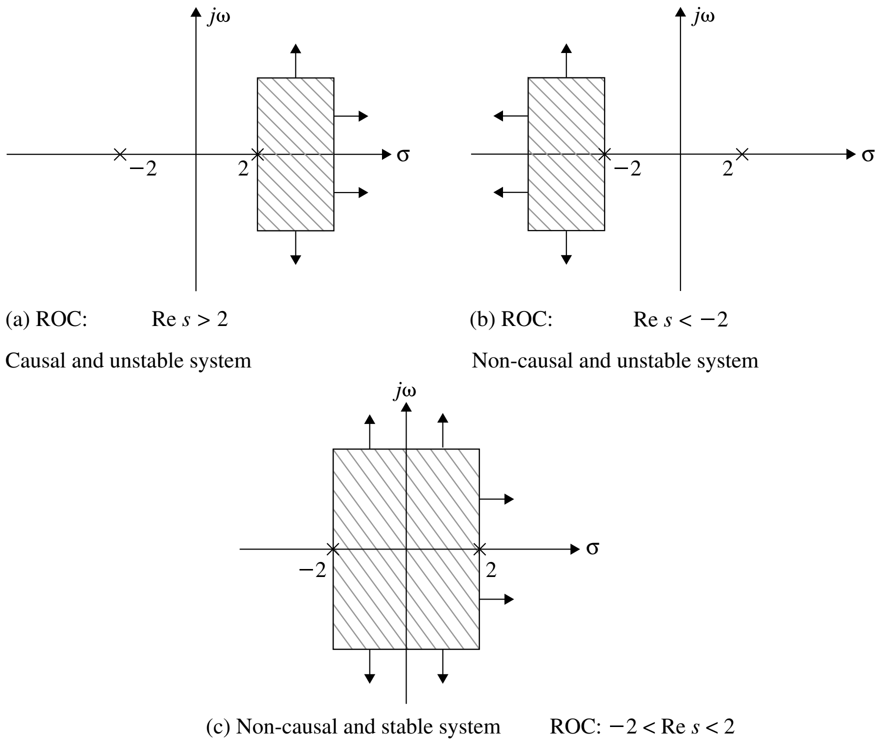


Fig. 4.55 ROC related to causality and stability

19. **Find the Laplace inverse of the following $X(s)$** (Fig. 4.55):

$$\boldsymbol{X(s) = \frac{(s+5)}{(s+2)(s+3)^3} \qquad \text{ROC: Re } s > -2}$$

$$x(t) = [3e^{-2t} - (t^2 + 3t + 3)e^{-3t}]u(t)$$

20. **Solve the following differential equation:**

$$\boldsymbol{\frac{d^2y(t)}{dt^2} + \frac{dy(t)}{dt} - 2y(t) = \frac{dx(t)}{dt} + x(t)}$$

The initial conditions are $y(0^-) = 2$; $\frac{dy(0^-)}{dt} = 1$. The input is

(a) $x(t) = \delta(t)$ an impulse
(b) $x(t) = u(t)$ unit step
(c) $x(t) = e^{-4t}u(t)$ an exponential decay.

(a) $x(t) = \left[\frac{2}{3}e^{-2t} + \frac{7}{3}e^{t}\right]u(t) \quad$ ROC: Re $s > 1$

(b) $x(t) = \left[-\frac{1}{2} + \frac{1}{6}e^{-2t} + \frac{7}{3}e^{t}\right]u(t) \quad$ ROC: Re $s > 1$

(c) $y(t) = \left[\frac{1}{2}e^{-2t} - \frac{3}{10}e^{-4t} + \frac{27}{15}e^{t}\right]u(t) \quad$ ROC: Re $s > 1$

21. **The unit step response of a certain LTIC system $y(t) = 10e^{-5t}$. Find (a) the impulse response? (b) the response due to the exponential decay $x(t) = e^{-3t}u(t)$?**

(a) $h(t) = 10\delta(t) - 50e^{-5t}u(t) \quad$ ROC: Re $s > -5$

(b) $y(t) = (25e^{-5t} - 15e^{-3t})u(t) \quad$ ROC: Re $s > -3$

22. **The impulse response of a certain system is $h_1(t) = e^{-3t}u(t)$ and the impulse response of another system is $h_2(t) = e^{-5t}u(t)$. These two systems are connected in cascade. Find (a) the impulse response of the cascade-connected system (b). Is the system BIBO stable?**

(a) $h(t) = \frac{1}{2}[e^{-3t} - e^{-5t}]u(t) \quad$ ROC: Re $s > -3$
(b) The system is BIBO stable since the ROC is to the right of right most pole at $s = -3$ which includes the $j\omega$ axis.

23. **The impulse response of a certain system is given by $h(t) = e^{-5t}$. The system is excited by $x(t) = e^{-3t}u(t) + e^{-2t}u(-t)$. Determine**

(a) The system transfer function
(b) Output of the system $y(t)$
(c) BIBO stability of the system.

(a) $H(s) = \dfrac{-1}{(s+2)(s+3)(s+5)} \quad$ ROC: $-3 < \text{Re } s < -2$

(b) $y(t) = \left(\frac{1}{2}e^{-3t} - \frac{1}{6}e^{-5t}\right)u(t) + \frac{1}{3}e^{-2t}u(-t)$

(c) The system is not BIBO stable since the ROC does not include the $j\omega$ axis.

24. **A certain LTIC system is described by the following differential equation**

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 30y(t) = \frac{dx(t)}{dt} + 4x(t)$$

The system is subjected to the following input.

$$x(t) = e^{-3t}u(t)$$

The initial conditions are $y(0^+) = 3$ and $\dot{y}(0^+) = 1$. Derive an expression for the output response as a function of time.

$$y(t) = \left[\frac{35}{22}e^{-5t} + \frac{145}{99}e^{6t} - \frac{1}{18}e^{-3t}\right]u(t) \qquad \text{ROC: } -3 < \text{Re}\, s < 6$$

25. **A certain LTIC system is described by the following differential equation:**

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 4x(t)$$

where $x(t) = e^{-3t}u(t)$. The initial conditions are $y(0^-) = 2$ and $\dot{y}(0^-) = 1$. Determine

(a) The characteristic polynomial
(b) The characteristic equation
(c) The eigen values
(d) The zero input response.
(e) The zero state response.
(f) Total response. Use Laplace transform method.

(a) The characteristic polynomial is $F(s) = s^2 + 3s + 2$.
(b) The characteristic equation is $\lambda^2 + 3\lambda + 2 = 0$.
(c) The eigen values are $\lambda_1 = -1$ and $\lambda_2 = -2$
(d) Zero input response is $y_s(t) = [5e^{-t} - 3e^{-2t}]u(t)$.
(e) Zero state response is

$$y_i(t) = \left[\frac{3}{2}e^{-t} - 2e^{-2t} + \frac{1}{2}e^{-3t}\right]u(t)$$

(f) Total response is $y(t) = y_i(t) + y_s(t)$

$$y(t) = \left[\frac{13}{2}e^{-t} - 5e^{-2t} + \frac{1}{2}e^{-3t}\right]u(t)$$

26. **An LTIC system has the following T.F**

$$H(s) = \frac{(s+10)}{s^3 + 5s^2 + 3s + 4}$$

Determine the differential equation.

$$\frac{d^3y(t)}{dt^3} + 5\frac{d^2y(t)}{dt^2} + \frac{3dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + 10x(t)$$

27. **An LTIC system is described by the following differential equation**

$$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{dx(t)}{dt}+4x(t)$$

The system is in the initial state of $y(0^-)=2$ and $\dot{y}(0^-)=1$. The system is excited with the input $x(t)=e^{-5t}$. Determine

(a) The natural response of the system.
(b) The forced response of the system.
(c) Total response of the system. Use Laplace transform method.

(a) The natural response of the system is

$$y_n(t)=\left(\frac{31}{8}e^{-t}-\frac{7}{4}e^{-3t}\right)u(t)$$

(b) The forced response of the system is

$$y_f(t)=\left(-\frac{1}{8}e^{-5t}\right)u(t)$$

(c) The total response of the system is

$$y(t)=\left[\frac{31}{8}e^{-t}-\frac{7}{4}e^{-3t}-\frac{1}{8}e^{-5t}\right]u(t)$$

28. **The impulse response of an LTIC system is given by $x(t)=e^{-2t}u(t)$. Is the system causal?** $X(s)=\frac{1}{s+2}$ and rational ROC: $Re\ s>-2$ which lies in RHP. Hence, the system is causal.

29. **The impulse response of an LTIC system is given by $h(t)=e^{-2|t|}$. Is the system causal.** $H(s)=\frac{-4}{(s-2)(s+2)}$ which is rational ROC is $-2<Re\ s<2$. The ROC is not to the right of the right most pole and hence, the system is not causal.

30. **Check the stability of an LTIC system whose impulse response is $h(t)=e^{-2}|t|$** $H(s)=\frac{-4}{(s-2)(s+2)}$ which is rational. The ROC is $-2<Re\,s<2$. This includes the imaginary axis. Hence, the system is stable.

31. **Consider the following transfer function.**

$$X(s)=\frac{1}{(s+2)(s-2)}$$

Identify all possible ROCs and in each case find the impulse response, stability, and causality. Also sketch the ROC. (1) ROC: Re $s>+2$

$$h(t) = \frac{1}{4}(e^{2t} - e^{-2t})u(t)$$

ROC does not include $j\omega$ axis. The system is unstable. The system is causal since ROC is right-sided and in RHP.
(2) ROC: Re $s < -2$

$$h(t) = \frac{1}{4}[-e^{2t} + e^{-2t}]u(-t)$$

ROC does not include $j\omega$ axis. The system is unstable and non-causal since the ROC is left-sided.
(3) ROC: $-2 <$ Re $s < 2$

$$h(t) = \frac{1}{4}[-e^{2t}u(-t) - e^{-2t}u(t)]$$

ROC includes the $j\omega$ axis and the system is stable. The system is non-causal since ROC is a strip.

32. **Find the bilateral LT of**

$$\boldsymbol{x(t) = e^{-10|t|}}$$

$$X(s) = \frac{-20}{(s^2 - 100)}$$

33. **Find the bilateral LT of**

$$\boldsymbol{x(t) = e^{t}u(t) - e^{3t}u(-t)}$$

$$X(s) = \frac{(2s - 4)}{(s - 1)(s - 3)}$$

34. **Find the bilateral LT of**

$$\boldsymbol{X(s) = \frac{(s - 5)}{(s + 2)(s + 5)} \qquad \text{ROC: } -5 < \text{Re } s < -2}$$

$$x(t) = \frac{1}{3}[10e^{-5t}u(t) + 7e^{-2t}u(-t)]$$

35. **Find the inverse bilateral LT of**

$$\boldsymbol{X(s) = \frac{(s + 2)}{(s - 2)(s - 5)} \qquad \text{ROC: } 2 < \text{Re } s < 5}$$

$$x(t) = -\frac{1}{3}[7e^{5t}u(-t) + 4e^{2t}u(t)]$$

36. **Find the inverse bilateral LT of**

$$\mathbf{X(s) = \frac{(s^2 - 2s - 3)}{(s+2)(s+4)(s-6)} \qquad ROC: -2 < Re\, s < 6}$$

$$x(t) = \left(\frac{-5}{16}e^{-2t} + \frac{21}{20}e^{-4t}\right)u(t) - \frac{21}{80}e^{6t}u(-t)]$$

Chapter 5
The z-Transform Analysis of Discrete Time Signals and Systems

Chapter Objectives

- To define the z-transform and the inverse z-transform.
- To find the z-transform and ROC of typical DT signals.
- To find the properties of ROC.
- To find the properties of z-transform.
- To find the inverse z-transform.
- To solve difference equation using the z-transform.
- To establish the relationship between the z-transform, Fourier transform and the Laplace transform.
- To find the causality and stability of DT system.
- To realize the structure of DT system.

5.1 Introduction

The z-transform is the discrete counterpart of Laplace transform. The Laplace transform converts integro-differential equations into algebraic equations. In the same way, the z-transform converts difference equations of discrete-time system to algebraic equations which simplifies the discrete-time system analysis. There are many connections between Laplace and z-transforms except for some minor differences. DTFT represents discrete-time signal in terms of complex sinusoids. When this sort of representation is generalized and represented in terms of complex exponential, it is termed as z-transform. This sort of representation has a broader characterization of system with signals. Further, the DTFT is applicable only for stable system whereas z-transform can be applied even to unstable systems which means that z-transform can be used to larger class of systems and signals. It is to be noted that many of the

S. Palani, *Basic System Analysis*,
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properties in DTFT, Laplace transform and z-transform are common except that the Laplace transform deals with continues time signals and systems.

5.2 The z-Transform

Let z^n be an everlasting exponential. Let $h(n)$ be the impulse response of the discrete-time system. The response of a linear, time invariant discrete-time system to the everlasting exponential z^n is given as $H(z)z^n$. That is, it is the same exponential within a multiplicative constant. Thus, the system response to the excitation $x[n]$ is the sum of the system's responses to all these exponentials. The tool that is used to represent an arbitrary discrete signal $x[n]$ as a sum of everlasting exponential of the form z^n is called the z-transform.

Let $x[n] = z^n$ be the input signal applied to an LTI discrete-time system whose impulse response is $h[n]$. The system output $y[n]$ is given by

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \end{aligned}$$

Substitute $x[n] = z^n$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \left[\sum_{k=-\infty}^{\infty} h[k]z^{-k} \right]$$

Define the transfer function

$$H[z] = \sum_{k=-\infty}^{\infty} h[k]z^{-k} \tag{5.1}$$

Equation (5.1) may be written as

$$H[z^n] = H[z]z^n$$

To represent any arbitrary signals as a weighted superposition of the Eigen function z^n, let us substitute $z = re^{j\Omega}$ into Eq. (5.1)

$$\begin{aligned} H[re^{j\Omega}] &= \sum_{n=-\infty}^{\infty} h[n][re^{j\Omega}]^{-n} \\ &= \sum_{n=-\infty}^{\infty} \left(h[n]r^{-n}\right) e^{-j\Omega n} \end{aligned} \tag{5.2}$$

Equation (5.2) corresponds to the DTFT of the signal $h[n]r^{-n}$. The inverse of $H[re^{j\Omega}]$, by mathematical manipulation of Eq. (5.2) can be obtained as

$$h[n] = \frac{1}{2\pi j} \oint H(z) z^{n-1} dz \tag{5.3}$$

More generally Eqs. (5.2) and (5.3) can be written as

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{5.4}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz \tag{5.5}$$

The above equations are called z-transform pair. Equation (5.4) is the z-transform of $x[n]$ and Eq. (5.5) is called inverse z-transform. In Eq. (5.4) the range of n is $-\infty < n < \infty$ and hence it is called bilateral z-transform. If $x[n] = 0$ for $n < 0$, Equation (5.4) can be written as

$$X[z] = \sum_{n=0}^{\infty} x[n] z^{-n} \tag{5.6}$$

Equation (5.6) is called unilateral or right-sided z-transform. Bilateral z-transform has limited practical applications. Unless otherwise it is specifically mentioned, z-transform means unilateral. z-transform and inverse z-transform are symbolically represented as given below:

$$\begin{aligned} Z[x[n]] &= X[z] \\ x[n] &\stackrel{Z}{\longleftrightarrow} X[z] \\ z^{-1}[X[z]] &= x[n] \\ X[z] &\stackrel{Z^{-1}}{\longleftrightarrow} x[n] \end{aligned} \tag{5.7}$$

5.3 Existence of the z-Transform

Consider the unilateral z-transform given by Eq. (5.6)

$$\begin{aligned} X[z] &= \sum_{n=0}^{\infty} x[n] z^{-n} \\ &= \sum_{n=0}^{\infty} \frac{x[n]}{z^n} \end{aligned}$$

For the existence for $X[z]$,

$$|X[z]| \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{|z|^n} < \infty \tag{5.8}$$

If the signal $x[n]$ is expressed in terms of an exponential signal r^n, then if $x[n] \leq r^n$ for some r, then

$$|x[n]| \leq r^n \tag{5.9}$$

Substitute Eq. (5.9) in Eq. (5.8)

$$\begin{aligned} |X[z]| &\leq \sum_{n=0}^{\infty} \left(\frac{r}{z}\right)^n \\ &= \frac{1}{\left[1 - \frac{r}{|z|}\right]} \qquad \text{iff } |z| > r \end{aligned} \tag{5.10}$$

From Eq. (5.10), it is evident that the z-transform of $x[n]$ which is $X(z)$ exists for $|z| > r$ and the signal is z-transformable. If the signal $x[n]$ grows faster than the exponential signal r^n for any r_0, Eq. (5.10) is not convergence and $x[n]$ is not z-transformable.

5.4 Connection Between Laplace Transform, z-Transform and Fourier Transform

Consider the Laplace transform of $x(t)$ which is represented below

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{5.11}$$

When $s = j\omega$, Equation (5.11) becomes

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{5.12}$$

Equation (5.12) represents the Fourier transform. **The Laplace transform reduces to the Fourier transform on the imaginary axis where $s = j\omega$.** The relationship between these two transforms can also be interpreted as follows. The complex variable s can be written as $(\sigma + j\omega)$. Equation (5.11) is written as

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$
$$= \int_{-\infty}^{\infty} \left[x(t)e^{-\sigma t}\right] e^{-j\omega t}dt \qquad (5.13)$$

Equation (5.13) can be recognized as the Fourier transform of $[x(t)e^{-\sigma t}]$. **Thus, the Laplace transform of $x(t)$ is the Fourier transform of $x(t)$ after multiplication by the real exponential $e^{-\sigma t}$ which may be growing or decay with respect to time.**

The complex variable z can be expressed in polar form as

$$z = re^{j\omega} \qquad (5.14)$$

where r is the magnitude of z and ω is the angle of z.

Substitute $z = re^{j\omega}$ in Eq. (5.6)

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} \{x[n]r^{-n}\}e^{-j\omega n}$$
$$= F[x[n]r^{-n}] \qquad (5.15)$$

Thus, $X(re^{j\omega})$ is the Fourier transform of the sequence $x[n]$ which is multiplied by a real exponential r^{-n} which may be growing or decaying with increasing n depending on whether r is greater or less than unity. If $r = 1$, then $|z| = 1$ and equation becomes

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = F[x[n]]$$

The z-transform reduces to Fourier transform in the complex z-plane on the contour of a circle with unit radius. The circle which is called unit circle plays the role in the z-transform similar to the role of the imaginary axis in the s-plane for Laplace transform. The unit circle in the z-plane is shown in Fig. 5.1.

5.5 The Region of Convergence (ROC)

In Eq. (5.4) which defines the z-transform $X(z)$ the sum may not coverage for all values of z. The values of z in the complex z-plane for which the sum in the z-transform equation converges is called the region of convergence which is written in abbreviated form as ROC. The concept of ROC is illustrated in the following examples.

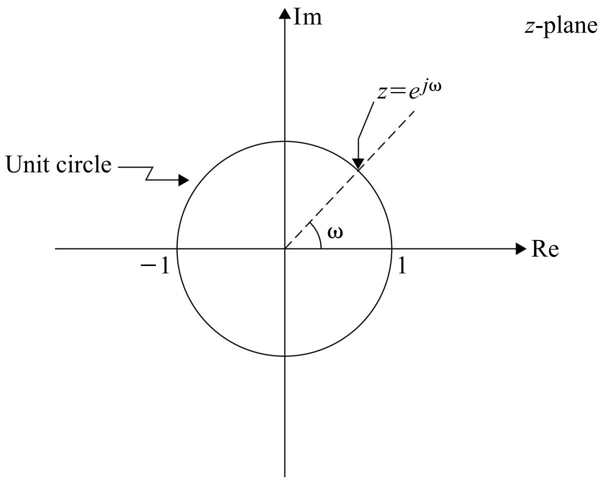


Fig. 5.1 z-transform reduces to FT on the unit circle

Example 5.1 Consider the following discrete-time signals:

(a) $x[n] = a^n u[n] \quad a < 1$

(b) $x[n] = -a^n u(-n-1) \quad a < 1$

(c) $x[n] = a^n u[n] - b^n u(-n-1) \quad b > a$ and $a > b$

Find the z-transform and the ROC in the z-plane.

Solution (a) $\boldsymbol{x[n] = a^n u[n]}$

The signal $x[n]$ is shown in Fig. 5.2a which is a right-sided signal.

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} a^n u[n] z^{-n} \\ &= \sum_{n=0}^{\infty} a^n z^{-n} \qquad [\because\ u[n] = 1 \text{ all } n \geq 0] \\ &= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n \end{aligned}$$

Using the power series we get

$$X(z) = \frac{1}{\left[1 - \frac{a}{z}\right]}$$

where $\frac{a}{z} < 1$ or $|z| > |a|$.

$$X(z) = \frac{z}{(z-a)} \tag{5.16}$$

$$X(z) = \frac{1}{1-az^{-1}} \tag{5.17}$$

Fourier transform is represented in the form as shown in Eq. (5.16) to identify poles and zero and system transfer function. Equation (5.17) form is used when inverse z-transform is taken and also for structure realization. z^{-1} is used as time delay operation. z-transform for the causal real exponential converges iff $|z| > |a|$. Thus, the ROC of $X(z)$ is to the exterior of the circle of radius a, which is shown in Fig. 5.2b in shaded area. The ROC includes the unit circle for $|a| < 1$.

(b) $\boldsymbol{x[n] = -a^n[u[-n-1]]}$
The signal $x[n]$ is shown in Fig. 5.3a which is a left-sided signal

$$\begin{aligned}
Z[-a^n u[-n-1]] &= \sum_{n=-\infty}^{-1} -a^n z^{-n} \qquad \because\ [u(-n-1)] = 1 \text{ for all } -n \\
&= \sum_{n=-\infty}^{-1} -\left[\frac{a}{z}\right]^n = \sum_{n=1}^{\infty} -\left[\frac{z}{a}\right]^n \\
&= -\left[\frac{z}{a} + \frac{z^2}{a^2} + \frac{z^3}{a^3} + \cdots\right] \\
&= 1 - \left[1 + \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \cdots\right] \\
&= 1 - \frac{1}{1-\frac{z}{a}} \qquad \text{if } \left|\frac{z}{a}\right| < 1
\end{aligned}$$

$$Z[-a^n u[-n-1]] = \frac{z}{(z-a)} \quad \text{ROC } |z| < a \tag{5.18}$$

The z-transforms of $x[n] = a^n u[n]$ which is causal and that of $x[n] = -a^n u[-n - 1]$ which is anti-causal are identical. In the former case the ROC is to the exterior of the circle passing through the outermost pole and in the letter case (anti-causal) the ROC is to the interior of the circle passing through the innermost pole. The ROC is shown in Fig. 5.3b.

(c) $\boldsymbol{x[n] = a^n u[n] - b^n u[-n-1]}$
From the results derived in Example 5.1a and b, we can find the z-transform of $x[n]$ as

$$X(z) = \frac{z}{(z-a)} + \frac{z}{(z-b)}$$

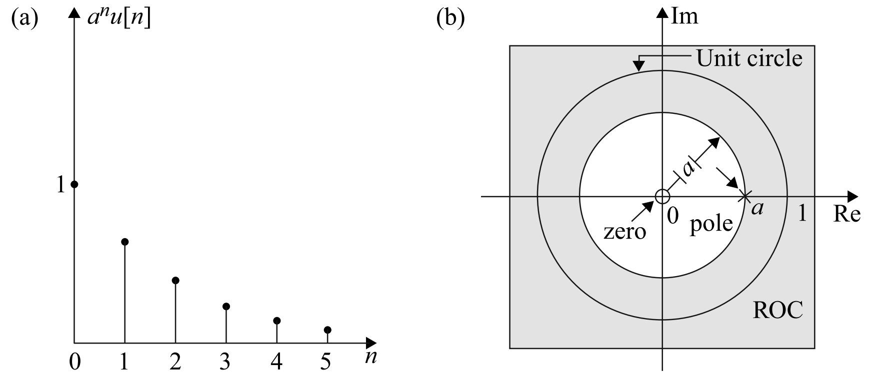


Fig. 5.2 **a** $x[n] = a^n u[n]$ and **b** ROC: $0 < a < 1$

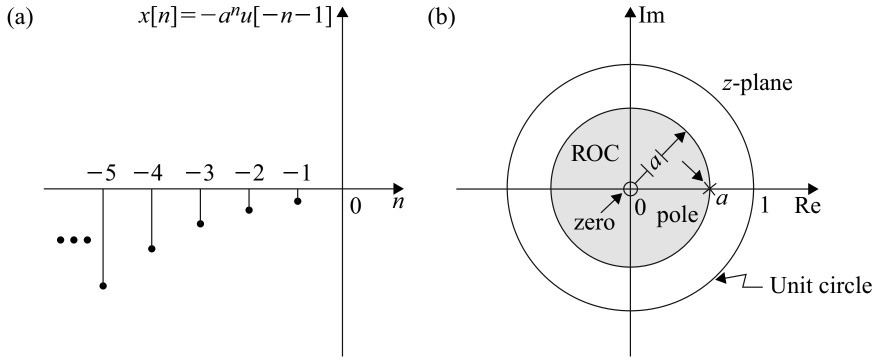


Fig. 5.3 **a** $x[n] = -a^n u[-n-1]$ and **b** ROC: $0 < a < 1$

The right-sided signal $a^n u[n]$ converges if $|z| > a$ and the left-sided signal $-b^n u[-n-1]$ converges if $|z| < b$. The ROC for $|a| > |b|$ and $|a| < |b|$ are shown in Fig. 5.4a and b respectively. From Fig. 5.4a it is observed that the two ROCs do not overlap and hence z-transform does not exist for this signal. Now consider Fig. 5.4b, it is observed that the two ROCs overlap and the overlapping area is shaded in the form of a ring. The z-transform exists in the case with ROC as $|a| < |z| < |b|$.

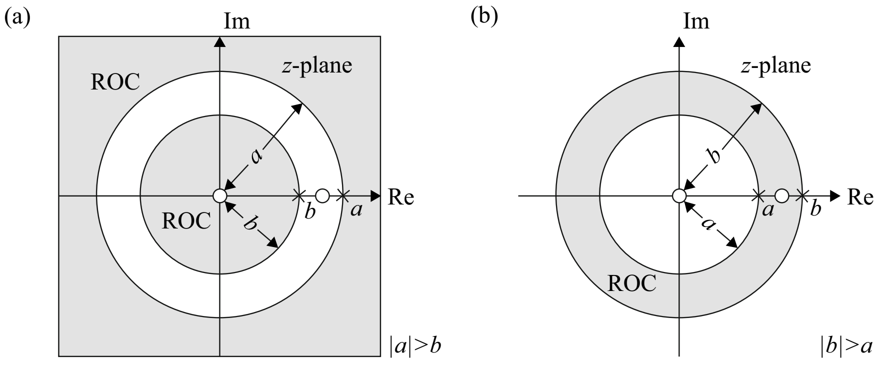


Fig. 5.4 ROC of a two-sided sequence

5.6 Properties of the ROC

Assuming that $X(z)$ is the rational function of z the properties of the ROC are summed up and given below:

1. The ROC is a concentric ring in the z-plane.
2. The ROC does not contain any pole.
3. If $x[n]$ is a finite sequence in a finite interval $N_1 \leq n \leq N_2$, then the ROC is the entire z-plane except $z = 0$ and $z = \infty$.
4. If $x[n]$ is a right-sided sequence (causal) then the ROC is the exterior of the circle $|z| = r_{\text{max}}$ where r_{max} is the radius of the outermost pole of $X(z)$.
5. If $x[n]$ is a left-sided sequence (non-causal) then the ROC is the interior of the circle $|z| = r_{\text{min}}$ where r_{min} is the radius of the innermost pole of $X(z)$.
6. If $x[n]$ is a two-sided sequence then the ROC is given by $r_1 < |z| < r_2$ where r_1 and r_2 are the magnitudes of the two poles of $X(z)$. Here ROC is an annular ring between the circle $|z| = r_1$ and $|z| = r_2$ which does not include any poles.

The following examples illustrate the method of finding z-transform $X(z)$ for the discrete-time sequence $x[n]$.

Example 5.2 Find the z-transform and the ROC for the sequences $x[n]$ given below:

1. $x[n] = \{2, \underset{\uparrow}{-1}, 0, 3, 4\}$

2. $x[n] = \{1, -2, 3, -1, \underset{\uparrow}{2}\}$

3. $x[n] = \{5, 3, \underset{\uparrow}{-2}, 0, 4, -3\}$

4. $x[n] = \delta[n]$

5. $x[n] = u[n]$
6. $x[n] = u[-n]$
7. $x[n] = a^{-n}u[-n]$
8. $x[n] = a^{-n}u[-n-1]$
9. $x[n] = (-a)^n u[-n]$
10. $x[n] = a^{|n|}$ for $|a| < 1$ and $|a| > 1$
11. $x[n] = e^{j\omega_0 n}u[n]$
12. $x[n] = \cos\omega_0 nu[n]$
13. $x[n] = \sin\omega_0 nu[n]$
14. $x[n] = u[n] - u[n-6]$
15. $x[n] = \left[\cos\left(\frac{\pi n}{3} + \frac{\pi}{4}\right)\right]u[n]$

(*Anna University, May, 2007*)

Solution 1. $\boldsymbol{x[n] = \{2, -1, 0, 3, 4\}}$

$$X[z] = \sum_{n=0}^{4} x[n]z^{-n}$$

$$X[z] = 2 - z^{-1} + 0 + 3z^{-3} + 4z^{-4}$$

$X[z]$ will not converges if $|z| = 0$. Hence, ROC is $|z| > 0$.

2. $\boldsymbol{x[n] = \{1, -2, 3, -1, \underset{\uparrow}{2}\}}$

$$X[z] = \sum_{n=-4}^{0} x[n]z^{-n}$$

$$X[z] = z^4 - 2z^3 + 3z^2 - z + 2$$

$X[z]$ will not converges if $|z| = \infty$. Hence, ROC is $|z| < \infty$.

3. $\boldsymbol{x[n] = \{5, 3, \underset{\uparrow}{-2}, 0, 4, -3\}}$

$$X[z] = \sum_{n=-2}^{3} x[n]z^{-n}$$

$$X[z] = 5z^2 + 3z - 2 + 0 + 4z^{-2} - 3z^{-3}$$

For $|z| = 0$ and $|z| = \infty$, $X[z]$ is infinity. Hence, ROC is $0 < |z| < \infty$.

4. $\boldsymbol{x[n] = \delta[n]}$

$$X[z] = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n}$$
$$\delta[n] = 1 \qquad n = 0$$
$$= 0 \qquad n \neq 0$$
$$X[z] = 1 \qquad \text{ROC is entire } z\text{-plane}$$

5. $\boldsymbol{x[n] = u[n]}$

$$X[z] = \sum_{n=0}^{\infty} z^{-n}$$
$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$
$$= \frac{1}{1 - \frac{1}{z}} \qquad \text{[By using summation formula]}$$
$$X[z] = \frac{z}{(z-1)}$$
$$X[z] = \frac{1}{(1 - z^{-1})} \qquad \text{ROC: } |z| > 1 \tag{5.19}$$

6. $\boldsymbol{x[n] = u[-n]}$

$$X[z] = \sum_{n=-\infty}^{0} z^{-n}$$
$$= \sum_{n=0}^{\infty} z^{n}$$
$$= 1 + z + z^2 + \cdots$$

$$X[z] = \frac{1}{1 - z} \qquad \text{ROC: } |z| < 1 \tag{5.20}$$

7. $\boldsymbol{x[n] = a^{-n} u[-n]}$

$$
\begin{aligned}
X[z] &= \sum_{n=-\infty}^{0} a^{-n} z^{-n} \\
&= \sum_{n=-\infty}^{0} (az)^{-n} \\
&= \sum_{n=0}^{\infty} (az)^{n} \\
&= 1 + (az) + (az)^2 + \cdots
\end{aligned}
$$

$$X[z] = \frac{1}{(1-az)} \qquad \text{ROC: } |z| < \frac{1}{a} \tag{5.21}$$

8. $\boldsymbol{x[n] = a^{-n}u[-n-1]}$

$$
\begin{aligned}
X[z] &= \sum_{n=-\infty}^{-1} a^{-n} z^{-n} \\
&= \sum_{n=-\infty}^{-1} (az)^{-n} \\
&= \sum_{n=1}^{\infty} (az)^{n} \\
&= az + (az)^2 + (az)^3 + \cdots
\end{aligned}
$$

$$
\begin{aligned}
X[z] &= az[1 + az + (az)^2 + \cdots] \\
&= \frac{az}{1-az}
\end{aligned}
$$

$$X[z] = \frac{-z}{\left(z - \frac{1}{a}\right)} \qquad \text{ROC: } |z| < \frac{1}{a} \tag{5.22}$$

9. $\boldsymbol{x[n] = (-a)^{n}u[-n]}$

$$X[z] = \sum_{n=-\infty}^{0} (-a)^n z^{-n}$$
$$= \sum_{n=-\infty}^{0} \left(\frac{z}{-a}\right)^{-n} = \sum_{n=0}^{\infty} \left(\frac{z}{-a}\right)^{n}$$
$$= 1 + \left(\frac{z}{-a}\right) + \left(\frac{z}{-a}\right)^2 + \left(\frac{z}{-a}\right)^3 + \cdots$$

$$X[z] = \frac{a}{(z+a)} \qquad \text{ROC} : |z| < |a| \tag{5.23}$$

10. $\boldsymbol{x[n] = a^{|n|};\ a < 1}$

$$x[n] = a^n u[n] + a^{-n} u[-n-1]$$
$$Z[a^n u[n]] = \frac{z}{(z-a)} \qquad \text{ROC: } |z| > a$$
$$Z[a^{-n} u[-n-1]] = \frac{-z}{\left(z - \frac{1}{a}\right)} \qquad \text{ROC: } |z| < \frac{1}{a}$$
$$X[z] = \frac{z}{(z-a)} - \frac{z}{\left(z - \frac{1}{a}\right)}$$

$$X[z] = \frac{(a^2-1)}{a} \frac{z}{(z-a)\left(z-\frac{1}{a}\right)} \tag{5.24}$$

ROC: $a < |z| < \frac{1}{a}$. The ROC is sketched and shown in Fig. 5.5a for $a < 1$.

$$x[n] = a^{|n|} \qquad a > 1$$

The ROC is sketched and shown in Fig. 5.5b. In Fig. 5.5b the two ROCs do not overlap and there is no common ROC. Hence, $x[n]$ does not have $X[z]$.

11. $\boldsymbol{x[n] = e^{j\omega_0 n} u[n]}$

$$X[z] = \sum_{n=0}^{\infty} e^{j\omega_0 n} z^{-n}$$
$$= \sum_{n=0}^{\infty} \left(\frac{e^{j\omega_0}}{z}\right)^n$$
$$= \left(\frac{1}{1 - \frac{e^{j\omega_0}}{z}}\right)$$

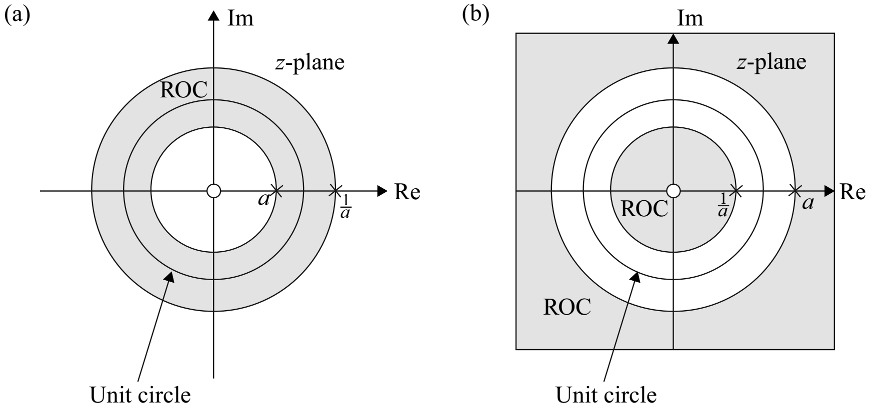


Fig. 5.5 ROC of $x[n] = a^{|n|}$. **a** $a < 1$ and **b** $a > 1$

$$X[z] = \frac{z}{(z - e^{j\omega_0})} \qquad \text{ROC: } |z| > |e^{j\omega_0}| \text{ or } |z| > 1 \tag{5.25}$$

12. $\boldsymbol{x[n] = \cos \omega_0 n u[n]}$

$$\begin{aligned} x[n] &= \frac{1}{2}[e^{j\omega_0 n} + e^{-j\omega_0 n}] \\ Z[e^{j\omega_0 n}] &= \frac{z}{(z - e^{j\omega_0})} \\ Z[e^{-j\omega_0 n}] &= \frac{z}{(z - e^{-j\omega_0})} \\ X[z] &= \frac{1}{2}\left[\frac{z}{(z - e^{j\omega_0})} + \frac{z}{(z - e^{-j\omega_0})}\right] \\ &= \frac{z}{2}\frac{\left[z - e^{-j\omega_0} + z - e^{j\omega_0}\right]}{\left[z^2 - z(e^{-j\omega_0} + e^{j\omega_0}) + 1\right]} \end{aligned}$$

$$X[z] = \frac{z}{2}\frac{[2z - 2\cos\omega_0]}{\left[z^2 - 2z\cos\omega_0 + 1\right]}$$

$$X[z] = \frac{(1 - z^{-1}\cos\omega_0)}{(1 - z^{-1}2\cos\omega_0 + z^{-2})} \qquad \text{ROC: } |z| > 1 \tag{5.26}$$

13. $\boldsymbol{x[n] = \sin \omega_0 n u[n]}$

$$x[n] = \frac{1}{2j}[e^{j\omega_0 n} - e^{-j\omega_0 n}]$$
$$Z[e^{j\omega_0 n}u[n]] = \frac{z}{(z - e^{j\omega_0})}$$
$$Z[e^{-j\omega_0 n}u[n]] = \frac{z}{(z - e^{-j\omega_0})}$$
$$X[z] = \frac{z}{2j}\left[\frac{1}{(z - e^{j\omega_0})} - \frac{1}{(z - e^{-j\omega_0})}\right]$$
$$= \frac{z}{2j}\frac{\left[z - e^{-j\omega_0} - z + e^{j\omega_0}\right]}{\left[z^2 - 2z\cos\omega_0 + 1\right]}$$
$$= \frac{z\sin\omega_0}{(z^2 - 2z\cos\omega_0 + 1)}$$

$$X[z] = \frac{z^{-1}\sin\omega_0}{(1 - 2z^{-1}\cos\omega_0 + z^{-2})} \qquad \text{ROC: } |z| > 1 \tag{5.27}$$

14. $\boldsymbol{x[n] = u[n] - u[n-6]}$

$$x[n] = \{1,\ 1,\ 1,\ 1,\ 1,\ 1\}$$
$$X[z] = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$
$$= \left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5}\right]$$

$$X[z] = \frac{[z^5 + z^4 + z^3 + z^2 + z + 1]}{[z^5]} \qquad \text{ROC: all } z \text{ except } z \neq 0 \tag{5.28}$$

The above result can be represented in a compact form as

$$X[z] = \sum_{n=0}^{5} z^{-n}$$
$$= \sum_{n=0}^{5}\left(\frac{1}{z}\right)^n$$

The following summation formula is used to simplify this (Fig. 5.6).

$$\sum_{k=m}^{n} a^k = \frac{a^{n+1} - a^m}{(a-1)}$$

where $a = \frac{1}{z}$; $k = 0$ and $n = 5$

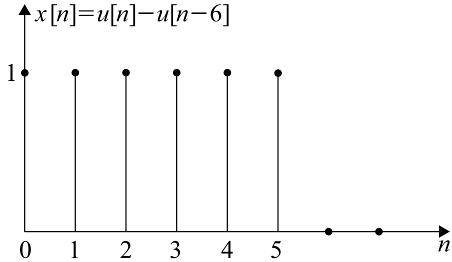


Fig. 5.6 Representation of $x[n] = u[n] - u[n-6]$

$$X[z] = \frac{\left(\frac{1}{z}\right)^6 - \left(\frac{1}{z}\right)^0}{\left(\frac{1}{z} - 1\right)}$$

$$X[z] = \frac{z}{(z-1)}(1 - z^{-6})$$

15. $\boldsymbol{x[n] = \left[\cos\left(\frac{\pi n}{3} + \frac{\pi}{4}\right)\right] u[n]}$

$$\begin{aligned}
x[n] &= \frac{1}{2}\left[e^{j\left(\frac{\pi n}{3}+\frac{\pi}{4}\right)} + e^{-j\left(\frac{\pi n}{3}+\frac{\pi}{4}\right)}\right] \\
&= \frac{1}{2}\left[e^{j\frac{\pi}{4}}e^{j\frac{\pi n}{3}} + e^{-j\frac{\pi}{4}}e^{-j\frac{\pi}{4}}\right] \\
X[z] &= \frac{1}{2}\left[e^{j\frac{\pi}{4}}\frac{z}{(z - e^{j\frac{\pi}{3}})} + e^{-j\frac{\pi}{4}}\frac{z}{(z - e^{-j\frac{\pi}{3}})}\right] \\
X[z] &= \frac{z}{2}\frac{\left[ze^{j\frac{\pi}{4}} - e^{-j\frac{\pi}{12}} + ze^{-j\frac{\pi}{4}} - e^{-j\frac{\pi}{12}}\right]}{z^2 - z(e^{j\frac{\pi}{3}} + e^{-j\frac{\pi}{3}}) + 1} \\
&= \frac{z}{2}\frac{\left[2z\cos\frac{\pi}{4} - 2\cos\frac{\pi}{12}\right]}{\left(z^2 - 2z\cos\frac{\pi}{3} + 1\right)}
\end{aligned}$$

$$X[z] = \frac{z[0.707z - 0.966]}{(z^2 - z + 1)} \qquad \text{ROC: } |z| > 1$$

5.7 Properties of z-Transform

The transformations of $x(t)$ and $x[n]$ to $X(s)$, and $X(j\omega)$ using Laplace transform and Fourier transform respectively as seen from Chapter 6 and Chapter 8 becomes easier if the properties of these transforms are directly applied. Similarly if the properties of z-transform are applied directly to $x[n]$, then $X[z]$ can be easily derived. Hence, some of the important properties of z-transform which are applied to signals and systems are derived and the applications illustrated. The following properties are derived:

1. Linearity;
2. Time shifting;
3. Time reversal;
4. Multiplication by n;
5. Multiplication by an exponential;
6. Time expansion;
7. Convolution theorem;
8. Initial value theorem;
9. Final value theorem.

5.7.1 Linearity

If

$$x_1[n] \overset{Z}{\longleftrightarrow} X_1[z] \quad \text{and} \quad x_2[n] \overset{Z}{\longleftrightarrow} X_2[z]$$

then

$$\{a_1x_1[n] + a_2x_2[n]\} \overset{Z}{\longleftrightarrow} [a_1X_1[z] + a_2X_2[z]] \tag{5.29}$$

Proof Let

$$\begin{aligned} x[n] &= a_1x_1[n] + a_2x_2[n] \\ X[z] &= \sum_{n=-\infty}^{\infty} [a_1x_1[n] + a_2x_2[n]]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} a_1x_1[n]z^{-n} + \sum_{n=-\infty}^{\infty} a_2x_2[n]z^{-n} \end{aligned}$$

$$X[z] = a_1x_1[z] + a_2x_2[z]$$

5.7.2 Time Shifting

If

$$x[n] \overset{Z}{\longleftrightarrow} X[z]$$

then

$$x[n-k] \overset{Z}{\longleftrightarrow} z^{-k}X[z]$$

Proof Let

$$Z[x[n-k]] = \sum_{n=-\infty}^{\infty} x[n-k]z^{-n}$$

Substitute $(n-k) = m$

$$\begin{aligned} Z[x[n-k]] &= \sum_{m=-\infty}^{\infty} x[m]z^{-(k+m)} \\ &= \sum z^{-k}x[m]z^{-m} \end{aligned}$$

$$Z[x[n-k]] = z^{-k}X[z] \tag{5.30}$$

5.7.3 Time Reversal

If

$$x[n] \overset{Z}{\longleftrightarrow} X[z] \qquad \text{ROC: } r_1 < |z| < r_2$$

then

$$x[-n] \overset{Z}{\longleftrightarrow} X[z^{-1}] \qquad \text{ROC: } \frac{1}{r_1} < |z| < \frac{1}{r_2}$$

Proof Let

$$Z[x[-n]] = \sum_{n=-\infty}^{\infty} x[-n]z^{-n}$$

Substitute $-n = m$

$$\begin{aligned} Z[x[-n]] &= \sum_{n=\infty}^{-\infty} x[m]z^{m} \\ &= \sum_{m=-\infty}^{\infty} x[m](z^{-1})^{m} \end{aligned}$$

$$Z[x[-n]] = X[z^{-1}] \tag{5.31}$$

Thus, according to time reversal property, folding the signal in the time domain is equivalent to replacing z by z^{-1}. Further the ROC of $X[z]$ which is $r_1 < |z| < r_2$ becomes $r_1 < |z^{-1}| < r_2$ which is $\frac{1}{r_2} < |z| < \frac{1}{r_1}$.

5.7.4 Multiplication by n

If

$$Z[x[n]] = X[z]$$

then

$$Z[nx[n]] = -z\frac{d}{dz}X[z]$$

Proof Let

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$Z[nx[n]] = \sum_{n=-\infty}^{\infty} nx[n]z^{-n}$$

$$= z\sum_{n=-\infty}^{\infty} nx[n]z^{-n-1}$$

$$= z\sum_{n=-\infty}^{\infty} x[n][nz^{-n-1}]$$

$$Z[nx[n]] = z\sum_{n=-\infty}^{\infty} -x[n]\frac{d}{dz}[z^{-n}]$$

$$= -z\frac{d}{dz}\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$Z[nx[n]] = -z\frac{d}{dz}X[z] \tag{5.32}$$

5.7.5 Multiplication by an Exponential

If

$$Z[x[n]] = X[z]$$

then

$$Z[a^n x[n]] = X[a^{-1}z]$$

Proof Let

$$Z[a^n x[n]] = \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n}$$
$$= \sum_{n=-\infty}^{\infty} x[n][a^{-1}z]^{-n}$$

$$Z[a^n x[n]] = X[a^{-1}z] \tag{5.33}$$

ROC: $r_1 < |a^{-1}z| < r_2$ or $ar_1 < |z| < ar_2$. In $X[z]$, z is replaced by $\frac{z}{a}$.

5.7.6 Time Expansion

If

$$Z[x[n]] = X[z]$$

then

$$Z[x_k[n]] = X[z^k]$$

Proof

$$Z[x_k[n]] = \sum_{n=-\infty}^{\infty} x\left[\frac{n}{k}\right] z^{-n}$$

where n is multiple of k. Substitute $\frac{n}{k} = l$

$$Z[x_k[n]] = \sum_{l=-\infty}^{\infty} x[l]z^{-kl}$$
$$= \sum_{l=-\infty}^{\infty} x[l][z^k]^{-l} = X[z^k]$$

$$Z[x_k[n]] = X[z^k] \tag{5.34}$$

5.7.7 Convolution Theorem

If

$$y[n] = x[n] * h[n]$$

then

$$Y[z] = X[z]H[z]$$

Proof

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$
$$Y[Z] = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x[k]h[n-k] \right] z^{-n}$$
$$= \sum_{k=-\infty}^{\infty} x[k]z^{-k} \sum_{n=-\infty}^{\infty} h[n-k]z^{-(n-k)}$$

Substitute $(n-k) = l$

$$Y[z] = \sum_{k=-\infty}^{\infty} x[k]z^{-k} \sum_{l=-\infty}^{\infty} h[l]z^{-l}$$

$$Y[z] = X[z]Y[z] \tag{5.35}$$

5.7.8 Initial Value Theorem

If

$$X[z] = Z[x[n]]$$

where $x[n]$ is causal, then

$$x[0] = \underset{z\to\infty}{Lt} X[z]$$

Proof For a causal signal $x[n]$

$$\begin{aligned} X[z] &= \sum_{n=0}^{\infty} x[n]z^{-n} \\ &= x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \end{aligned}$$

Taking $z \to \infty$ on both sides we get

$$\begin{aligned} \underset{z\to\infty}{Lt} X[z] &= \underset{z\to\infty}{Lt} [x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots] \\ &= x[0] \end{aligned}$$

$$x[0] = \underset{z\to\infty}{Lt} X[z] \tag{5.36}$$

5.7.9 Final Value Theorem

If $Z[x[n]] = X[z]$ where $x[n]$ is a causal signal and the ROC of $X[z]$ has no poles on or outside the unit circle then

$$x[\infty] = \underset{z\to 1}{Lt} (z-1)X[z]$$

Proof

$$Z[x[n+1]] - Z[x[n]] = \underset{k\to\infty}{Lt} \sum_{n=0}^{k} [x[n+1] - x[n]]z^{-n}$$

$$x[\infty] = \underset{k\to\infty}{Lt} \sum_{n=0}^{k} [x[n+1] - x[n]]z^{-n}$$

$$zX[z] - x[0] - X[z] = \underset{k\to\infty}{Lt} \sum_{n=0}^{k} [x[n+1] - x[n]]z^{-n}$$

$$(z-1)X[z] - x[0] = \underset{k\to\infty}{Lt} \sum_{n=0}^{k} [x[n+1] - x[n]]z^{-n}$$

Taking $\underset{z\to\infty}{Lt}$ on both sides we get

$$\begin{aligned}
&\underset{z\to\infty}{Lt}(z-1)X[z] - x[0] \\
&= \underset{k\to\infty}{Lt}\ [x[1] - x[0]] + [x[2] - x[-1]] + [x[3] - x[2]] + \cdots + [x[k+1] - x[k]] \\
&= x[\infty] - x[0]
\end{aligned}$$

$$x[\infty] = \underset{z\to 1}{Lt}(z-1)X[z] \tag{5.37}$$

Example 5.3 Find the z-transform of the following sequences and also ROC using the properties of z-transform:

1. $x[n] = \delta[n - n_0]$
2. $x[n] = u[n - n_0]$
3. $x[n] = a^{n+1}u[n+1]$
4. $x[n] = a^{n-1}u[n-1]$
5. $x[n] = \left(\frac{1}{2}\right)^n u[-n]$

   $(AnnaUniversity, December, 2007)$
6. $x[n] = u[n-6] - u[n-10]$
7. $x[n] = nu[n]$
8. $x[n] = n[u[n] - u[n-8]]$
9. $x[n] = a^n \cos\omega_0 n u[n]$
10. $x[n] = a^n \sin\omega_0 n u[n]$
11. Show that $u[n] * u[n-1] = nu[n]$
12. $x[n] = n\left(-\frac{1}{4}\right)^n u[n] * \left(\frac{1}{6}\right)^{-n} u[-n]$
13. $x[n] = \left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u[n]$

    Find$X[z]$ and plot the poles and zeros. $(AnnaUniversity, December, 2007)$

14. $x[n] = 1 \quad n \geq 0$
$\quad = z^n \quad n < 0$

$(Anna University, April, 2005)$

15. (a) $x[n] = \left[\left(-\frac{1}{3}\right)^n + 3\left(\frac{1}{6}\right)^n\right] u[n]$

(b) $x[n] = \left[\left(-\frac{1}{3}\right)^n u[-n] + 3\left(\frac{1}{6}\right)^n\right] u[n]$

(c) $x[n] = \left[\left(-\frac{1}{3}\right)^n + 3\left(\frac{1}{6}\right)^n\right] u[-n]$

16. (a) $x[n] = \left[\left(\frac{1}{4}\right)^n + \left(\frac{1}{5}\right)^n\right] u[n]$

(b) $x[n] = \left[\left(\frac{1}{5}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[-n-1]\right]$

(c) $x[n] = \left(\frac{1}{4}\right)^n u[n] + \left(\frac{1}{5}\right)^n u[-n-1]$

17. $x[n] = \delta[n] + \frac{1}{2}\delta(n+1) + \delta(n-3)$ $(Anna University, December, 2006)$

18. $x[n] = 4^n \cos\left[\frac{2\pi n}{6} + \frac{\pi}{4}\right] u[-n-1]$.Sketch the pole-zero plot and indicate the ROC. $(Anna University, April, 2008)$

19. $x[n] = nu[n-1]$ $(Anna University, December, 2006)$

20. $x[n] = (4)^n \quad n < 0$
$\quad = \left(\frac{1}{4}\right)^n \quad n = 0, 2, 4, \ldots$
$\quad = \left(\frac{1}{5}\right)^n \quad n = 1, 3, 5, \ldots$

Solution 1. $\boldsymbol{x[n] = \delta[n - n_0]}$

$$\delta[n] \xleftrightarrow{Z} 1 \qquad \text{ROC: } |z| > 0$$

By applying time shifting property we get

$$Z[\delta[n - n_0]] = z^{-n_0} \tag{5.38}$$

ROC: all z excluding $|z| = 0$.

2. $\boldsymbol{x[n] = u[n - n_0]}$

$$u[n] \xleftrightarrow{Z} \frac{z}{(z-1)}$$

By applying time shifting (right shifted) property we get

$$Z[u[n-n_0]] = \frac{z^{-n_0}z}{(z-1)} = \frac{z^{-(n_0-1)}}{(z-1)}$$

$$X[z] = \frac{z^{-(n_0-1)}}{(z-1)} \qquad \text{ROC: } 1 < |z| < \infty \tag{5.39}$$

3. $\boldsymbol{x[n] = a^{n+1}u[n+1]}$

$$a^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{(z-a)}$$

By applying time shifting (left shifted) property we get

$$Z[a^{n+1}u[n+1]] = z\frac{z}{(z-a)}$$

$$X[z] = \frac{z^2}{(z-a)} \qquad \text{ROC: } |a| < |z| < \infty \tag{5.40}$$

4. $\boldsymbol{x[n] = a^{n-1}u[n-1]}$

$$a^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{(z-a)}$$

Applying time shifting (right shifted) property we get

$$Z[a^{n-1}u[n-1]] = \frac{z^{-1}z}{(z-a)}$$

$$X[z] = \frac{1}{(z-a)} \qquad \text{ROC: } a < |z| < \infty \tag{5.41}$$

5. $\boldsymbol{x[n] = \left(\frac{1}{2}\right)^n u[-n]}$

$$u[-n] \overset{Z}{\longleftrightarrow} \frac{1}{(1-z)}$$
$$x[n] = \left(\frac{1}{2}\right)^n u[-n]$$

By using multiplication property (replacing z by $(\frac{1}{2})^{-1}z$) we get

$$X[z] = \frac{1}{(1-2z)} \qquad \text{ROC: } |z| < \frac{1}{2}$$

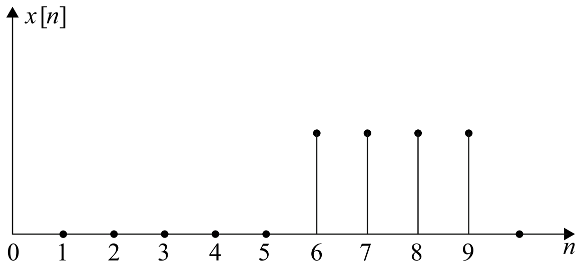


Fig. 5.7 $x[n] = u[n-6] - u[n-10]$

6. $\boldsymbol{x[n] = u[n-6] - u[n-10]}$
   The signal is represented in Fig. 5.7.

$$X[z] = z^{-6} + z^{-7} + z^{-8} + z^{-9} = \frac{1}{z^6} + \frac{1}{z^7} + \frac{1}{z^8} + \frac{1}{z^9}$$

$$X[z] = \frac{z^8 + z^2 + z + 1}{z^9}$$

ROC: all z except $z \neq 0$. The above result can be simplified using the summation formula as

$$\begin{aligned} X[z] &= \sum_{n=6}^{9} \left(\frac{1}{z}\right)^n \\ &= \frac{\left(\frac{1}{z}\right)^{10} - \left(\frac{1}{z}\right)^6}{\left(\frac{1}{z} - 1\right)} \end{aligned}$$

$$X[z] = \frac{z}{(z-1)}[z^{-6} - z^{-10}]$$

The above result can be obtained by the time shifting property of the unit step sequence.

$$\begin{aligned} Z[u[n-6]] &= \frac{z}{(z-1)} z^{-6} \\ Z[u[n-10]] &= \frac{z}{(z-1)} z^{-10} \\ X[z] &= \frac{z}{(z-1)}[z^{-6} - z^{-10}] \end{aligned}$$

$$X[z] = \frac{(z^{-5} - z^{-9})}{(z-1)}$$

7. $\boldsymbol{x[n] = nu[n]}$

$$Z[u[n]] = \frac{z}{(z-1)}$$

Applying the differentiation property in z

$$Z[nu[n]] = -z\frac{dX[z]}{dz}$$
$$Z[nu[n]] = -z\frac{d}{dz}\left[\frac{z}{(z-1)}\right]$$

$$X[z] = \frac{z}{(z-1)^2} \tag{5.42}$$

8. $\boldsymbol{x[n] = n[u[n] - u[n-8]]}$
By using shift theorem we get

$$Z[u[n] - u[n-8]] = \frac{z}{(z-1)}[1 - z^{-8}]$$
$$= \frac{(z - z^{-7})}{(z-1)}$$
$$Z[n[u[n] - u[n-8]]] = -z\frac{d}{dz}\frac{[z - z^{-7}]}{z-1}$$
$$X[z] = -z\frac{[(z-1)(1+7z^{-8}) - (z - z^{-7})]}{(z-1)^2}$$
$$X[z] = \frac{(-8z^{-6} + 7z^{-7} + z)}{(z-1)^2}$$

$$X[z] = \frac{[z^8 - 8z + 7]}{z^7(z-1)^2}$$

9. $\boldsymbol{x[n] = a^n \cos\omega_0 nu[n]}$
For Example 5.2.12 we get

$$Z[\cos\omega_0 nu[n]] = \frac{[1 - z^{-1}\cos\omega_0]}{[1 - 2\cos\omega_0 z^{-1} + z^{-2}]}$$

To apply multiplication property, replace z by $|\frac{z}{a}|$ or $z^{-1} = |\frac{z}{a}|^{-1} = az^{-1}$

$$\therefore\ Z[a^n\cos\omega_0 nu[n]] = \frac{[1 - az^{-1}\cos\omega_0]}{[1 - 2a\cos\omega_0 z^{-1} + a^2z^{-2}]}$$

$$X[z] = \frac{[1 - az^{-1}\cos\omega_0]}{[1 - 2a\cos\omega_0 z^{-1} + a^2 z^{-2}]} \tag{5.43}$$

10. $\boldsymbol{x[n] = a^n \sin\omega_0 n u[n]}$
For Example 5.2.13 we get

$$Z[\sin\omega_0 n u[n]] = \frac{z^{-1}\sin\omega_0}{[1 - 2\cos\omega_0 z^{-1} + z^{-2}]}$$

To apply multiplication property, as in the previous example, replace z^{-1} by a z^{-1} and $z^{-2} = a^2 z^{-2}$

$$Z[a^n \sin\omega_0 n u[n]] = \frac{az^{-1}\sin\omega_0}{[1 - 2a\cos\omega_0 z^{-1} + a^2 z^{-2}]}$$

$$X[z] = \frac{[az^{-1}\sin\omega_0]}{[1 - 2a\cos\omega_0 z^{-1} + a^2 z^{-2}]} \tag{5.44}$$

11. **Show that** $\boldsymbol{u[n] * u[n-1] = nu[n]}$

$$\begin{aligned}
Z[u[n]] &= \frac{z}{(z-1)} \\
Z[u[n-1]] &= \frac{1}{(z-1)} \\
Z[u[n] * u[n-1]] &= Z[u[n]]Z[u[n-1]] \\
&= \frac{z}{(z-1)}\frac{1}{(z-1)} \\
&= \frac{z}{(z-1)^2}
\end{aligned}$$

Multiplying by Z^{-1} both sides we get

$$u[n] * u[n-1] = Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$u[n] * u[n-1] = n[u[n]]$$

12. $\boldsymbol{x[n] = n\left(-\frac{1}{4}\right)^n u[n] * \left(-\frac{1}{6}\right)^{-n} u[-n]}$

$$x_1[n] = \left(-\frac{1}{4}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{\left(z+\frac{1}{4}\right)}$$

$$n\left[\left(-\frac{1}{4}\right)^n u[n]\right] \overset{Z}{\longleftrightarrow} -z\frac{d}{dz}\frac{z}{\left(z+\frac{1}{4}\right)} = -z\frac{\left[z+\frac{1}{4}-z\right]}{\left(z+\frac{1}{4}\right)^2}$$

$$= \frac{\left[-\frac{z}{4}\right]}{\left(z+\frac{1}{4}\right)^2} \qquad \text{ROC: } |z| > \frac{1}{4}$$

$$x_2[n] = \left(\frac{1}{6}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{\left(z-\frac{1}{6}\right)} \qquad \text{ROC: } |z| > \frac{1}{6}$$

If time reversal property is used. z is to be replaced by z^{-1}

$$\left(\frac{1}{6}\right)^{-n} u[-n] \overset{Z}{\longleftrightarrow} \frac{z^{-1}}{\left(z^{-1}-\frac{1}{6}\right)}$$

$$X_1[z] = -\frac{6}{z-6} \qquad \text{ROC: } |z| < 6$$

$$X[z] = X_1[z]X_2[z] = \frac{\frac{z}{4}6}{\left(z+\frac{1}{4}\right)^2(z-6)}$$

$$X[z] = \frac{1.5z}{\left(z+\frac{1}{4}\right)(z-6)} \qquad \text{ROC: } \frac{1}{4} < |z| < 6$$

13. $\boldsymbol{x[n] = \left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u[n]}$
Find $X[z]$ and plot the poles and zeros. (*Anna University, December, 2007*)

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{\left(z-\frac{1}{2}\right)}$$

$$x_2[n] = \left(\frac{1}{4}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{\left(z-\frac{1}{4}\right)}$$

$$x[n] = x_1[n] - x_2[n]$$

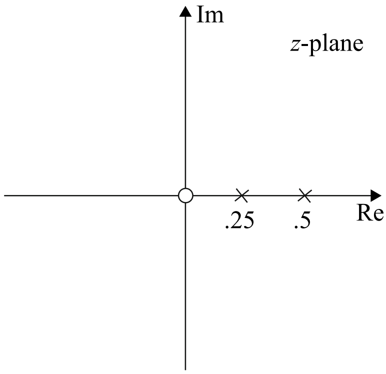
$$X[z] = X_1[z] - X_2[z]$$

$$= \frac{z}{(z-0.5)} - \frac{z}{(z-0.25)}$$

$$X[z] = \frac{z0.25}{(z-0.5)(z-0.25)}$$

The pole-zero plot is shown in Fig. 5.8.

Fig. 5.8 Pole-zero plot



14.

$$\begin{aligned} \boldsymbol{x[n]} &= \mathbf{1} \qquad \boldsymbol{n \geq 0} \\ &= \mathbf{3^n} \qquad \boldsymbol{n < 0} \end{aligned}$$

(*Anna University, April, 2005*)

$$\begin{aligned} x[n] &= u[n] + 3^n u[-n-1] \\ &= x_1[n] + x_2[n] \\ X_1[z] &= \frac{z}{(z-1)} \qquad \text{ROC: } |z| > 1 \\ x_2[n] &= (3)^n u[-n-1] \end{aligned}$$

Using time reversal and multiplication properties we get

$$\begin{aligned} X_2[z] &= -\frac{z}{(z-3)} \qquad \text{ROC: } |z| < 3 \\ X[z] &= X_1(z) + X_2(z) \\ &= \frac{z}{(z-1)} - \frac{z}{(z-3)} \end{aligned}$$

$$X[z] = \frac{-2z}{(z-1)(z-3)} \qquad \text{ROC: } 1 < z < 3$$

15.

$$\text{(a)}\quad \boldsymbol{x[n] = \left(-\frac{1}{3}\right)^n u[n] + 3\left(\frac{1}{6}\right)^n u[n]}$$
$$\text{(b)}\quad \boldsymbol{x[n] = \left[\left(-\frac{1}{3}\right)^n u[-n] + 3\left(\frac{1}{6}\right)^n\right] u[n]}$$
$$\text{(c)}\quad \boldsymbol{x[n] = \left[\left(-\frac{1}{3}\right)^n + 3\left(\frac{1}{6}\right)^n\right] u[-n]}$$

(a)

$$\begin{aligned} x[n] &= \left(-\frac{1}{3}\right)^n + 3\left(\frac{1}{6}\right)^n u[n] \\ &= x_1[n] + x_2[n] \\ X_1[z] &= \frac{z}{\left(z+\frac{1}{3}\right)} \qquad \text{ROC: } |z| > -\frac{1}{3} \\ X_2[z] &= 3\frac{z}{\left(z-\frac{1}{6}\right)} \qquad \text{ROC: } |z| > \frac{1}{6} \end{aligned}$$

$$\begin{aligned} X[z] &= X_1[z] + X_2[z] \\ &= z\left[\frac{1}{\left(z+\frac{1}{3}\right)} + \frac{3}{\left(z-\frac{1}{6}\right)}\right] \qquad \text{ROC: } |z| > \frac{1}{6} \end{aligned}$$

(b)

$$\begin{aligned} x[n] &= \left[\left(-\frac{1}{3}\right)^n u[-n] + 3\left(\frac{1}{6}\right)^n\right] u[n] = x_1[n] + x_2[n] \\ x_1[n] &= \left(-\frac{1}{3}\right)^n u[-n] \end{aligned}$$

Applying the properties of time reversal and multiplication we get

$$\begin{aligned} X_1[z] &= \frac{1}{(1+3z)} \quad \textit{See}\text{ Example 5.3.12; ROC: } |z| < \frac{1}{3} \\ x_2[n] &= 3\left(\frac{1}{6}\right)^n u[n] \\ X_2[z] &= \frac{3z}{\left(z-\frac{1}{6}\right)} \qquad \text{ROC: } |z| > \frac{1}{6} \\ X[z] &= X_1[z] + X_2[z] \end{aligned}$$

$$X[z] = \left[\frac{1}{(1+3z)} + \frac{3z}{\left(z-\frac{1}{6}\right)}\right] \quad \text{ROC: } \frac{1}{6} < |z| < \frac{1}{3}$$

(c)

$$\begin{aligned} x[n] &= \left[\left(-\frac{1}{3}\right)^n + 3\left(\frac{1}{6}\right)^n\right]u[-n] \\ &= x_1[n] + x_2[n] \\ X_1[z] &= \frac{1}{(1+3z)} \qquad \text{ROC: } |z| < \frac{1}{3} \end{aligned}$$

The derivation is given in Example 5.3.15(b)

$$\begin{aligned} x_2[n] &= 3\left(\frac{1}{6}\right)^n u[-n] \\ u[-n] &\overset{Z}{\longleftrightarrow} \frac{1}{(1-z)} \end{aligned}$$

Applying multiplication property we get

$$Z\left(\frac{1}{6}\right)^n u[-n] \overset{Z}{\longleftrightarrow} \frac{1}{(1-6z)} \qquad \text{ROC: } |z| > \frac{1}{6}$$

$$X[z] = \left[\frac{1}{(1+3z)} + \frac{3}{(1-6z)}\right] \quad \text{ROC: } \frac{1}{6} < |z| < \frac{1}{3}$$

16. (a)

$$\boldsymbol{x[n] = \left[\left(\frac{1}{4}\right)^n + \left(\frac{1}{5}\right)^n\right]u[n]}$$

Applying results of Eq. (5.16) we get

$$X[z] = \left[\frac{z}{\left(z-\frac{1}{4}\right)} + \frac{z}{\left(z-\frac{1}{5}\right)}\right] \qquad \text{ROC: } |z| > \frac{1}{4}$$

(b)

$$\boldsymbol{x[n] = \left[\left(\frac{1}{5}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[-n-1]\right]}$$

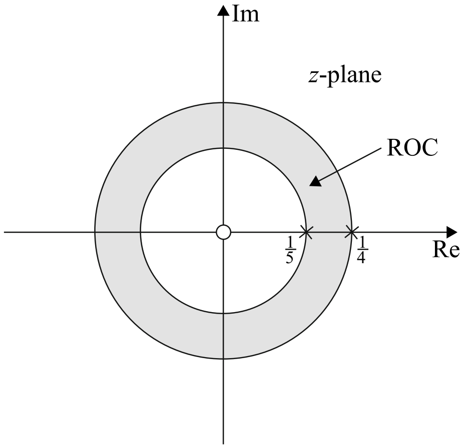


Fig. 5.9 $X[z]$ and its ROC of Example 5.16b

$$\left(\frac{1}{5}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{\left(z-\frac{1}{5}\right)} \qquad \text{ROC: } |z| > \frac{1}{5}$$

$$\left(\frac{1}{4}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{-z}{\left(z-\frac{1}{4}\right)} \qquad \text{ROC: } |z| < \frac{1}{4}$$

$$\left(\frac{1}{5}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{z}{\left(z-\frac{1}{5}\right)} - \frac{z}{\left(z-\frac{1}{4}\right)}$$

$$X[z] = \frac{-\frac{z}{20}}{\left(z-\frac{1}{5}\right)\left(z-\frac{1}{4}\right)} \quad \text{ROC: } \frac{1}{5} < |z| < \frac{1}{4}$$

The poles and zero and the ROC are marked in Fig. 5.9.

(c)

$$\boldsymbol{x[n] = \left[\left(\frac{1}{4}\right)^n u[n] + \left(\frac{1}{5}\right)^n\right] u[-n-1]}$$

$$\left(\frac{1}{4}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{\left(z-\frac{1}{4}\right)} \qquad \text{ROC: } |z| > \frac{1}{4}$$

$$\left(\frac{1}{5}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{-z}{\left(z-\frac{1}{5}\right)} \qquad \text{ROC: } |z| < \frac{1}{5}$$

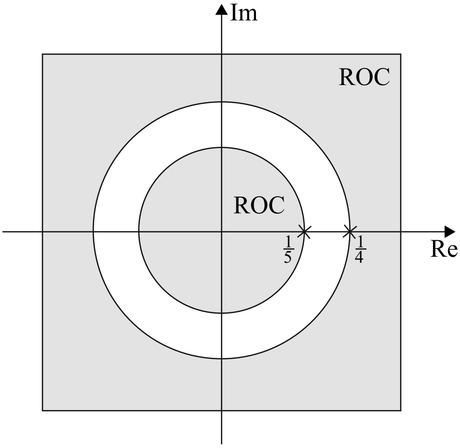


Fig. 5.10 ROC of Example 5.16c

The ROCs of the above two equations are shown in Fig. 5.10 and it is seen that they do not overlap and thus the given $x[n]$ does not have $X[z]$.

17. $\boldsymbol{x[n] = \delta[n] + \frac{1}{2}\delta(n+1) + \delta(n-3)}$

$$\delta[n] \overset{Z}{\longleftrightarrow} 1$$
$$\frac{1}{2}\delta[n+1] \overset{Z}{\longleftrightarrow} \frac{1}{2}z$$
$$\delta[n-3] \overset{Z}{\longleftrightarrow} z^{-3}$$

$$X[z] = 1 + \frac{1}{2}z + z^{-3}$$

18. $\boldsymbol{x[n] = 4^n \cos\left[\frac{2\pi n}{6} + \frac{\pi}{4}\right] u[-n-1]}$

$$\cos\left(\frac{2\pi n}{6}\right) = \cos\frac{\pi n}{3}$$
$$\cos\left(\frac{2\pi n}{6} + \frac{\pi}{4}\right) = \frac{e^{j\left(\frac{\pi}{4}+\frac{\pi n}{3}\right)} + e^{-j\left(\frac{\pi}{4}+\frac{\pi n}{3}\right)}}{2}$$
$$= \frac{1}{2}e^{j\left(\frac{\pi}{4}\right)}e^{j\left(\frac{\pi n}{3}\right)} + \frac{1}{2}e^{-j\left(\frac{\pi}{4}\right)}e^{-j\left(\frac{\pi n}{3}\right)}$$
$$4^n \cos\left(\frac{2\pi n}{6} + \frac{\pi}{n}\right) = \frac{1}{2}e^{j\left(\frac{\pi}{4}\right)}\left(4e^{j\frac{\pi}{3}}\right)^n + \frac{1}{2}e^{-j\frac{\pi}{4}}\left(4e^{-j\frac{\pi}{3}}\right)^n$$

From Eq. (5.18)

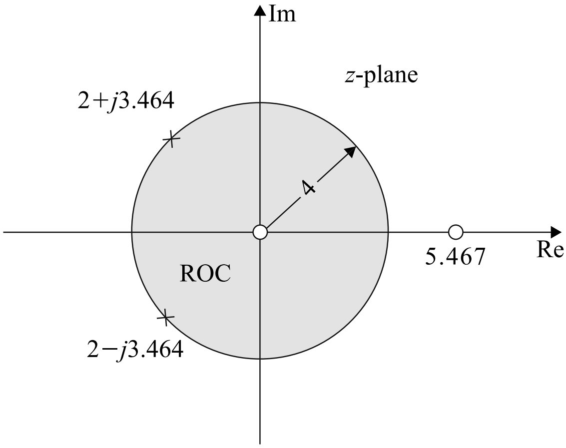


Fig. 5.11 Poles and zeros and ROC of Example 5.3.18

$$\left(4e^{j\frac{\pi}{3}}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{-z}{(z-4e^{j\frac{\pi}{3}})} \qquad \text{ROC: } |z| < 4$$

$$\left(4e^{-j\frac{\pi}{3}}\right)^n u[-n-1] \overset{Z}{\longleftrightarrow} \frac{-z}{(z-4e^{-j\frac{\pi}{3}})} \qquad \text{ROC: } |z| < 4$$

$$\begin{aligned} X[z] &= -\frac{1}{2}z\left[\frac{e^{j\frac{\pi}{4}}}{(z-4e^{j\frac{\pi}{3}})} + \frac{e^{-j\frac{\pi}{4}}}{(z-4e^{-j\frac{\pi}{3}})}\right] \\ &= -\frac{1}{2}z\left[\frac{ze^{j\frac{\pi}{4}} - 4e^{-j\frac{\pi}{12}} - 4e^{j\frac{\pi}{12}} + ze^{-j\frac{\pi}{4}}}{z^2 - z4(e^{j\frac{\pi}{3}} + e^{-j\frac{\pi}{3}}) + 16}\right] \\ &= \frac{-\frac{1}{2}z[\sqrt{2}z - 7.73]}{(z^2 - 4z + 16)} \end{aligned}$$

$$X[z] = \frac{-0.707z[z-5.467]}{(z-2+j3.464)(z-2-j3.464)} \quad \text{ROC: } |z| < 4$$

The pole-zero diagram is shown in Fig. 5.11. The ROC is the interior of the circle.

19. $\boldsymbol{x[n] = nu[n-1]}$

Method 1

$$u[n-1] \overset{Z}{\longleftrightarrow} \frac{1}{(z-1)}$$

Using differential property we get

$$nu[n-1] \overset{Z}{\longleftrightarrow} -z\frac{d}{dz}\frac{1}{(z-1)}$$

$$z[nu[n-1]] = \frac{z}{(z-1)^2}$$

Method 2

$$\begin{aligned} nu[n-1] &= (n-1)u[n-1] + u[n-1] \\ (n-1)u[n-1] &\overset{Z}{\longleftrightarrow} \frac{zz^{-1}}{(z-1)^2} = \frac{1}{(z-1)^2} \end{aligned}$$

$$\begin{aligned} u[n-1] &\overset{Z}{\longleftrightarrow} \frac{1}{(z-1)} \\ nu[n-1] &\overset{Z}{\longleftrightarrow} \frac{1}{(z-1)^2} + \frac{1}{(z-1)} \end{aligned}$$

$$Z[nu[n-1]] = \frac{z}{(z-1)^2}$$

20.

$$\boldsymbol{x[n] = \begin{cases} (4)^n & n < 0 \\ \left(\frac{1}{4}\right)^n & n = 0, 2, 4, \ldots \\ \left(\frac{1}{5}\right)^n & n = 1, 3, 5, \ldots \end{cases}}$$

$$\begin{aligned} X[z] &= \sum_{n=-\infty}^{-1} (4)^n z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n z^{-n} \\ &= X_1[z] + X_2[z] + X_3[z] \\ X_1[z] &= \sum_{n=-\infty}^{-1} (4)^n z^{-n} \\ &= \sum_{n=-\infty}^{-1} \left(\frac{z}{4}\right)^{-n} \\ &= \sum_{n=1}^{\infty} \left(\frac{z}{4}\right)^n \\ &= \frac{z}{4} + \left(\frac{z}{4}\right)^2 + \cdots \\ &= \frac{z}{4}\left[1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + \cdots\right] \\ &= \frac{z}{4}\frac{1}{\left(1 - \frac{z}{4}\right)} \\ &= \frac{-z}{(z-4)} \qquad \text{ROC: } |z| < 4 \end{aligned}$$

$$X_2[z] = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n z^{-n}$$
$$= \sum_{n=0}^{\infty} (4z)^{-n}$$
$$= \sum_{p=0}^{\infty} (4z)^{-2p} \quad \text{where } n = 2p \text{ and } p = 0, 1, 2, \ldots$$

$$X_2[z] = \sum_{p=0}^{\infty} \left(16z^2\right)^{-p}$$
$$= \frac{1}{\left(1 - \frac{1}{16z^2}\right)}$$
$$= \frac{z^2}{\left(z^2 - \frac{1}{16}\right)} \qquad \text{ROC: } |z| > \frac{1}{4}$$
$$X_3[z] = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n z^{-n}$$
$$= \sum_{n=0}^{\infty} (5z)^{-n}$$
$$= \sum_{q=0}^{\infty} (5z)^{-(2q+1)} \quad \text{where } n = 2q + 1$$
$$= \frac{1}{5z} \sum_{q=0}^{\infty} \left(25z^2\right)^{-q}$$
$$= \frac{1}{5z} \frac{1}{\left(1 - \frac{1}{25z^2}\right)}$$
$$= \frac{z/5}{\left(z^2 - \frac{1}{25}\right)} \qquad \text{ROC: } |z| > \frac{1}{5}$$

$$X[z] = \left[-\frac{z}{(z-4)} + \frac{z^2}{\left(z^2 - \frac{1}{16}\right)} + \frac{z/5}{\left(z^2 - \frac{1}{25}\right)} \right]$$

ROC: $\frac{1}{4} < |z| < 4$.

Example 5.4 Find the initial and final values of the following functions:

$$\text{(a)} \qquad X[z] = \frac{z}{(4z^2 - 5z - 1)} \qquad \text{ROC: } |z| > 1$$

$$\text{(b)} \qquad X[z] = \frac{10z(z - 0.4)}{(z - 0.5)(z - 0.3)} \qquad \text{ROC: } |z| > 0.5$$

Solution (a) $\boldsymbol{X[z] = \frac{z}{(4z^2 - 5z - 1)}}$ **Initial Value**

$$\begin{aligned} x[0] &= \underset{z\to\infty}{Lt} X[z] \\ &= \underset{z\to\infty}{Lt} \frac{z}{z^2\left(4 - \frac{5}{z} - \frac{1}{z^2}\right)} \\ &= \underset{z\to\infty}{Lt} \frac{1}{z\left(4 - \frac{5}{z} - \frac{1}{z^2}\right)} \end{aligned}$$

$$x[0] = 0$$

Final Value

$$x[\infty] = \underset{z\to 1}{Lt} \frac{(z-1)}{z}$$

Provided all the poles are inside the unit circle and possibly one pole on the unit circle.

$$(4z^2 - 5z + 1) = 4(z - 1)\left(z - \frac{1}{4}\right)$$

$$X[z] = \frac{z}{4(z-1)\left(z - \frac{1}{4}\right)}$$

The poles $(z - 1)$ is on the unit circle and $z = \frac{1}{4}$ within unit circle. $X[z]$ is valid to apply final value theorem.

$$x[\infty] = \underset{z\to 1}{Lt} \frac{(z-1)}{z} \frac{z}{4(z-1)\left(z - \frac{1}{4}\right)}$$

$$x[\infty] = \frac{1}{3}$$

(b) $\boldsymbol{X[z] = \frac{10z(z-0.4)}{(z-0.5)(z-0.3)}}$

$$x[0] = \underset{z\to\infty}{Lt} \frac{10z^2\left(1 - \frac{0.4}{z}\right)}{z^2\left(1 - \frac{0.5}{z}\right)\left(1 - \frac{0.3}{z}\right)}$$

$$x[0] = 10$$

To find the final value $x[\infty]$, the poles of $X[z]$ are all inside the unit circle and hence is valid to apply final value theorem.

$$x[\infty] = \underset{z\to 1}{Lt} \frac{10z(z-1)(z-0.4)}{z(z-0.5)(z-0.3)}$$

$$x[\infty] = 0$$

Example 5.5

$$X[z] = \frac{\left[1 - \frac{1}{4}z^{-2}\right]}{\left[1 + \frac{1}{4}z^{-2}\right]\left[1 + \frac{5}{4}z^{-1} + \frac{3}{8}z^{-2}\right]}$$

How many different regions of convergence could correspond to $X[z]$?

(*Anna University, May, 2008*)

Solution

$$X[z] = \frac{z^2\left[z^2 - \frac{1}{4}\right]}{\left(z^2 + \frac{1}{4}\right)\left(z^2 + \frac{5}{4}z + \frac{3}{8}\right)} = \frac{z^2\left(z + \frac{1}{2}\right)\left(z - \frac{1}{2}\right)}{\left(z - \frac{j}{2}\right)\left(z + \frac{j}{2}\right)\left(z + \frac{3}{4}\right)\left(z + \frac{1}{2}\right)}$$

$$X[z] = \frac{z^2\left[z - \frac{1}{2}\right]}{\left(z - \frac{j}{2}\right)\left(z + \frac{j}{2}\right)\left(z + \frac{3}{4}\right)}$$

The poles and zeros are located in Fig. 5.12. From Fig. 5.12 circle passing through $|z| = \frac{3}{4}$ and $|z| = \frac{1}{2}$ are drawn. $X[z]$ exists from the following ROCs.

1. $|z| > \frac{3}{4}$. ROC is the exterior of the outer most pole $z = -\frac{3}{4}$. The system is causal and $X[z]$ exits (Fig. 5.12a).
2. $|z| < \frac{1}{2}$. ROC is the interior of the inner most pole $\pm\frac{j}{2}$. The system is anti-causal and $X[z]$ exits (Fig. 5.12b).
3. $\frac{1}{2} < |z| < \frac{3}{4}$. The ROC is a ring between the two circles of radius $r_1 = \frac{3}{4}$ and $r_1 = \frac{1}{2}$. Here $X[z]$ exits. The system is both causal and anti-causal (Fig. 5.12c).

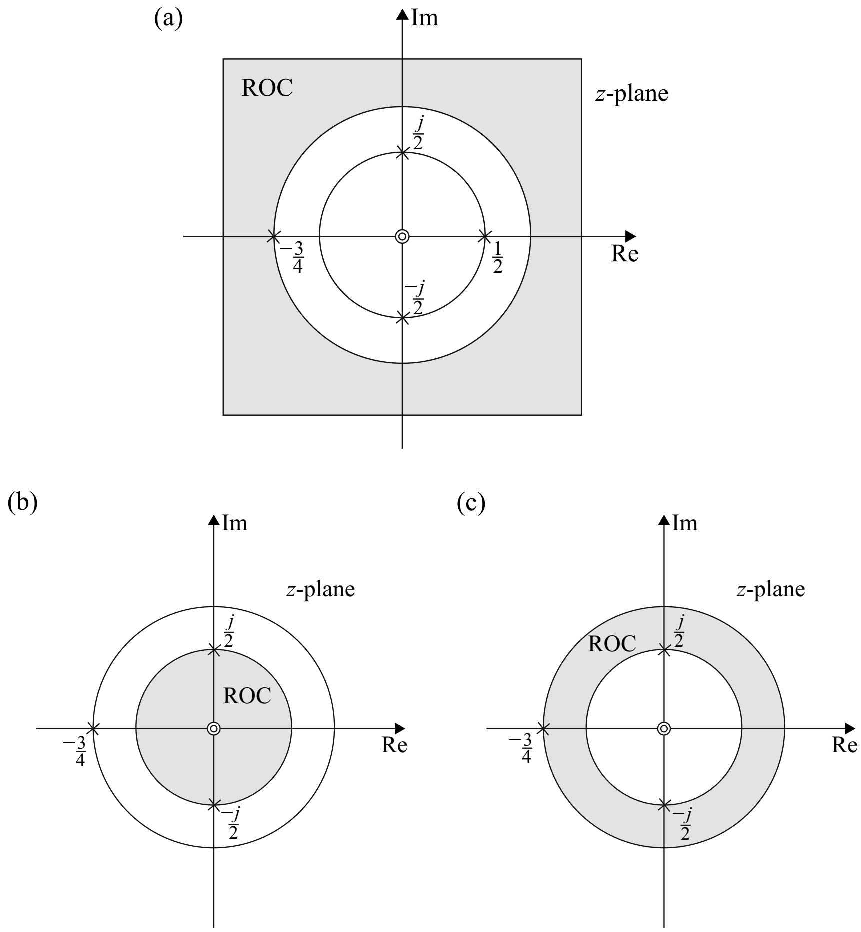


Fig. 5.12 Pole-zero diagram and ROC of $X[z]$

The unilateral z-transform pairs are given in Table 5.1, The properties of z-transform are given in Table 5.2.

5.8 Inverse z-Transform

If $X[z]$ is given then the sequence $x[n]$ is determined. This is called inverse z-transform. As in the Laplace transform, in inverse z-transform also, the integration in the complex z-plane using Eq. (5.5) is avoided since it is tedious. Instead the following methods are used. They are:

Table 5.1 Unilateral z-transform pairs

No.	$x[n]$	$X[z]$
1	$\delta[n]$	1
2	$u[n]$	$\frac{z}{(z-1)}$
3	$nu[n]$	$\frac{z}{(z-1)^2}$
4	$n^2u[n]$	$\frac{z(z+1)}{(z-1)^3}$
5	$a^nu[n]$	$\frac{z}{(z-a)}$
6	$a^{n-1}u[n-1]$	$\frac{1}{(z-a)}$
7	$na^nu[n]$	$\frac{az}{(z-a)^2}$
8	$\cos\omega_0 nu[n]$	$\frac{1-\cos\omega_0 z^{-1}}{1-2\cos\omega_0 z^{-1}+z^{-2}}$
9	$\sin\omega_0 nu[n]$	$\frac{z^{-1}\sin\omega_0}{1-2\cos\omega_0 z^{-1}+z^{-2}}$
10	$a^n\cos\omega_0 nu[n]$	$\frac{1-az^{-1}\cos\omega_0}{1-2a\cos\omega_0 z^{-1}+a^2z^{-2}}$
11	$a^n\sin\omega_0 nu[n]$	$\frac{az^{-1}\sin\omega_0}{1-2a\cos\omega_0 z^{-1}+a^2z^{-2}}$

Table 5.2 z-transform-properties (operations)

Operation	$x[n]$	$X[z]$
Linearity	$a_1x_1[n]+a_2x_2[n]$	$a_1X_1[z]+a_2X_2[z]$
Multiplication by a^n	$a^nx[n]u[n]$	$X\left[\frac{z}{a}\right]$
Multiplication by n	$nx[n]u[n]$	$-z\frac{d}{dz}X[z]$
Time shifting	$x[n-n_0]$	$z^{-n_0}X[z]$
Multiplication by $e^{j\omega_0 n}$	$e^{j\omega_0 n}x[n]$	$X[e^{-j\omega_0}z]$
Time reversal	$x[-n]$	$X\left[\frac{1}{z}\right]$
Accumulation	$\sum_{k=-\infty}^{n}x[n]$	$\frac{z}{(z-1)}X[z]$
Convolution	$x_1[n]*x_2[n]$	$X_1[z]X_2[z]$
Initial value	$x[0]$	$\underset{z\to\infty}{Lt}X[z]$
Final value	$x[\infty]$	$\underset{z\to 1}{Lt}\frac{(z-1)}{z}X[z]$ poles of $(z-1)X[z]$ are inside the unit circle
Right shifting	$x[n-m]u[n-m]$	$\frac{1}{z^m}X[z]$
	$x[n-m]u[n]$	$\frac{1}{z^m}X[z]+\frac{1}{z^m}\sum_{n=1}^{m}x(-m)z^n$
	$x[n-1]u[n]$	$\frac{1}{z}X[z]+x[-1]$
	$x[n-2]u[n]$	$\frac{1}{z^2}X[z]+\frac{1}{z}x[-1]+x[-2]$
Left shifting	$x[n+m]u[n]$	$z^mX[z]-z^m\sum_{n=0}^{m-1}x[n]z^{-n}$
	$x[n+1]u[n]$	$zX[z]-zx(0)$
	$x[n+2]u[n]$	$z^2X[z]-z^2x[0]-zx[1]$

1. Partial fraction method;
2. Power series expansion;
3. Residue method.

Of these, the partial fraction method is very easy to apply as was done in determining inverse Laplace transform.

5.8.1 Partial Fraction Method

If $X[z]$ is a rational function of z then it can be expressed as follows:

$$X[z] = \frac{N[z]}{D[z]} = \frac{K(z - z_1)(z - z_2)\ldots(z - z_m)}{(z - p_1)(z - p_2)\ldots(z - p_n)} \tag{5.45}$$

where $n \geq m$ and all the poles are simple.

$$\begin{aligned}\frac{X[z]}{z} &= \frac{K(z - z_1)(z - z_2)\ldots(z - z_m)}{z(z - p_1)(z - p_2)\ldots(z - p_n)} \\ &= \frac{A_0}{z} + \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \ldots + \frac{A_n}{z - p_n}\end{aligned}$$

where

$$\begin{aligned}A_0 &= X[z]|_{z=0} \\ A_1 &= (z - p_1)\frac{X[z]}{z}\bigg|_{z=p_1}\end{aligned}$$

$$X[z] = A_0 + A_1\frac{z}{z - p_1} + \cdots + \frac{A_n z}{z - p_n} \tag{5.46}$$

Using z-transform pair table, $x[n]$ can be determined. The following examples illustrate the above method. For repeated poles, the z-transform pairs given in Table 5.3 may be referred to.

Table 5.3 z-transform pairs of repeated poles

$X[z]$	$x[n]$ ROC: $\lvert z\rvert > \lvert a\rvert$
1. $\frac{z}{z-a}$	$a^n u[n]$
2. $\frac{z}{(z-a)^2}$	$na^{n-1}u[n]$
3. $\frac{z}{(z-a)^3}$	$\frac{n(n-1)a^{n-2}}{\angle 2}u[n]$
4. $\frac{z}{(z-a)^k}$	$\frac{n(n-1)(n-2)\ldots(n-(k-2))a^{n-k+1}}{\angle(k-1)}u[n]$

Example 5.6 Find the inverse z-transform of

$$X[z] = \frac{1 - \frac{1}{3}z^{-1}}{(1 - z^{-1})(1 + 2z^{-1})} \qquad \text{ROC: } |z| > 2$$

(*Anna University, April, 2004*)

Solution

$$\begin{aligned}
X[z] &= \frac{1 - \frac{1}{3}z^{-1}}{(1 - z^{-1})(1 + 2z^{-1})} \\
&= \frac{z\left(z - \frac{1}{3}\right)}{(z - 1)(z + 2)} \\
\frac{X[z]}{z} &= \frac{A_1}{(z - 1)} + \frac{A_2}{(z + 2)} \\
\left(z - \frac{1}{3}\right) &= A_1(z + 2) + A_2(z - 1)
\end{aligned}$$

Substitute $z = 1$

$$A_1 = \frac{2}{9}$$

Substitute $z = -2$

$$A_2 = \frac{7}{9}$$

$$X[z] = \frac{1}{9}\left[\frac{2z}{z - 1} + \frac{7z}{z + 2}\right]$$

$$x[n] = \frac{1}{9}\left[2(1)^n + 7(-2)^n\right]u[n]$$

Example 5.7 Find the inverse z-transform of

$$X[z] = \frac{1}{1024}\left[\frac{1024 - z^{-10}}{1 - \frac{1}{2}z^{-1}}\right] \qquad \text{ROC: } |z| > 0$$

(*Anna University, April, 2008*)

Solution

$$\begin{aligned}
X[z] &= \frac{1}{1024}\left[\frac{1024 - z^{-10}}{1 - \frac{1}{2}z^{-1}}\right] \\
&= \frac{z}{\left(z - \frac{1}{2}\right)} - \frac{z}{\left(z - \frac{1}{2}\right)}\frac{z^{-10}}{1024}
\end{aligned}$$

Taking inverse z-transform we get

$$\begin{aligned}
x[n] &= \left(\frac{1}{2}\right)^n u[n] - \frac{1}{1024}\left(\frac{1}{2}\right)^{n-10} u[n-10] \\
&= \left(\frac{1}{2}\right)^n u[n] - \frac{1}{1024}\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{-10} u[n-10] \\
&= \left(\frac{1}{2}\right)^n u[n] - \frac{1}{1024}\left(\frac{1}{2}\right)^{n} 1024u[n-10] \\
&= \left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{2}\right)^{n} u[n-10] \\
x[n] &= \left(\frac{1}{2}\right)^n - 0 \qquad 0 \le n \le 9 \\
&= \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^n \\
&= 0 \qquad n \ge 10
\end{aligned}$$

$$\begin{aligned}
x[n] &= \left(\frac{1}{2}\right)^n \qquad 0 \le n \le 9 \\
&= 0 \qquad \text{otherwise}
\end{aligned}$$

Example 5.8 Find the inverse z-transform of

$$X[z] = \frac{z^2}{(1-az)(z-a)}$$

(*Anna University, December, 2007*)

Solution

$$\begin{aligned}
X[z] &= \frac{z^2}{(1-az)(z-a)} \\
\frac{X[z]}{z} &= \frac{-z}{a\left[z-\frac{1}{a}\right][z-a]}
\end{aligned}$$

$$\begin{aligned}
\frac{X[z]}{z} &= \frac{A_1}{\left(z-\frac{1}{a}\right)} + \frac{A_2}{(z-a)} \\
-\frac{z}{a} &= A_1(z-a) + A_2\left(z-\frac{1}{a}\right)
\end{aligned}$$

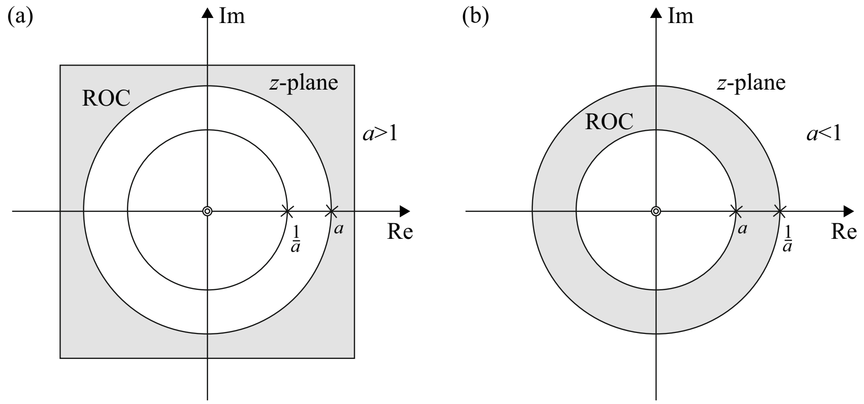


Fig. 5.13 ROC of Example 5.8

Substitute $z = \frac{1}{a}$

$$-\frac{1}{a^2} = A_1\left(\frac{1}{a} - a\right)$$
$$A_1 = \frac{-1}{a(1-a^2)}$$

Substitute $z = a$

$$-1 = A_2\left(a - \frac{1}{a}\right)$$
$$A_2 = \frac{a}{(1-a^2)}$$
$$X[z] = \frac{1}{(1-a^2)}\left[\frac{-1}{a}\frac{z}{\left(z-\frac{1}{a}\right)} + \frac{az}{(z-a)}\right]$$

For $a > 1$, the ROC is shown in Fig. 5.13a. For $a < 1$, the ROC is shown in Fig. 5.13b.

For $a > 1$, the ROC is exterior of the outermost pole. Hence, the function is casual.

$$x[n] = \frac{1}{(1-a^2)}\left[\frac{-1}{a}\frac{1}{(a)^n} + a(a)^n\right]u[n]$$

$$x[n] = \frac{1}{(1-a^2)}\left[-\left(\frac{1}{a}\right)^{n+1} + (a)^{n+1}\right]u[n]$$

For $a < 1$, the ROC is $a < |z| < \frac{1}{a}$ and it is a concentric strip. The pole at $|z| = \frac{1}{a}$ is anti-causal and $z = a$ is causal.

$$x[n] = \frac{1}{(1-a^2)}\left[\left(\frac{1}{a}\right)^{n+1} u[-n-1] + (a)^{n+1}u[n]\right]$$

Example 5.9

$$X[z] = \frac{(7z-23)}{(z-3)(z-4)}$$

Find $x[n]$. ROC: $|z| > 4$.

Solution Method 1: Dividing both sides by z we get

$$\begin{aligned}\frac{X[z]}{z} &= \frac{(7z-23)}{z(z-3)(z-4)} \\ &= \frac{A_1}{z} + \frac{A_2}{(z-3)} + \frac{A_3}{(z-4)} \\ (7z-23) &= A_1(z-3)(z-4) + A_2 z(z-4) + A_3 z(z-3)\end{aligned}$$

Substitute $z = 0$

$$-23 = 12A_1; \qquad A_1 = -\frac{23}{12}$$

Substitute $z = 3$

$$-2 = A_2(3)(-1); \qquad A_2 = \frac{2}{3}$$

Substitute $z = 4$

$$5 = 4A_3; \qquad A_3 = \frac{5}{4}$$

$$X[z] = -\frac{23}{12} + \frac{2}{3}\frac{z}{(z-3)} + \frac{5}{4}\frac{z}{(z-4)}$$

$$X[n] = \left[-\frac{23}{12}\delta[n] + \frac{2}{3}(3)^n + \frac{5}{4}(4)^n\right]u[n]$$

Method 2:

$$\begin{aligned} X[z] &= \frac{(7z-23)}{(z-3)(z-4)} \\ &= \frac{A_1}{(z-3)} + \frac{A_2}{(z-4)} \\ 7z - 23 &= A_1(z-4) + A_2(z-3) \end{aligned}$$

Substitute $z = 3$

$$-2 = -A_1; \qquad A_1 = 2$$

Substitute $z = 4$

$$\begin{aligned} 5 &= A_2 \\ X[z] &= \frac{2}{(z-3)} + \frac{5}{(z-4)} \\ \frac{2}{(z-3)} &\stackrel{Z^{-1}}{\longleftrightarrow} 2(3)^{n-1}u[n-1] \\ \frac{5}{(z-4)} &\stackrel{Z^{-1}}{\longleftrightarrow} 5(4)^{n-1}u[n-1] \end{aligned}$$

$$x[n] = [2(3)^{n-1} + 5(4)^{n-1}]u[n-1]$$

The results of the above two methods are the same even though they are expressed in different forms.

Example 5.10

$$X[z] = \frac{10z}{(z+2)(z+4)^2} \qquad \text{ROC: } |z| > 4$$

Find $x[n]$ using partial fraction method.

Solution This is the case with poles repeated twice

$$\begin{aligned} X[z] &= \frac{10z}{(z+2)(z+4)^2} \\ \frac{X[z]}{z} &= \frac{10}{(z+2)(z+4)^2} = \frac{A_1}{(z+2)} + \frac{A_2}{(z+4)} + \frac{A_3}{(z+4)^2} \\ 10 &= A_1(z+4)^2 + A_2(z+2)(z+4) + A_3(z+2) \end{aligned}$$

Substitute $z = -2$

$$10 = 4A_1; \qquad A_1 = \frac{5}{2}$$

Substitute $z = -4$

$$10 = -2A_3; \qquad A_3 = -5$$

Compare the coefficients of free terms

$$\begin{aligned}10 &= 16A_1 + 8A_2 + 2A_3 \\ &= 16\frac{5}{2} + 8A_2 - 10 \\ A_2 &= -\frac{5}{2}\end{aligned}$$

$$X[z] = \frac{5}{2}\frac{z}{(z+2)} - \frac{5}{2}\frac{z}{(z+4)} - \frac{5}{(z+4)^2}$$

$$x[n] = \left[\frac{5}{2}(-2)^n - \frac{5}{2}(-4)^n - 5n(-4)^n\right]u[n]$$

Example 5.11

$$X[z] = \frac{z(z^2+z-30)}{(z-2)(z-4)^3} \qquad \text{ROC: } |z| > 4$$

Find $x[n]$ using partial fraction method.

Solution This is the case with poles repeated thrice

$$\begin{aligned}X[z] &= \frac{z(z^2+z-30)}{(z-2)(z-4)^3} \\ \frac{X[z]}{z} &= \frac{(z^2+z-30)}{(z-2)(z-4)^3} = \frac{(z-5)(z+6)}{(z-2)(z-4)^3} \\ &= \frac{A_1}{(z-2)} + \frac{A_2}{(z-4)^3} + \frac{A_3}{(z-4)^2} + \frac{A_4}{(z-4)} \\ (z^2+z-30) &= A_1(z-4)^3 + A_2(z-2) + A_3(z-2)(z-4) + A_4(z-2)(z-4)^2\end{aligned}$$

Substitute $z = 2$

$$(-3)(8) = -8A_1; \qquad A_1 = 3$$

Substitute $z = 4$

$$(-1)(10) = 2A_2; \qquad A_2 = -5$$

$$(z^2 + z - 30) = 3(z^3 - 12z^2 + 48z - 64) - 5(z - 2) + A_3(z^2 - 6z + 8) + A_4(z^3 - 10z^2 + 32z - 32)$$

Compare the coefficients of z^2

$$1 = -36 + A_3 - 10A_4$$
$$A_3 - 10A_4 = 37$$

Compare the coefficients of z

$$1 = 144 - 5 - 6A_3 + 32A_4$$
$$6A_3 - 32A_4 = 138$$

Solving the above equation we get

$$A_3 = 7; \qquad A_4 = -3$$

$$X[z] = \frac{3z}{(z-2)} - \frac{5z}{(z-4)^3} + \frac{7z}{(z-4)^2} - \frac{3z}{(z-4)}$$
$$\frac{z}{(z-4)^3} \overset{Z^{-1}}{\longleftrightarrow} \frac{n(n-1)}{\angle 2}(4)^{n-2}u[n] = \frac{n(n-1)}{32}(4)^n u[n]$$
$$\frac{z}{(z-4)^2} \overset{Z^{-1}}{\longleftrightarrow} n(4)^{n-1}u[n] = \frac{1}{4}n(4)^n u[n]$$

$$x[n] = \left[3(2)^n + \left\{-\frac{5}{32}n(n-1) + \frac{7}{4}n - 3\right\}(4)^n\right]u[n]$$

The values of A_1, A_2 and A_3 determined are checked for their correctness as follows:

$$\frac{X[z]}{z} = \frac{(z-5)(z+6)}{(z-2)(z-4)^3}$$

Substitute $z = 0$

$$\left.\frac{X[z]}{z}\right|_{z=0} = \frac{(-5)(6)}{(-2)(-4)^3} = -\frac{15}{64}$$

$$\frac{X[z]}{z} = \frac{3}{z-2} - \frac{5}{(z-4)^3} + \frac{7}{(z-4)^2} - \frac{3}{(z-4)}$$

Substitute $z = 0$

$$\left.\frac{X[z]}{z}\right|_{z=0} = -\frac{3}{2} + \frac{5}{64} + \frac{7}{16} + \frac{3}{4} = -\frac{15}{64}$$

Hence the values of A_1, A_2, A_3 and A_4 are found to be correct.

Example 5.12

$$X[z] = \frac{z(z+10)}{(z-1)(z^2-8z+20)}$$

Find $x[n]$ using partial fraction method.

Solution This is the case with complex poles

$$\begin{aligned}
X[z] &= \frac{z(z+10)}{(z-1)(z^2-8z+20)} \\
\frac{X[z]}{z} &= \frac{(z+10)}{(z-1)(z-4+j2)(z-4-j2)} \\
&= \frac{A_1}{(z-1)} + \frac{A_2}{(z-4+j2)} + \frac{A_3}{(z-4-j2)} \\
(z+10) &= A_1(z^2-8z+20) + A_2(z-1)(z-4-j2) + A_3(z-1)(z-4+j2)
\end{aligned}$$

Substitute $z = 1$

$$11 = A_1(13); \qquad A_1 = \frac{11}{13}$$

Substitute $z = 4 + j2$

$$\begin{aligned}
(14+j2) &= A_3(4+j2-4+j2)(4+j2-1) \\
A_3 &= \frac{(14+j2)}{j4(3+j2)} = \frac{14.142\angle 8.13^\circ}{4\sqrt{13}\angle 123.69^\circ} \\
&= 0.98\angle -115.56^\circ = 0.98e^{-j115.56^\circ} \\
A_2 &= \text{conjugate of } A_3 \\
&= 0.98\angle 115.56^\circ = 0.98e^{j115.56^\circ} \\
X[z] &= \frac{11}{13}\frac{z}{(z-1)} + \frac{0.98e^{j115.56^\circ}}{(z-4+j2)} + \frac{0.98e^{-j115.56^\circ}}{(z-4-j2)}
\end{aligned}$$

Using z-transform pair we get the following inverse z-transform

$$x[n] = \frac{11}{13}u[n] + [0.98e^{j115.56^\circ}(4 - j2)^n + 0.98e^{-j115.56^\circ}(4 + j2)^n]u[n]$$

$115.56^\circ = 2$ radians

$$\begin{aligned}(4 + j2)^n &= (4.47)^n e^{j0.4636n}\\ (4 - j2)^n &= (4.47)^n e^{-j0.4636n}\end{aligned}$$

$$\begin{aligned}x[n] &= \frac{11}{13}u[n] + [0.98e^{j2}e^{-j0.4636n}(4.47)^n + 0.98(4.47)^n e^{-j2}e^{j0.4636n}]u[n]\\ &= \frac{11}{13}u[n] + 0.98 * (4.47)^n[e^{j(2-.4636n)} + e^{-j(2-.4636n)}]u[n]\end{aligned}$$

$$x[n] = \left[\frac{11}{13} + 1.96(4.47)^n \cos(2 - 0.4636n)\right]u[n]$$

Example 5.13

$$X[z] = \frac{(5z^3 - 29z^2 + 8z + 60)}{(z^2 - 7z + 10)}$$

Find $x[n]$ by partial fraction method.

Solution This is the case with irrational system function. The solution of $x[n]$ will have forward and backward shifts. Dividing the numerator polynomial by the denominator polynomial we get

$$\begin{array}{r|l} & 5z + 6 \\ z^2 - 7z + 10 & 5z^3 - 29z^2 + 8z + 60 \\ & \underline{5z^3 - 35z^2 + 50z} \\ & \quad 6z^2 - 42z + 60 \\ & \quad \underline{6z^2 - 42z + 60} \end{array}$$

$$(z^2 - 7z + 10) = (z - 2)(z - 5)$$

$$X[z] = (5z + 6) + \frac{1}{(z - 2)(z - 5)} = X_1[z] + X_2[z]$$

where

$$\begin{aligned}
X_1[z] &= (5z+6) \\
X_2[z] &= \frac{1}{(z-2)(z-5)} \\
\frac{X_2[z]}{z} &= \frac{1}{z(z-2)(z-5)} \\
&= \frac{A_1}{z} + \frac{A_2}{z-2} + \frac{A_3}{z-5} \\
1 &= A_1(z-2)(z-5) + A_2 z(z-5) + A_3 z(z-2)
\end{aligned}$$

Substitute $z = 0$

$$A_1 = \frac{1}{10}$$

Substitute $z = 2$

$$1 = A_2(2)(-3); \qquad A_2 = -\frac{1}{6}$$

Substitute $z = 5$

$$1 = A_3(5)(3); \qquad A_3 = \frac{1}{15}$$

$$X[z] = 5z + 6 + \frac{1}{10} - \frac{1}{6}\frac{z}{(z-2)} + \frac{1}{15}\frac{z}{(z-5)}$$

$$x[n] = \left[\delta(n+1) + 6.1\delta[n] - \frac{1}{6}(2)^n + \frac{1}{15}(5)^n\right]u[n]$$

Example 5.14 Find the inverse z-transform of

$$X[z] = \frac{(5 + z^{-2} + 4z^{-3})}{(z^2 + 7z + 10)}$$

Solution

$$\begin{aligned}
X[z] &= \frac{(5 + z^{-2} + 4z^{-3})}{(z^2 + 7z + 10)} = \frac{(5 + z^{-2} + 4z^{-3})}{z}\frac{z}{(z+2)(z+5)} \\
&= [5z^{-1} + z^{-3} + 4z^{-4}]\frac{z}{(z+2)(z+5)} \\
\frac{z}{(z+2)(z+5)} &= \frac{1}{3}\left[\frac{z}{z+2} - \frac{z}{z+5}\right] \\
\frac{z}{(z+2)(z+5)} &\overset{Z^{-1}}{\longleftrightarrow} \frac{1}{3}\left[(-2)^n - (-5)^n\right]u[n]
\end{aligned}$$

Now

$$x[n] = [5z^{-1} + z^{-3} + 4z^{-4}]\frac{1}{3}[(-2)^n - (-5)^n]u[n]$$

Using the time shifting property we get

$$x[n] = \frac{5}{3}\left[(-2)^{n-1} - (-5)^{n-1}\right]u[n-1] + \frac{1}{3}\left[(-2)^{n-3} - (-5)^{n-3}\right]u[n-3]$$
$$+ \frac{4}{3}\left[(-2)^{n-4} - (-5)^{n-4}\right]u[n-4]$$

Example 5.15 Find the inverse z-transform for the following system functions:

(a) $X[z] = \dfrac{4}{(z-5)}$ ROC: $|z| < 5$

(b) $X[z] = z(1 - z^{-1})(1 + 2z^{-1})$ ROC: $0 < |z| < \infty$

Solution (a) $\boldsymbol{X[z] = \frac{4}{(z-5)}}$

$$X[z] = \frac{4}{(z-5)} = 4z^{-1}\frac{z}{z-5}$$
$$x[n] = 4z^{-1}[(5)^n]u[n]]$$

$$x[n] = 4(5)^{n-1}u[n-1]$$

(b) $\boldsymbol{X[z] = z(1 - z^{-1})(1 + 2z^{-1})}$

$$X[z] = z(1 - z^{-1})(1 + 2z^{-1})$$
$$X[z] = z[1 + 2z^{-1} - z^{-1} - 2z^{-2})$$
$$= [z + 1 - 2z^{-1}]$$

$$x[n] = \{1, \underset{\uparrow}{1}, -2\}$$

5.8.2 *Inverse z-Transform using Power Series Expansion*

The z-transform Eq. (5.4)

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

can be expressed in power series form and the coefficients of $z^{|n|}$ give the values of the sequence. Equation (5.4) can be express as,

$$X[z] = \cdots + x[-3]z^3 + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \cdots \tag{5.47}$$

Equation (5.47) does not give closed form. However if $X[z]$ is not in a simpler form other than the polynomial in z^{-1}, using power series method $x[n]$ is easily obtained. If $X[z]$ is rational, the power series is obtained by long division. The following examples illustrate the above method.

Example 5.16 Using power series expansion, find the inverse z-transform of the following $X[z]$:

(a) $X[z] = \dfrac{4z}{(z^2 - 3z + 2)}$ ROC: $|z| > 2$

(b) $X[z] = \dfrac{4z}{(z^2 - 3z + 2)}$ ROC: $|z| < 1$

(c) $X[z] = \dfrac{1}{(1 - az^{-1})}$ ROC: $|z| > |a|$ and ROC: $|z| < |a|$

(*Anna University, December, 2006*)

Solution (a) $\boldsymbol{X[z] = \frac{4z}{(z^2-3z+2)}}$**; ROC:** $\boldsymbol{|z| > 2}$

$$X[z] = \frac{4z}{(z^2 - 3z + 2)} = \frac{4z}{(z-1)(z-2)}$$

For ROC: $|z| > 2$, $x[n]$ is a right-sided sequence where $n \geq 0$. Hence, the long division is done in such a way that $X[z]$ is expressed in power of z^{-1}.

$$\begin{array}{r|l} & 4z^{-1} + 12z^{-2} + 28z^{-3} + \cdots \\ \hline z^2 - 3z + 2 & 4z \\ & \underline{4z - 12 + 8z^{-1}} \\ & \quad 12 - 8z^{-1} \\ & \quad \underline{12 - 36z^{-1} + 24z^{-2}} \\ & \qquad 28z^{-1} - 24z^{-2} \\ & \qquad \underline{28z^{-1} - 84z^{-2} + 56z^{-3}} \end{array}$$

$$X[z] = 4z^{-1} + 12z^{-2} + 28z^{-3} + \cdots$$

$$x[n] = \{\underset{\uparrow}{0},\ 4,\ 12,\ 28, \ldots\}$$

(b) $\boldsymbol{X[z] = \frac{4z}{(z^2-3z+2)}}$; **ROC:** $\boldsymbol{|z| < 1}$

For ROC: $|z| < 1$, $x[n]$ sequence is negative where $n \leq 0$. The long division is done in such a way that $X[z]$ is expressed in power of z.

$$\begin{array}{r|l} & 2z + 3z^2 + \frac{7}{2}z^3 \\ \hline 2 - 3z + z^2 & 4z \\ & \underline{4z - 6z^2 + 2z^3} \\ & \quad 6z^2 - 2z^3 \\ & \quad \underline{6z^2 - 9z^3 + 3z^4} \\ & \qquad 7z^3 - 3z^4 \\ & \qquad \underline{7z^3 - \frac{21}{2}z^4 + \frac{7}{2}z^5} \end{array}$$

$$X[z] = 2z + 3z^2 + \frac{7}{2}z^3 + \cdots$$

$$x[n] = \left\{\cdots \frac{7}{2},\ 3,\ 2,\ \underset{\uparrow}{0}\right\}$$

(c) $\boldsymbol{X[z] = \frac{1}{(1-az^{-1})}}$; **ROC:** $\boldsymbol{|z| > |a|}$

$$X[z] = \frac{z}{(z-a)}$$

The ROC: $|z| > a$, and it is the exterior of the circle of radius $|a|$. Hence, $x[n]$ is a right-sided sequence where $n \geq 0$. The long division is done in such that $X[z]$ is expressed in terms of power of z^{-1} as shown below:

$$\begin{array}{r l}
 & 1 + az^{-1} + a^2z^{-2} + a^3a^{-3} + \cdots \\
z - a) & z \\
 & \underline{z - a} \\
 & a \\
 & \underline{a - a^2z^{-1}} \\
 & a^2z^{-1} \\
 & \underline{a^2z^{-1} - a^3z^{-2}} \\
 & a^3z^{-2} \\
 & \underline{a^3z^{-2} - a^4z^{-3}}
\end{array}$$

$$X[z] = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$$

$$x[n] = \{\underset{\uparrow}{1},\ a,\ a^2,\ a^3, \cdots\}$$

$$x[n] = a^n u[n]$$

For ROC: $|z| < |a|$, $x[n]$ sequence is left sided

$$\begin{array}{r l}
 & -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 \cdots \\
-a + z) & z \\
 & \underline{z - a^{-1}z^2} \\
 & a^{-1}z^2 \\
 & \underline{a^{-1}z^2 - a^{-2}z^3} \\
 & a^{-2}z^3 \\
 & \underline{a^{-2}z^3 - a^{-3}z^4}
\end{array}$$

$$X[z] = -a^{-1}z - a^{-2}z^2 - a^{-3}z^3 + \cdots$$

$$x[n] = \left\{\cdots,\ \frac{1}{a^3},\ -\frac{1}{a^2},\ -\frac{1}{a},\ \underset{\uparrow}{0}\right\}$$

$$x[n] = -a^n u[-n-1]$$

Example 5.17 Determine the inverse z-transform of

$$X[z] = \log(1 - 2z), \qquad |z| < \frac{1}{2}$$

by using the power series

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad |x| < 1$$

and by first differentiating $X[z]$ and then using this to recover $x[n]$.

(*Anna University, December, 2007*)

Solution (a) **Using Power Series**

$$X[z] = \log(1-2z)$$
$$= -\sum_{n=1}^{\infty} \frac{1}{n}(2z)^n$$

Replace $n = -n$

$$X[z] = \sum_{n=-1}^{-\infty} \frac{1}{n}(2z)^{-n} = \sum_{n=-1}^{-\infty} \left(\frac{1}{2}\right)^n \frac{1}{n} z^{-n}$$

By z-transform definition, it is a left-sided signal

$$X[n] = \frac{1}{n}\left(\frac{1}{2}\right)^n u(-n-1) \qquad n \le -1$$
$$= 0 \qquad n \ge 0.$$

(b) **Using Differentiation Property**

$$X[z] = \log(1-2z)$$
$$\frac{d}{dz}X[z] = \frac{-2}{(1-2z)}$$

Multiplying both sides by $-z$ we get

$$-z\frac{d}{dz}X[z] = \frac{2z}{(1-2z)}$$
$$= \frac{-z}{z-\frac{1}{2}}$$

$$-z\frac{d}{dz}X[z] \overset{Z^{-1}}{\longleftrightarrow} nx[n]$$

$$\frac{-z}{z-\frac{1}{2}} \overset{Z^{-1}}{\longleftrightarrow} \left(\frac{1}{2}\right)^n u(-n-1) \qquad \text{ROC: } |z| < \frac{1}{2}$$

$$nx[n] = \left(\frac{1}{2}\right)^n u(-n-1)$$

$$x[n] = \left(\frac{1}{2}\right)^n \frac{1}{n} u(-n-1)$$

Example 5.18 Find the inverse z-transform of

$$\text{(a)} \qquad X[z] = \log(1 + az^{-1}) \qquad |z| > |a|$$

$$\text{(b)} \qquad X[z] = \log(1 - az^{-1}) \qquad |z| > |a|$$

(*Madras University, October, 1998*)

Solution (a) The power series expansion for $\log(1+x)$ is

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \qquad \text{for } x < 1$$

$$\log(1+az^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(az^{-1})^n}{n} \qquad |az^{-1}| < 1 \text{ or } |z| > |a|$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

Since the summation is from $n = 1$, using time shifting property we get

$$x[n] = \frac{(-1)^{n+1} a^n}{n} u[n-1]$$

(b) The power series expansion for $\log(1-x)$ is

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n \qquad |x| < 1$$

$$\log(1-az^{-1}) = -\sum_{n=1}^{\infty} \frac{1}{n} (az^{-1})^n$$

$$\log(1 + az^{-1}) = -\sum_{n=1}^{\infty} \frac{a^n}{n} z^{-n}$$

$$x[n] = -\frac{a^n}{n} u[n-1]$$

,

5.8.3 Inverse z-Transform using Contour Integration or the Method of Residue

The inverse z-transform can be obtained from Eq. (5.2) which is given by

$$x[n] = \frac{1}{2\pi j} \oint_c X[z] z^{n-1} dz \tag{5.48}$$

The above integral can be evaluated by summing up all the residues of the poles which are inside the circle c of Eq. (5.48) which can be expressed as

$$\begin{aligned} x[n] &= \sum \text{ (Residues of } X[z]z^{-n} \text{ at the poles inside } (c) \\ &= \sum_i (z - z_i) X[z] z^{-n-1} \Big|_{z=z_i} \end{aligned} \tag{5.49}$$

For multiples poles of order k, and $z = \alpha$, the residue is written as,

$$\text{Residue} = \frac{1}{\angle(k-1)} \underset{z \to \alpha}{Lt} \left\{ \frac{d^{k-1}}{dz^{k-1}} (z - \alpha)^k X[z] z^{n-1} \right\} \tag{5.50}$$

Example 5.19 Find the inverse z-transform of the following $X[z]$ using Residue method

(a) $X[z] = \dfrac{(1 + z^{-1})}{(1 + 8z^{-1} + 15z^{-2})} \qquad |z| > 5$

(b) $X[z] = \dfrac{z^{-1}}{(1 - 10z^{-1} + 24z^{-2})} \qquad 4 < |z| < 6$

(c) $X[z] = \dfrac{z}{\left(z - \frac{1}{2}\right)^2}$

Solution (a) $\boldsymbol{X[z] = \frac{(1+z^{-1})}{(1+8z^{-1}+15z^{-2})}; \; |z| > 5}$

$$X[z] = \frac{z(z+1)}{(z^2+8z+15)}$$

$$\begin{aligned}
X[z] &= \frac{z(z+1)}{(z+3)(z+5)} \\
x[n] &= \sum \text{ Residue of } \frac{z(z+1)}{(z+3)(z+5)} z^{n-1} \\
&= \text{ Residue of } (z+3)\frac{z(z+1)}{(z+3)(z+5)} z^{n-1}\Big|_{z=-3} \\
&\quad + \text{Residue of } (z+5)\frac{z(z+1)z^{n-1}}{(z+3)(z+5)}\Big|_{z=-5}
\end{aligned}$$

$$x[n] = -(-3)^n + 2(-5)^n$$

(b) $\boldsymbol{X[z] = \frac{z^{-1}}{(1-10z^{-1}+24z^{-2})}; \; 4 < |z| < 6}$

$$X[z] = \frac{z}{(z^2-10z+24)} = \frac{z}{(z-4)(z-6)}$$

For $n \geq 0$

$$\begin{aligned}
x[n] &= \text{ Residue of } X[z]z^{n-1}\Big|_{z=4} \\
&= (z-4)\frac{z(z^{n-1})}{(z-4)(z-6)}\Big|_{z=4} \\
&= -\frac{1}{2}(4)^n u[n]
\end{aligned}$$

For $n < 0$

$$x[n] = -\left[(z-6)\frac{zz^{n-1}}{(z-4)(z-6)}\right]_{z=6} = -\frac{1}{2}(6)^n u(-n-1)$$

$$x[n] = -\frac{1}{2}[(4)^n u[n] + (6)^n u(-n-1)]$$

(c) $\boldsymbol{X[z] = \frac{z}{(z-\frac{1}{2})^2}}$

$$x[n] = \frac{d}{dz}\left[\left(z - \frac{1}{2}\right)^2 \frac{zz^{n-1}}{\left(z - \frac{1}{2}\right)}\right]_{z=\frac{1}{2}}$$

$$= \frac{d}{dz} z^n \bigg|_{z=1/2} = nz^{n-1}\bigg|_{z=1/2}$$

$$x[n] = 2n\left(\frac{1}{2}\right)^n u[n]$$

5.9 The System Function of DT Systems

Let

1. $x[n]$ = Input of the system;
2. $y[n]$ = Output of the system;
3. $h[n]$ = Impulse response of the system.

The output $y[n]$ can be expressed as the convolution of $x[n]$ with $h[n]$ as

$$y[n] = x[n] * h[n] \tag{5.51}$$

By applying convolution property of z-transform we obtain

$$Y[z] = X[z]H[z] \tag{5.52}$$

where $Y[z]$, $X[z]$ and $H[z]$ are the z-transforms of $y[n]$, $x[n]$ and $h[n]$ respectively. Equation (5.52) can be expressed as

$$H[z] = \frac{Y[z]}{X[z]} \tag{5.53}$$

In Eq. (5.53), $H[z]$ is referred to as the system function or the transfer function. System function is defined as the ratio of the z-transforms of the output $y[n]$ and the input $x[n]$. The system function completely depends on the system characteristic. Equations (5.51) and (5.52) are illustrated in Fig. 5.14a and b respectively.

Fig. 5.14 System impulse response and system function

(a) $x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$

(b) $X[z] \longrightarrow \boxed{H[z]} \longrightarrow Y[z] = X[z]H[z]$

5.10 Causality of DT Systems

An linear time invariant discrete-time system is said to be causal if the impulse response $h[n] = 0$ for $n < 0$ and it is therefore right sided. The ROC of such a system $H[z]$ is the exterior of a circle. If $H[z]$ is rational then the system is said to be causal if the ROC lies exterior of the circle passing through the outermost pole and includes infinity area. A DT system which is linear time invariant with its system function $H[z]$ rational is said to be causal iff the ROC is the exterior of a circle which passes through the outermost pole of $H[z]$. Further, the degree of the numerator polynomial of $H[z]$ should be less than or equal to the degree of the denominator polynomial.

5.11 Stability of DT System

As we discussed in Chap. 2, an LTI discrete-time system is said to be BIBO stable if the impulse response $h[n]$ is summable. This is expressed as

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{5.54}$$

The corresponding requirement on $H[z]$ is that the ROC of $H[z]$ contains unit circle. By definition of z-transform

$$H[z] = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

Let $z = e^{j\Omega}$

$$\begin{aligned} |z| &= |e^{j\Omega}| \\ &= 1 \\ |H[e^{j\Omega}]| &= \left| \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n} \right| \end{aligned}$$

$$\leq \sum_{n=-\infty}^{\infty} \left|h[n]e^{-j\Omega n}\right|$$

$$= \sum_{n=-\infty}^{\infty} |h[n]| < \infty \tag{5.55}$$

From Eq. (5.55) we see that the stability condition given by Eq. (5.54) is satisfied if $z = e^{j\Omega}$. Thus implies that $H[z]$ must contain unit circle $|z| = 1$.

An LTI system is stable iff the ROC of its system function $H[z]$ contains the unit circle $|z| = 1$.

5.12 Causality and Stability of DT System

For a causal system whose $H[z]$ is rational the ROC is outside the outermost pole. For the BIBO stability the ROC should include the unit circle $|z| = 1$. For the system to be causal and stable the above requirements are satisfied if all the poles are within the unit circle in the z-plane.

An LTID system with the system function $H[z]$ is said to be both causal and stable iff all the poles of $H[z]$ lie inside the unit circle.

The above characteristics of LTI discrete-time systems are illustrated in Fig. 5.15 for a causal system.

Example 5.20 The input to the causal LTI system is

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n]$$

The z-transform of the output of the system is

$$Y[z] = \frac{-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+z^{-1}\right)}$$

Determine $H[z]$, the z-transform of the impulse response and also determine the output $y[n]$. (*Anna University, December, 2007*)

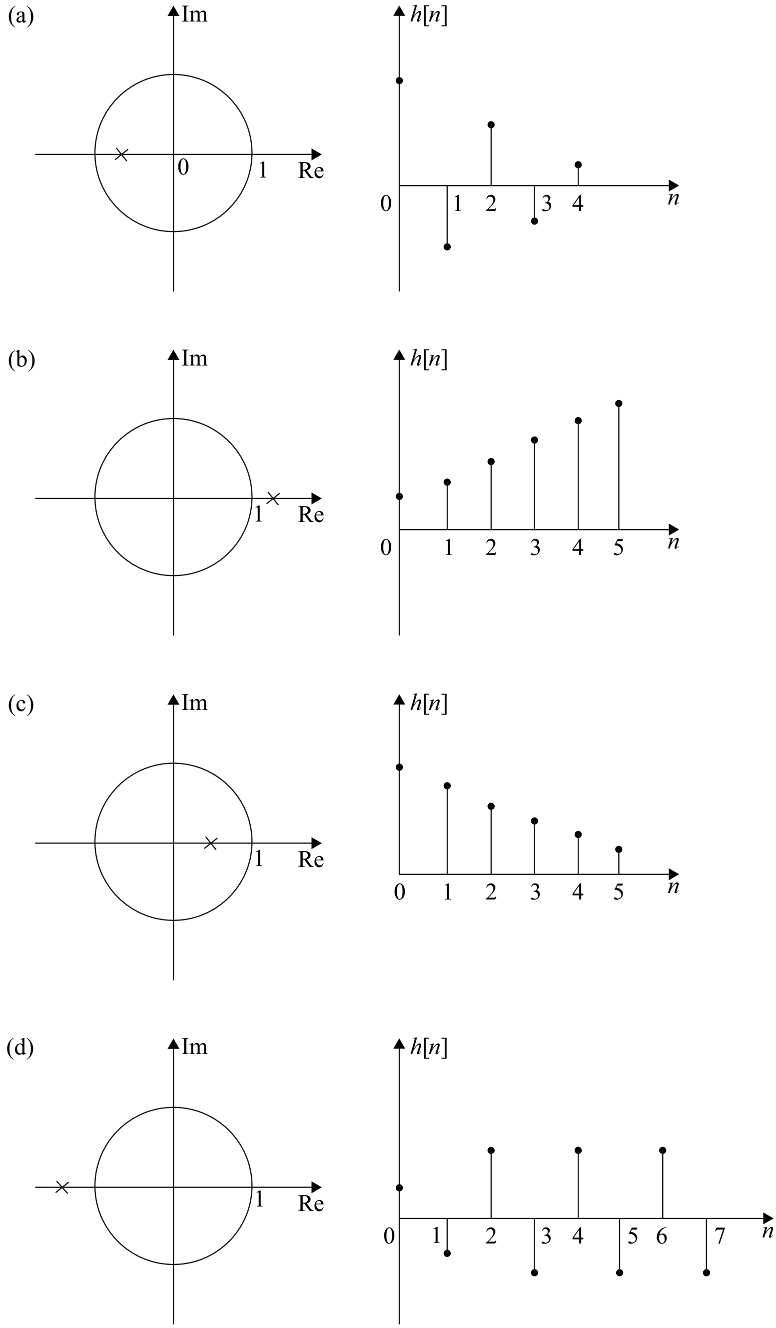


Fig. 5.15 Pole location and impulse response of a causal system

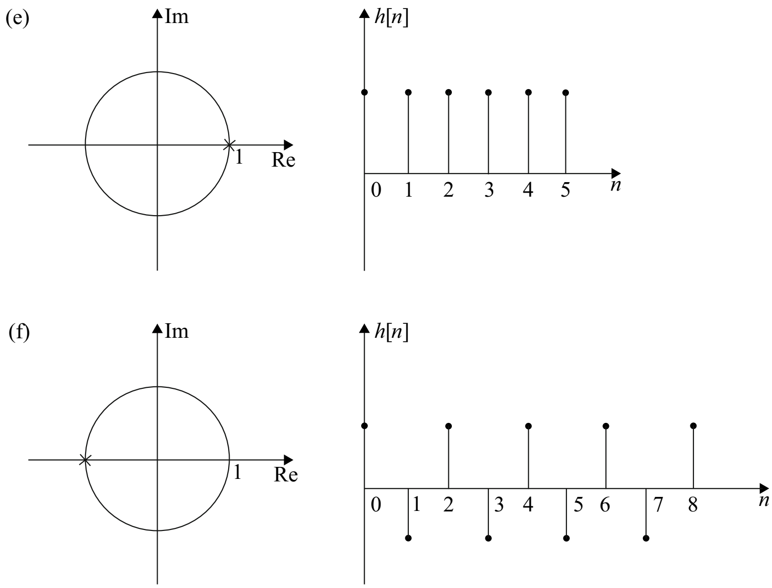


Fig. 5.15 (continued)

Solution

$$X[z] = -\frac{z}{(z-1)} + \frac{z}{(z-0.5)}$$
$$= \frac{-0.5z}{(z-1)(z-0.5)}$$

$$Y[z] = \frac{-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1+z^{-1})}$$
$$= \frac{-\frac{1}{2}z}{(z-0.5)(z+1)}$$
$$H[z] = \frac{Y[z]}{X[z]}$$
$$= \frac{(-0.5)z(z-1)(z-0.5)}{(z-0.5)(z+1)(-0.5)z}$$
$$= \frac{(z-1)}{(z+1)}$$

$$\begin{aligned}\frac{H[z]}{z} &= \frac{(z-1)}{z(z+1)} \\ &= \frac{A_1}{z} + \frac{A_2}{z+1} \\ (z-1) &= A_1(z+1) + A_2 z\end{aligned}$$

Substitute $z = 0$

$$-1 = A_1$$

Substitute $z = -1$

$$-2 = -A_2; \qquad A_2 = 2$$

$$H[z] = -1 + \frac{2z}{(z+1)}$$

$$h[n] = -\delta[n] + (-1)^n 2u[n]$$

$$\begin{aligned}Y[z] &= \frac{-\frac{1}{2}z}{(z-0.5)(z+1)} \\ \frac{Y[z]}{z} &= \frac{-\frac{1}{2}}{(z-0.5)(z+1)} \\ &= \frac{A_1}{z-0.5} + \frac{A_2}{z+1} \\ -\frac{1}{2} &= A_1(z+1) + A_2(z-0.5)\end{aligned}$$

Substitute $z = 0.5$

$$-\frac{1}{2} = \frac{3}{2}A_1; \qquad A_1 = -\frac{1}{3}$$

Substitute $z = -1$

$$-\frac{1}{2} = -\frac{3}{2}A_2; \qquad A_2 = \frac{1}{3}$$

$$Y[z] = \frac{1}{3}\left[-\frac{1}{(z-0.5)} + \frac{1}{(z+1)}\right]$$

$$y[n] = \frac{1}{3}\left[-\left(\frac{1}{2}\right)^n + (-1)^n\right]u[n]$$

Example 5.21 A certain LTI system is described by the following system function:

$$H[z] = \frac{\left(z+\frac{1}{2}\right)}{(z-1)\left(z-\frac{1}{2}\right)}$$

Find the system response to the input $x[n] = 4^{-(n+2)}u[n]$.

Solution

$$\begin{aligned}
x[n] &= 4^{-(n+2)}u[n] \\
&= \frac{1}{16}(4)^{-n}u[n] \\
X[z] &= \frac{1}{16}\frac{z}{\left(z-\frac{1}{4}\right)} \\
Y[z] &= H[z]X[z] \\
&= \frac{1\left(z+\frac{1}{2}\right)z}{16(z-1)\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} \\
\frac{Y[z]}{z} &= \frac{\left(z+\frac{1}{2}\right)}{16(z-1)\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} \\
&= \frac{A_1}{z-1} + \frac{A_2}{\left(z-\frac{1}{2}\right)} + \frac{A_3}{\left(z-\frac{1}{4}\right)}
\end{aligned}$$

$$\frac{1}{16}\left(z+\frac{1}{2}\right) = A_1\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right) + A_2(z-1)\left(z-\frac{1}{4}\right) + A_3(z-1)\left(z-\frac{1}{2}\right)$$

Substitute $z = 1$

$$\left(\frac{1}{16}\right)\left(\frac{3}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)A_1; \qquad A_1 = \frac{1}{4}$$

Substitute $z = \frac{1}{2}$

$$\frac{1}{16} = -\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)A_2; \qquad A_2 = -\frac{1}{2}$$

Substitute $z = \frac{1}{4}$

$$\frac{1}{16}\frac{3}{4} = -\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)A_3; \qquad A_3 = \frac{1}{4}$$

$$Y[z] = \left[\frac{1}{4}\frac{z}{(z-1)} - \frac{1}{2}\frac{z}{\left(z-\frac{1}{2}\right)} + \frac{1}{4}\frac{z}{\left(z-\frac{1}{4}\right)}\right]$$

$$y[n] = \left[\frac{1}{4}(1)^n - \frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{1}{4}\left(\frac{1}{4}\right)^n\right]u[n]$$

Example 5.22 Given

$$x[n] = \{2,\ -3,\ 1\}$$
$$h[n] = \{1,\ 2,\ -1\}$$

Find $y[n]$ using z-transform.

Solution

$$\begin{aligned} X[z] &= (2 - 3z^{-1} + z^{-2}) \\ H[z] &= 1 + 2z^{-1} - z^{-2} \\ Y[z] &= X[z]H[z] \\ &= [2 - 3z^{-1} + z^{-2}][1 + 2z^{-1} - z^{-2}] \\ &= 2 + z^{-1} - 7z^{-2} + 5z^{-3} - z^{-4} \end{aligned}$$

$$y[n] = \{2,\ 1,\ -7,\ 5,\ -1\}$$

Example 5.23 Given

$$x[n] = u[n]$$
$$y[n] = (2)^n u[n]$$

Find the system function and the impulse response.

Solution

$$\begin{aligned}
x[n] &= u[n] \\
X[z] &= \frac{z}{(z-1)} \qquad |z| > 1 \\
y[n] &= (2)^n u[n] \\
Y[z] &= \frac{z}{(z-2)} \qquad |z| > 2 \\
H[z] &= \frac{Y[z]}{X[z]} = \frac{(z-1)}{(z-2)} \qquad |z| > 2 \\
\frac{H[z]}{z} &= \frac{(z-1)}{z(z-2)} \\
&= \frac{A_1}{z} + \frac{A_2}{(z-2)} \\
z - 1 &= A_1(z-2) + A_2 z
\end{aligned}$$

Substitute $z = 0$

$$-1 = A_1(-2); \qquad A_1 = \frac{1}{2}$$

Substitute $z = 2$

$$1 = 2A_2; \qquad A_2 = \frac{1}{2}$$

$$H[z] = \frac{1}{2}\left[1 + \frac{z}{(z-2)}\right]$$

$$y[n] = \frac{1}{2}\left[\delta(n) + (2)^n u[n]\right]$$

Example 5.24 Given

$$\begin{aligned}
y[n] &= \left(\frac{1}{4}\right)^n u[n] \\
x[n] &= \left(\frac{1}{2}\right)^n u[-n-1]
\end{aligned}$$

Find the system function and hence the system impulse response.

Solution

$$y[n] = \left(\frac{1}{4}\right)^n u[n]$$
$$Y[z] = \frac{z}{\left(z - \frac{1}{4}\right)} \qquad |z| > \frac{1}{4}$$
$$x[n] = \left(\frac{1}{2}\right)^n u[-n-1]$$
$$X[z] = -\frac{z}{\left(z - \frac{1}{2}\right)} \qquad |z| < \frac{1}{2}$$
$$H[z] = \frac{Y[z]}{X[z]}$$

$$H[z] = \frac{-\left(z - \frac{1}{2}\right)}{\left(z - \frac{1}{4}\right)}$$

$$\frac{H[z]}{z} = \frac{-\left(z - \frac{1}{2}\right)}{z\left(z - \frac{1}{4}\right)}$$
$$= \frac{A_1}{z} + \frac{A_2}{z\left(z - \frac{1}{4}\right)}$$
$$\frac{1}{2} - z = A_1\left(z - \frac{1}{4}\right) + A_2 z$$

Substitute $z = 0$

$$\frac{1}{2} = -\frac{1}{4}A_1; \qquad A_1 = -2$$

Substitute $z = \frac{1}{4}$

$$\frac{1}{2} = \frac{1}{4}A_2; \qquad A_2 = 2$$

$$H[z] = 2\left[-1 + \frac{z}{\left(z - \frac{1}{4}\right)}\right]$$

$$h[n] = 2\left[-\delta[n] + \left(\frac{1}{4}\right)^n\right]u[n]$$

Example 5.25 Consider the following system functions:

$$\text{(a)} \qquad H[z] = \frac{(1 + 4z^{-1} + z^{-2})}{(2z^{-1} + 5z^{-2} + z^{-3})}$$

$$\text{(b)} \qquad H[z] = \frac{(z-1)(z+2)}{\left(z - \frac{1}{2}\right)\left(z - \frac{3}{4}\right)} \qquad \text{ROC: } |z| > \frac{3}{4}$$

$$\text{(c)} \qquad H[z] = \frac{(z-1)(z+2)}{\left(z - \frac{1}{2}\right)\left(z - \frac{3}{4}\right)} \qquad \text{ROC: } |z| < \frac{1}{2}$$

Determine whether these systems are causal or not.

Solution (a) $\boldsymbol{H[z] = \frac{(1+4z^{-1}+z^{-2})}{(2z^{-1}+5z^{-2}+z^{-3})}}$

$$H[z] = \frac{(z^3 + 4z^2 + z)}{(2z^2 + 5z + 1)}$$

$H[z]$ is irrational since the degree of the numerator polynomial is greater than the denominator polynomial.

The System is Non-causal.

(b) $\boldsymbol{H[z] = \frac{(z-1)(z+2)}{(z-\frac{1}{2})(z-\frac{3}{4})}}$**; ROC:** $\boldsymbol{|z| > \frac{3}{4}}$
The ROC is the exterior of the circle passing through the outermost pole of $H[z]$. Hence $h[n]$, the impulse response is right sided.

The System is Causal.

(c) $\boldsymbol{H[z] = \frac{(z-1)(z+2)}{(z-\frac{1}{2})(z-\frac{3}{4})}}$ **ROC :** $\boldsymbol{|z| < \frac{1}{2}}$
The ROC is the interior of the circle passing through the innermost pole of $H[z]$. Hence $h[n]$, the impulse response is left sided.

The System is Non-causal.

Example 5.26 Consider the following system function:

$$H[z] = \frac{\left(2 - \frac{13}{4}z^{-1}\right)}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - 3z^{-1}\right)}$$

Determine the causality and stability of the system for the following cases.

(a) ROC: $|z| > 3$;
(b) ROC: $|z| < \frac{1}{4}$;
(c) ROC: $\frac{1}{4} < |z| < 3$.

Solution

$$H[z] = \frac{\left(2 - \frac{13}{4}z^{-1}\right)}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - 3z^{-1}\right)}$$
$$= \frac{z\left(2z - \frac{13}{4}\right)}{\left(z - \frac{1}{4}\right)(z - 3)}$$

(a) **ROC: $|z| > 3$**
The ROC is the exterior of the circle passing through the outermost pole of $H[z]$ which is rational (the denominator and numerator polynomials have same order). The impulse response $h[n]$ is a right-sided sequence. Hence, $H[z]$ is causal. The ROC does not contain unit circle. Hence, $h[n]$ is not summable. The system is unstable. Refer to Fig. 5.16a.

The System is Causal and Unstable.

(b) **ROC: $|z| < \frac{1}{4}$**
The ROC is the interior of the circle passing through the innermost pole of $H[z]$. The impulse response is a left-sided sequence. $H[z]$ is therefore non-causal. The ROC does not include the unit circle. The $h[n]$ is growing exponential negative sequence. The system is unstable. Refer to Fig. 5.16b.

The System is Non-causal and Unstable.

(c) **ROC: $\frac{1}{4} < |z| < 3$**
The ROC is to the left of the outermost pole and to the right innermost pole. Hence $h[n]$ will have right and left-sided sequences, which is non-causal. The ROC includes unit circle, which means that the right and left side sequences of $h[n]$ will exponentially decay and the system is stable. Refer to Fig. 5.16c.

The System is Non-causal and Stable.

The system cannot be both Causal and Stable.

Example 5.27 Consider the following system function:

$$H[z] = \frac{z}{\left(z - \frac{1}{4}\right)\left(z + \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}$$

For different possible ROCs, determine the causality, stability, and the impulse response of the system.

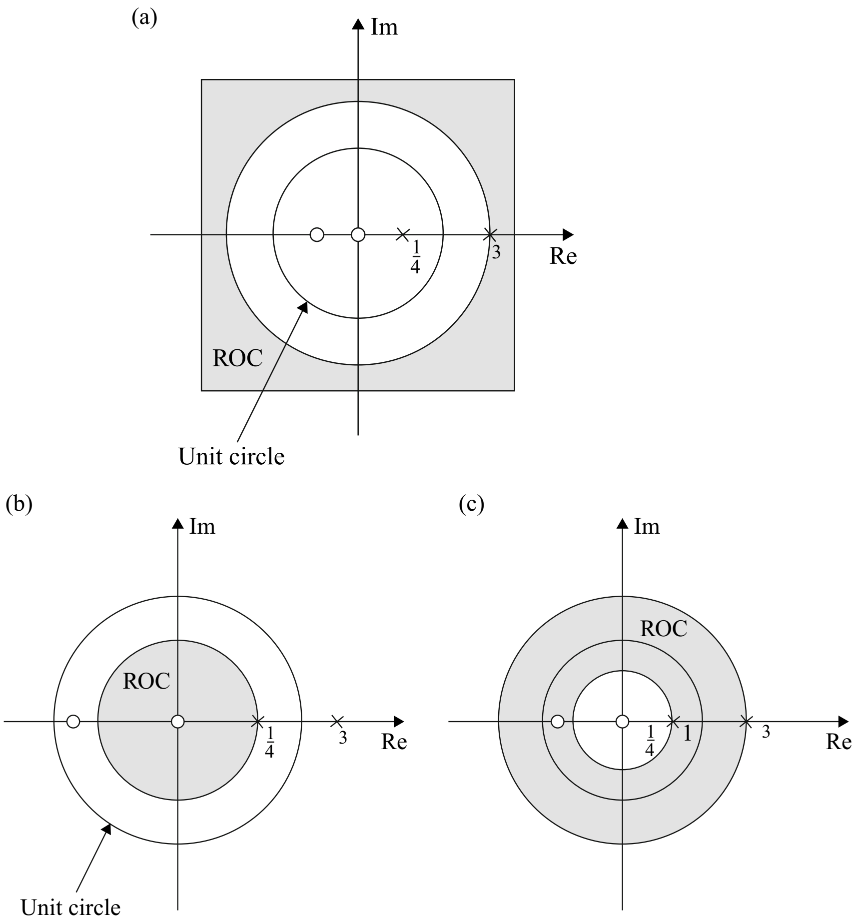


Fig. 5.16 **a** Causal and stable system. **b** Non-causal and unstable system and **c** Non-causal and stable system

Solution

$$H[z] = \frac{z}{\left(z - \frac{1}{4}\right)\left(z + \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}$$

The possible ROCs for $H[z]$ to exist are (a) ROC: $|z| > \frac{1}{2}$, (b) ROC: $|z| < \frac{1}{4}$ and (c) ROC: $\frac{1}{4} < |z| < \frac{1}{2}$.

$$\frac{H[z]}{z} = \frac{1}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}$$
$$= \frac{A_1}{\left(z-\frac{1}{4}\right)} + \frac{A_2}{\left(z+\frac{1}{4}\right)} + \frac{A_3}{\left(z-\frac{1}{2}\right)}$$
$$1 = A_1\left(z+\frac{1}{4}\right)\left(z-\frac{1}{2}\right) + A_2\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right) + A_3\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)$$

Substitute $z = \frac{1}{4}$

$$1 = A_1\left(\frac{1}{4}+\frac{1}{4}\right)\left(\frac{1}{4}-\frac{1}{2}\right); \qquad A_1 = -8$$

Substitute $z = -\frac{1}{4}$

$$1 = A_2\left(-\frac{1}{4}-\frac{1}{4}\right)\left(-\frac{1}{4}-\frac{1}{2}\right); \qquad A_2 = \frac{8}{3}$$

Substitute $z = \frac{1}{2}$

$$1 = A_3\left(\frac{1}{2}-\frac{1}{4}\right)\left(\frac{1}{2}+\frac{1}{4}\right); \qquad A_3 = \frac{16}{3}$$

$$H[z] = -\frac{8z}{\left(z-\frac{1}{4}\right)} + \frac{8}{3}\frac{z}{\left(z+\frac{1}{4}\right)} + \frac{16}{3}\frac{z}{\left(z-\frac{1}{2}\right)}$$

(a) **ROC: $|z| > \frac{1}{2}$**
The pole-zero diagram and the ROC are shown in Fig. 5.17a. From Fig. 5.17a the ROC is the exterior of the outermost pole $z = \frac{1}{2}$. Further, ROC includes unit circle. Thus $h[n]$ is a right-sided sequence and hence $H[z]$ is causal. Since ROC includes unit circle and all the poles are within unit circle, the system is stable. Now,

$$H[z] = -\frac{8z}{\left(z-\frac{1}{4}\right)} + \frac{8}{3}\frac{z}{\left(z+\frac{1}{4}\right)} + \frac{16}{3}\frac{z}{\left(z-\frac{1}{2}\right)}$$

$$h[n] = \left[-8\left(\frac{1}{4}\right)^n + \frac{8}{3}\left(-\frac{1}{4}\right)^n + \frac{16}{3}\left(\frac{1}{2}\right)^n\right]u[n]$$

The System is Causal and Stable.

(b) **ROC: $|z| < \frac{1}{4}$**
For ROC: $|z| < \frac{1}{4}$, the pole-zero diagram is shown in Fig. 5.17b. The ROC is interior of the circle passing through the innermost pole. Hence, the system is non-causal. The condition that the ROC does not include unit circle implies that the system is unstable. The sequence $h[n]$ is left sided. This is obtained as follows.

$$H[z] = -\frac{8z}{\left(z - \frac{1}{4}\right)} + \frac{8}{3}\frac{z}{\left(z + \frac{1}{4}\right)} + \frac{16}{3}\frac{z}{\left(z - \frac{1}{2}\right)}$$

$$h[n] = \left[8\left(\frac{1}{4}\right)^n - \frac{8}{3}\left(-\frac{1}{4}\right)^n - \frac{16}{3}\left(\frac{1}{2}\right)^n\right]u[-n-1]$$

The left-sided sequence $u[-n-1]$ will exponentially increase for $n < 0$ and makes the system unstable.

The System is Non-causal and Unstable.

(c) **ROC: $\frac{1}{4} < |z| < \frac{1}{2}$** The pole-zero diagram and ROC of $H[z]$ are shown in Fig. 5.17c. The ROC is concentric ring for $\frac{1}{4} < |z| < \frac{1}{2}$. The $h[n]$ sequences die to the poles at $z = \frac{1}{4}$ and $z = -\frac{1}{4}$ are right sided and the sequence due to the pole $z = \frac{1}{2}$ is left sided. Hence the system is non-causal. The ROC does not include the unit circle and hence the system is unstable. The impulse response is obtained as follows.

$$H[z] = -\frac{8z}{\left(z - \frac{1}{4}\right)} + \frac{8}{3}\frac{z}{\left(z + \frac{1}{4}\right)} + \frac{16}{3}\frac{z}{\left(z - \frac{1}{2}\right)}$$

$$h[n] = \left[-8\left(\frac{1}{4}\right)^n + \frac{8}{3}\left(-\frac{1}{4}\right)^n\right]u[n] - \frac{16}{3}\left(\frac{1}{2}\right)^n u[-n-1]$$

The term $-\frac{16}{3}(1/2)^n u[-n-1]$ for $n < 0$ yields exponentially increasing sequence.

The System is Non-causal and Unstable.

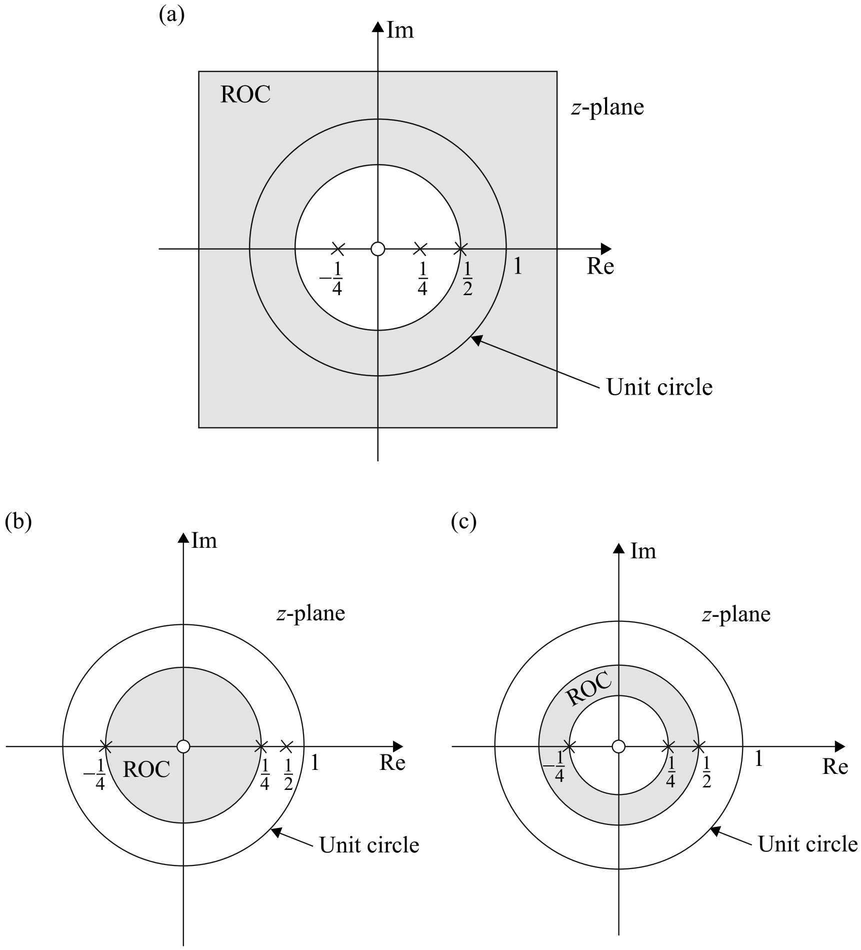


Fig. 5.17 **a** Pole-zero diagram and ROC: $|z| > \frac{1}{2}$ of Example 5.27. **b** Pole zero diagram and ROC: $|z| < \frac{1}{4}$ and **c** Pole-zero diagram and ROC: $\frac{1}{4} < |z| < \frac{1}{2}$

5.13 *z*-Transform Solution of Linear Difference Equations

As in the case of Laplace transform with differential equation, to get the solution in time domain *z*-transform is used to solve difference equation to get the output sequence as a function of n. By using the time shift property of *z*-transform, the difference equation is converted into algebraic equation, taking into account the initial conditions. by taking z-inverse transform, the time domain solution is obtained.

5.13.1 Right Shift (Delay)

If

$$x[n]u[n] \overset{Z}{\longleftrightarrow} X[z]$$

then

$$x[n-1]u[n-1] \overset{Z}{\longleftrightarrow} \frac{1}{z}X[z]$$
$$x[n-1]u[n] \overset{Z}{\longleftrightarrow} \frac{1}{z}X[z] + x[-1]$$
$$x[n-2]u[n] \overset{Z}{\longleftrightarrow} \frac{1}{z^2}X[z] + \frac{1}{z}x[-1] + x[-2]$$

In general,

$$x[n-m]u[n] \overset{Z}{\longleftrightarrow} z^{-m}X[z] + z^{-m}\sum_{n=1}^{m} x[-n]z^{n} \tag{5.56}$$

5.13.2 Left Shift (Advance)

If

$$x[n]u[n] \overset{Z}{\longleftrightarrow} X[z]$$
$$x[n+1]u[n] \overset{Z}{\longleftrightarrow} zX[z] - zx[0]$$
$$x[n+2]u[n] \overset{Z}{\longleftrightarrow} z^2X[z] - z^2x[0] - zx[1]$$

In general,

$$x[n+m]u[n] \overset{Z}{\longleftrightarrow} z^{m}x[z] - z^{m}\sum_{n=0}^{m-1} x[n]z^{-n} \tag{5.57}$$

Equations (5.56) and (5.57) are used to convert difference equations with initial conditions to algebraic equations in z. Application of Eq. (5.56), the delay shift is more common. The following examples illustrate the above procedure.

Example 5.28 Consider the following linear constant coefficient difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n-1]$$

Determine $y[n]$ when $x[n] = \delta[n]$ and $y[n] = 0,\ n < 0$.

(*Anna University, May and December, 2007*)

Solution If $y[n] = 0,\ n = 0$ implies the initial conditions are zero. Taking z-transform on both sides of the given equation we get

$$\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] Y[z] = 2z^{-1}X[z]$$

For $\delta[n],\ X[z] = 1$

$$Y[z] = \frac{2z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{2z}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

$$\frac{Y[z]}{z} = \frac{2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}$$

$$= \frac{A_1}{\left(z - \frac{1}{2}\right)} + \frac{A_2}{\left(z - \frac{1}{4}\right)}$$

$$2 = A_1\left(z - \frac{1}{4}\right) + A_2\left(z - \frac{1}{2}\right)$$

Substitute $z = \frac{1}{2}$

$$2 = A_1\frac{1}{4}; \qquad A_1 = 8$$

Substitute $z = \frac{1}{4}$

$$2 = A_2\left(-\frac{1}{4}\right); \qquad A_2 = -8$$

$$Y[z] = 8\left[\frac{z}{\left(z - \frac{1}{2}\right)} - \frac{z}{\left(z - \frac{1}{4}\right)}\right]$$

$$y[n] = 8\left[\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u[n] \qquad \text{ROC: } |z| > \frac{1}{2}$$

Example 5.29

$$y[n+2] + 1.1y[n+1] + 0.3y[n] = x[n+1] + x[n]$$

where $x[n] = (-4)^{-n}u[n]$. Find $y[n]$ if the initial condition is zero.

Solution Taking z-transform using left shift property we get

$$\begin{aligned}[z^2 + 1.1z + 0.3]Y[z] &= [z+1]X[z] \\ x[n] &= (-4)^{-n}u[n]\end{aligned}$$

$$\begin{aligned}X[z] &= \frac{z}{\left(z+\frac{1}{4}\right)} \\ Y[z] &= \frac{z(z+1)}{\left(z+\frac{1}{4}\right)(z^2+1.1z+0.3)} \\ &= \frac{z(z+1)}{\left(z+\frac{1}{4}\right)(z+0.5)(z+0.6)} \\ \frac{Y[z]}{z} &= \frac{(z+1)}{\left(z+\frac{1}{4}\right)(z+0.5)(z+0.6)} \\ &= \frac{A_1}{\left(z+\frac{1}{4}\right)} + \frac{A_2}{(z+0.5)} + \frac{A_3}{(z+0.6)} \\ (z+1) &= A_1(z+0.5)(z+0.6) + A_2\left(z+\frac{1}{4}\right)(z+0.6) \\ &\quad + A_3\left(z+\frac{1}{4}\right)(z+0.5)\end{aligned}$$

Substitute $z = -\frac{1}{4}$

$$\left(-\frac{1}{4}+1\right) = A_1\left(-\frac{1}{4}+0.5\right)\left(-\frac{1}{4}+0.6\right); \qquad A_1 = 8.57$$

Substitute $z = -0.5$

$$(-0.5+1) = A_2\left(-0.5+\frac{1}{4}\right)(-0.5+0.6); \qquad A_2 = -20$$

Substitute $z = -0.6$

$$(-0.6+1) = A_3\left(-0.6+\frac{1}{4}\right)(-0.6+0.5); \qquad A_3 = 11.43$$

$$Y[z] = \frac{8.57z}{\left(z+\frac{1}{4}\right)} - \frac{20z}{(z+0.5)} + \frac{11.43}{(z+0.6)}$$

$$y[n] = \left[8.57\left(-\frac{1}{4}\right)^n - 20\,(-0.5)^n + 11.43(-0.6)^n\right]u[n]$$

Example 5.30 A causal LTI system is described by the difference equation

$$y[n] = y[n-1] + y[n-2] + x[n-1]$$

Find (a) System function for this system and (b) Unit impulse response of the system.
(*Anna University, April, 2008*)

Solution Taking z-transform on both sides of the equation and making use of right shift property we get

$$[1 - z^{-1} - z^{-2}]Y[z] = z^{-1}X[z]$$

(a)

$$H[z] = \frac{Y[z]}{X[z]}$$

$$H[z] = \frac{z^{-1}}{(1 - z^{-1} - z^{-2})}$$

(b)

$$\begin{aligned} H[z] &= \frac{z}{(z^{-2} - z - 1)} \\ \frac{H[z]}{z} &= \frac{1}{(z-1.618)(z+0.618)} \\ &= \frac{A_1}{(z-1.618)} + \frac{A_2}{(z+1.618)} \\ 1 &= A_1(z+0.618) + A_2(z-1.618) \end{aligned}$$

Substitute $z = 1.618$

$$1 = A_1\,(1.618 + 0.618)\,; \qquad A_1 = 0.447$$

Substitute $z = -0.618$

$$1 = A_2\,(-0.618 - 1.618)\,; \qquad A_2 = -0.447$$

$$H[z] = 0.447\left[\frac{z}{z-1.618} - \frac{z}{z+0.618}\right]$$

$$h[n] = 0.447\left[(1.618)^n - (-0.618)^n\right]u[n]$$

Example 5.31 Find the impulse response of the discrete-time system described by the difference equation

$$y[n-2] - 3y[n-1] + 2y[n] = x[n-1]$$

(*Anna University, April, 2005*)

Solution

$$\begin{aligned}
[z^{-2} - 3z^{-1} + 2]Y[z] &= z^{-1}X[z] \\
H[z] &= \frac{Y[z]}{X[z]} \\
&= \frac{z^{-1}}{(z^{-2} - 3z^{-1} + 2)} \\
&= \frac{z}{(2z^2 - 3z + 1)} \\
\frac{H[z]}{z} &= \frac{0.5}{(z-1)(z-0.5)} \\
&= \frac{1}{(z-1)} - \frac{1}{(z-0.5)} \\
H[z] &= \frac{z}{z-1} - \frac{z}{z-0.5}
\end{aligned}$$

$$h[n] = \left[(1)^n - \left(\frac{1}{2}\right)^n\right]u[n]$$

Example 5.32 Determine the impulse response and frequency response of the system described by the difference equation

$$y[n] - \left(\frac{1}{6}\right)y[n-1] - \frac{1}{6}y[n-2] = x[n-1]$$

(*Anna University, May, 2007*)

Solution To obtain Impulse Response

$$\left[1-\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}\right]Y[z]=z^{-1}X[z]$$

$$H[z]=\frac{Y[z]}{X[z]}$$

$$=\frac{z}{\left(z^2-\frac{1}{6}z-\frac{1}{6}\right)}$$

$$=\frac{z}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

$$\frac{H[z]}{z}=\frac{1}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

$$=\frac{6}{5}\left[\frac{1}{\left(z-\frac{1}{2}\right)}-\frac{1}{\left(z+\frac{1}{3}\right)}\right]$$

$$H[z]=\frac{6}{5}\left[\frac{z}{\left(z-\frac{1}{2}\right)}-\frac{1}{\left(z+\frac{1}{3}\right)}\right]$$

$$h[n]=\frac{6}{5}\left[\left(\frac{1}{2}\right)^n-\left(\frac{1}{3}\right)^n\right]u[n]$$

To obtain Frequency Response
Substitute $z=e^{j\omega}$ in $H[z]$

$$H[e^{j\omega}]=\frac{e^{j\omega}}{\left(e^{j\omega}-\frac{1}{2}\right)\left(e^{j\omega}+\frac{1}{3}\right)}$$

This can be expressed in terms of amplitude and phase as follows:

$$H[e^{j\omega}]=\frac{e^{j\omega}}{\left(\cos\omega+j\sin\omega-\frac{1}{2}\right)\left(\cos\omega+j\sin\omega+\frac{1}{3}\right)}$$

since $|e^{j\omega}|=1$

$$|H(e^{j\omega})|=\frac{1}{\left[\left\{\left(\cos\omega-\frac{1}{2}\right)^2+\sin^2\omega\right\}\left\{\left(\cos\omega+\frac{1}{3}\right)^2+\sin^2\omega\right\}\right]^{\frac{1}{2}}}$$

Since $\angle e^{j\omega} = \omega$

$$\angle H(e^{j\omega}) = \omega - \tan^{-1}\frac{\sin\omega}{\left(\cos\omega - \frac{1}{2}\right)} - \tan^{-1}\frac{\sin\omega}{\left(\cos\omega + \frac{1}{3}\right)}$$

Example 5.33 A causal system is represented by the difference equation

$$y[n] + \frac{1}{4}y[n-1] = x[n] + \frac{1}{2}x[n-1]$$

use z-transform to determine the

(1) System function;
(2) Unit sample response of the system;
(3) Frequency response of the system.

Solution (1)

$$\left[1 + \frac{1}{4}z^{-1}\right]Y[z] = \left[1 + \frac{1}{2}z^{-1}\right]X[z]$$

$$H[z] = \frac{Y[z]}{X[z]}$$

$$H[z] = \frac{\left[1 + \frac{1}{2}z^{-1}\right]}{\left[1 + \frac{1}{4}z^{-1}\right]}$$

(2)

$$H[z] = \frac{\left(z + \frac{1}{2}\right)}{\left(z + \frac{1}{4}\right)}$$

$$\frac{H[z]}{z} = \frac{\left(z + \frac{1}{2}\right)}{z\left(z + \frac{1}{4}\right)} = \frac{A_1}{z} + \frac{A_2}{\left(z + \frac{1}{4}\right)}$$

$$\left(z + \frac{1}{2}\right) = A_1\left(z + \frac{1}{4}\right) + A_2 z$$

Substitute $z = 0$

$$A_1 = 2$$

Substitute $z = -\frac{1}{4}$

$$\left[-\frac{1}{4}+\frac{1}{2}\right] = A_2\left(-\frac{1}{4}\right); \qquad A_2 = -1$$

$$H[z] = 2 - \frac{z}{\left(z+\frac{1}{4}\right)}$$

$$h[n] = 2\delta[n] - \left(\frac{1}{4}\right)^n u[n]$$

(3)

$$H[z] = \frac{\left(z+\frac{1}{2}\right)}{\left(z+\frac{1}{4}\right)}$$

$$H[e^{j\omega}] = \frac{\left(e^{j\omega}+\frac{1}{2}\right)}{\left(e^{j\omega}+\frac{1}{4}\right)} = \frac{\left(\cos\omega+\frac{1}{2}\right)+j\sin\omega}{\left(\cos\omega+\frac{1}{4}\right)+j\sin\omega}$$

$$|H(e^{j\omega})| = \frac{\left[\left(\cos\omega+\frac{1}{2}\right)^2+\sin^2\omega\right]^{1/2}}{\left[\left(\cos\omega+\frac{1}{4}\right)^2+\sin\omega\right]^{1/2}}$$

$$\angle H(j\omega) = \tan^{-1}\frac{\sin\omega}{\left(\cos\omega+\frac{1}{2}\right)} - \tan^{-1}\frac{\sin\omega}{\left(\cos\omega+\frac{1}{4}\right)}$$

Example 5.34 Find the output of the system whose input-output is related by the difference equation

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] - \frac{1}{2}x[n-1]$$

for the step input. Assume initial conditions to be zero.

Solution

$$\left[1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}\right]Y[z] = \left[1-\frac{1}{2}z^{-1}\right]X[z]$$

$$Y[z] = \frac{\left[1-\frac{1}{2}z^{-1}\right]}{\left[1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}\right]}X[z]$$

For unit step input, $X[z] = \frac{z}{z-1}$

$$Y[z] = \frac{z^2\left[z - \frac{1}{2}\right]}{(z-1)\left(z^2 - \frac{5}{6}z + \frac{1}{6}\right)}$$

$$\frac{Y[z]}{z} = \frac{z\left[z - \frac{1}{2}\right]}{(z-1)\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$$

$$= \frac{z}{(z-1)\left(z - \frac{1}{3}\right)}$$

$$= \frac{A_1}{(z-1)} + \frac{A_2}{\left(z - \frac{1}{3}\right)}$$

$$z = A_1\left(z - \frac{1}{3}\right) + A_2(z-1)$$

Substrate $z = 1$

$$1 = A_1\left(1 - \frac{1}{3}\right); \qquad A_1 = \frac{3}{2}$$

Substrate $z = \frac{1}{3}$

$$\frac{1}{3} = A_2\left(\frac{1}{3} - 1\right); \qquad A_2 = -\frac{1}{2}$$

$$Y[z] = \frac{3}{2}\frac{z}{(z-1)} - \frac{1}{2}\frac{z}{\left(z - \frac{1}{3}\right)}$$

$$y[n] = \left[\frac{3}{2}(1)^n - \frac{1}{2}\left(\frac{1}{3}\right)^n\right]u[n]$$

Example 5.35 Find the output response of the discrete-time system described by the following difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$$

the initial conditions are $y[-1] = 0$ and $y = [-2] = 1$. The input $x[n] = \left(\frac{1}{5}\right)^n u[n]$.

Solution Taking z-transform on both sides of the above equation we get

$$
\begin{aligned}
Y[z]-\frac{3}{4}[z^{-1}Y[z]+y[-1]]+\frac{1}{8}[z^{-2}Y[z]\\
+z^{-1}y[-1]+y[-2]]=X[z]\\
\left[1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right]Y[z]=-\frac{1}{8}+\frac{z}{\left(z-\frac{1}{5}\right)}\\
\frac{\left[z^2-\frac{3}{4}z+\frac{1}{8}\right]}{z^2}Y[z]=-\frac{1}{8}+\frac{z}{\left(z-\frac{1}{5}\right)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{Y[z]}{z}&=-\frac{z}{8\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}+\frac{z^2}{\left(z-\frac{1}{5}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}\\
&=Y_1[z]+Y_2[z]\\
Y_1[z]&=-\frac{z}{8\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}\\
&=-\frac{A_1}{\left(z-\frac{1}{4}\right)}+\frac{A_2}{\left(z-\frac{1}{2}\right)}\\
-\frac{z}{8}&=A_1\left(z-\frac{1}{2}\right)+A_2\left(z-\frac{1}{4}\right)
\end{aligned}
$$

Substitute $z=\frac{1}{4}$

$$-\frac{1}{4}\frac{1}{8}=-A_1\frac{1}{4};\qquad A_1=\frac{1}{8}$$

Substrate $z=\frac{1}{2}$

$$-\frac{1}{2}\frac{1}{8}=A_2\frac{1}{4};\qquad A_2=-\frac{1}{4}$$

$$
\begin{aligned}
Y_1[z]&=\frac{1}{8\left(z-\frac{1}{4}\right)}-\frac{1}{4\left(z-\frac{1}{2}\right)}\\
Y_2[z]&=\frac{z^2}{\left(z-\frac{1}{5}\right)\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}\\
z^2&=A_1\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)+A_2\left(z-\frac{1}{5}\right)\left(z-\frac{1}{2}\right)+A_3\left(z-\frac{1}{5}\right)\left(z-\frac{1}{4}\right)
\end{aligned}
$$

Substitute $z=\frac{1}{5}$

$$\frac{1}{25}=A_1\left(\frac{1}{5}-\frac{1}{4}\right)\left(\frac{1}{5}-\frac{1}{2}\right);\qquad A_1=\frac{8}{3}$$

Substitute $z = \frac{1}{4}$

$$\frac{1}{16} = A_2\left(\frac{1}{4} - \frac{1}{5}\right)\left(\frac{1}{4} - \frac{1}{2}\right); \qquad A_2 = -5$$

Substitute $z = \frac{1}{2}$

$$\frac{1}{4} = A_3\left(\frac{1}{2} - \frac{1}{5}\right)\left(\frac{1}{2} - \frac{1}{4}\right); \qquad A_3 = \frac{10}{3}$$

$$\begin{aligned}
Y_2[z] &= \frac{8}{3}\frac{1}{\left(z-\frac{1}{5}\right)} - \frac{5}{\left(z-\frac{1}{4}\right)} + \frac{10}{3}\frac{1}{\left(z-\frac{1}{2}\right)}\\
Y[z] &= \frac{z}{8\left(z-\frac{1}{4}\right)} - \frac{z}{4\left(z-\frac{1}{2}\right)} + \frac{8}{3}\frac{z}{\left(z-\frac{1}{5}\right)} - \frac{5z}{\left(z-\frac{1}{4}\right)} + \frac{10}{3}\frac{z}{\left(z-\frac{1}{2}\right)}\\
&= -\frac{39}{8}\frac{z}{\left(z-\frac{1}{4}\right)} + \frac{37}{12}\frac{z}{\left(z-\frac{1}{2}\right)} + \frac{8}{3}\frac{z}{\left(z-\frac{1}{5}\right)}
\end{aligned}$$

$$y[n] = \left[-\frac{39}{8}\left(\frac{1}{4}\right)^n + \frac{37}{12}\left(\frac{1}{2}\right)^n + \frac{8}{3}\left(\frac{1}{5}\right)^n\right]u[n]$$

Example 5.36 Consider the following difference equation

$$y[n] + 2y[n-1] + 2y[n-2] = x[n]$$

The initial conditions are $y[-1] = 0$ and $y = [-2] = 2$. Find the step response of the system.

Solution Taking z-transform on both sides of the above equation we get

$$\begin{aligned}
X[z] &= Y[z] + 2[z^{-1}Y[z] + y[-1]] + 2[z^{-2}Y[z] + z^{-1}y[-1] + y[-2]]\\
-4 + X[z] &= [1 + 2z^{-1} + 2z^{-2}]Y[z]\\
-4 + X[z] &= \frac{(z^2 + 2z + 2)}{z^2}Y[z]
\end{aligned}$$

For step input $X[z] = \frac{z}{(z-1)}$

$$
\begin{aligned}
z^2+2z+2 &= (z+1+j)(z+1-j)\\
\frac{(z+1+j)(z+1-j)}{z^2}Y[z] &= -4+\frac{z}{z-1}\\
&= \frac{(4-3z)}{z-1}\\
\frac{Y[z]}{z} &= \frac{z(4-3z)}{(z-1)(z+1+j)(z+1-j)}\\
&= \frac{A_1}{(z-1)}+\frac{A_2}{(z+1+j)}+\frac{A_3}{(z+1-j)}\\
z[4-3z] &= A_1(z^2+2z+2)+A_2(z-1)(z+1-j)\\
&\quad +A_3(z-1)(z+1+j)
\end{aligned}
$$

Substitute $z=1$

$$1=A_15;\qquad A_1=\frac{1}{5}$$

Substitute $z=-1+j$

$$
\begin{aligned}
(-1+j)(4-3+j3) &= A_3(-1+j-1)(-1+1+j+j)\\
(-1+j)(1+j3) &= A_3(-2+j)j2\\
\sqrt{2}\angle 135^\circ\sqrt{10}\angle 71.56^\circ &= A_3\sqrt{5}\angle 153.43^\circ\sqrt{2}\angle 90^\circ\\
A_3 &= \frac{\sqrt{2}\angle 135^\circ\sqrt{10}\angle 71.56^\circ}{\sqrt{5}\angle 153.43^\circ\sqrt{2}\angle 90^\circ}\\
&= 1\angle -36.87^\circ = 1e^{-j0.643}
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= \text{conjugate of } A_3\\
&= 1e^{j0.643}
\end{aligned}
$$

The exponentials of A_1 and A_2 are expressed in radians using $57.3^\circ = 1$ radian.

$$
\begin{aligned}
Y[z] &= \frac{1}{5}\frac{z}{(z-1)}+\frac{e^{j0.643}z}{z+1+j}+\frac{e^{-j0.643}z}{z+1-j}\\
Y[z] &= \frac{1}{5}\frac{z}{(z-1)}+\frac{e^{j0.643}z}{(z+\sqrt{2}e^{j\frac{\pi}{4}})}+\frac{e^{-j0.643}z}{(z+\sqrt{2}e^{-j\frac{\pi}{2}})}
\end{aligned}
$$

Taking inverse z-transform we get

$$y[n] = \frac{1}{5} + e^{j0.643}(-\sqrt{2}e^{j\frac{\pi}{4}})^n + e^{-j0.643}(-\sqrt{2}e^{-j\frac{\pi}{4}})^n$$
$$= \frac{1}{5} + (-\sqrt{2})^n \left[e^{j(0.643+\frac{\pi}{4}n)} + e^{-j(0.643+\frac{\pi}{4}n)}\right]$$

$$y[n] = \left[\frac{1}{5} + 2(-\sqrt{2})^n \cos\left(\frac{\pi}{4}n + 0.643\right)\right] u[n]$$

Example 5.37 Solve the following difference equation:

$$y[n] + 6y[n-1] + 8y[n-2] = 5x[n-1] + x[n-2]$$

The initial conditions are $y[-1] = 1$ and $y[-2] = 2$. The input $x[n] = u[n]$.

Solution Taking z-transform on both sides we get

$$1 + 6(z^{-1}Y[z] + y[-1]) + 8(z^2Y[z] + z^{-1}y[-1] + y[-2]) = [5z^{-1} + z^{-2}]X[z]$$

For a causal signal $u[n]$, $x[-2]$, $x[-1]$ are zero.

$$[1 + 6z^{-1} + 8z^{-2}]Y[z] + (6 + 8z^{-1} + 16) = [5z^{-1} + z^{-2}]\frac{z}{(z-1)}$$
$$\frac{(z+2)(z+4)}{z^2}Y[z] = -(22 + 8z^{-1}) + (5z^{-1} + z^{-2})\frac{z}{(z-1)}$$
$$= \frac{(-22z^2 + 19z + 9)}{z(z-1)}$$
$$\frac{Y[z]}{z} = \frac{(-22z^2 + 19z + 9)}{(z-1)(z+2)(z+4)}$$
$$= \frac{A_1}{(z-1)} + \frac{A_2}{(z+2)} + \frac{A_3}{(z+4)}$$
$$-22z^2 + 19z + 9 = A_1(z+2)(z+4) + A_2(z-1)(z+4) + A_3(z-1)(z+2)$$

Substitute $z = 1$

$$-22 + 19 + 9 = A_1(3)(5); \qquad A_1 = 0.4$$

Substitute $z = -2$

$$-88 - 38 + 9 = A_2(-3)(2); \qquad A_2 = 19.5$$

Substitute $z = -4$

$$-352 - 76 + 9 = A_3(-5)(-2); \qquad A_3 = -41.9$$

$$Y[z] = \frac{0.4z}{(z-1)} + 19.5\frac{z}{(z+2)} - 41.9\frac{z}{(z+4)}$$

$$y[n] = \left[0.4 + 19.5(-2)^n - 41.9(-4)^n\right]u[n]$$

Example 5.38 Find the response of the LTID system described by the following difference equation

$$y[n+2] + y[n+1] + 0.24y[n] = x[n+1] + 2x[n]$$

where $x[n] = (\frac{1}{2})^n u[n]$ and all the initial conditions are zero.

Solution When the initial conditions are zero.

$$\begin{aligned} y[n+2] &\overset{Z}{\longleftrightarrow} z^2Y[z] \\ y[n+1] &\overset{Z}{\longleftrightarrow} zY[z] \\ x[n+1] &\overset{Z}{\longleftrightarrow} zX[z] \\ \left(\frac{1}{2}\right)^2 u[n] &\overset{Z}{\longleftrightarrow} \frac{z}{(z-0.5)} \end{aligned}$$

The given difference equation can be written in the following form after taking z-transform on both sides.

$$\begin{aligned} [z^2 + z + 0.24]Y[z] &= [z+2]\frac{z}{(z-0.5)} \\ (z^2 + z + 0.24) &= (z+0.6)(z+0.4) \\ \frac{Y[z]}{z} &= \frac{(z+2)}{(z-0.5)(z+0.6)(z+0.4)} \\ &= \frac{A_1}{(z-0.5)} + \frac{A_2}{(z+0.6)} + \frac{A_3}{(z+0.4)} \\ (z+2) &= A_1(z+0.6)(z+0.4) + A_2(z-0.5)(z+0.4) \\ &\quad + A_3(z-0.5)(z+0.6) \end{aligned}$$

Substitute $z = 0.5$

$$2.5 = A_1(1.1)(0.9); \qquad A_1 = 2.525$$

Substitute $z = -0.6$

$$1.4 = A_2(-1.1)(-0.2); \qquad A_2 = 6.36$$

Substitute $z = -0.4$

$$1.6 = A_3(-0.9)(-0.2); \qquad A_3 = -8.89$$

$$Y[z] = 2.525\frac{z}{(z-0.5)} + 6.36\frac{z}{(z+0.6)} - 8.89\frac{z}{(z+0.4)}$$

$$y[n] = \left[2.525(0.5)^n + 6.36(-0.6)^n - 8.89(-0.4)^n\right]u[n]$$

Example 5.39 Consider the following difference equation

$$y[n+2] - 5y[n+1] + 6y[n] = x[n+1] + 4x[n]$$

The auxiliary conditions are as follows $y[0] = 1$ and $y[1] = 2$ and the input $x[n] = u[n]$. Solve for $y[n]$.

Solution

$$\begin{aligned}
y[n+2] &\overset{Z}{\longleftrightarrow} z^2Y[z] - z^2y(0) - zy(1) \\
&= z^2Y[z] - z^2 - 2z \\
y[n+1] &\overset{Z}{\longleftrightarrow} zY[z] - zy[0] \\
&= zY[z] - z \\
x[n+1] &\overset{Z}{\longleftrightarrow} zX[z] - zx[0] \\
&= zX[z] - z
\end{aligned}$$

Taking z-transform on both sides of the above equation and substituting $X[z] = \frac{z}{(z-1)}$ we get

$$\begin{aligned}
[z^2 - 5z + 6]Y[z] &= z^2 + 2z - 5z + (z+4)\frac{z}{(z-1)} - z \\
(z-2)(z-3)Y[z] &= \frac{z(z-4)(z-1) + z(z+4)}{(z-1)} \\
\frac{Y[z]}{z} &= \frac{(z^2 - 4z + 8)}{(z-1)(z-2)(z-3)} \\
&= \frac{A_1}{(z-1)} + \frac{A_2}{(z-2)} + \frac{A_3}{(z-3)}
\end{aligned}$$

$$(z^{-2} - 4z + 8) = A_1(z-2)(z-3) + A_2(z-1)(z-3) + A_3(z-1)(z-2)$$

Substitute $z = 1$

$$1 - 4 + 8 = A_1(-1)(-2); \qquad A_1 = 2.5$$

Substitute $z = 2$

$$4 - 8 + 8 = A_2(-1); \qquad A_2 = -4$$

Substitute $z = 3$

$$9 - 12 + 8 = A_3(2)(1); \qquad A_3 = 2.5$$

$$Y[z] = 2.5\frac{z}{(z-1)} - 4\frac{z}{(z-2)} + 2.5\frac{z}{(z-3)}$$

$$y[n] = \left[2.5 - 4(2)^n + 2.5(3)^n\right] u[n]$$

Example 5.40 Solve the following difference equation

$$y[n+2] - 9y[n+1] + 20y[n] = 4x[n+1] + 2x[n]$$

The input $x[n] = (\frac{1}{2})^n u[n]$. The initial conditions are $y[-1] = 2$ and $y[-2] = 1$.

Solution The given difference equation is in advanced operator form which requires the knowledge of $y[1]$ and $y[2]$. Therefore, the given equation is converted into delay operator form as described below and the given initial condition is applied. Replacing n with $(n-2)$, the given difference equation is converted as

$$y[n] - 9y[n-1] + 20y[n-2] = 4x[n-1] + 2x[n-2]$$

Since the input is causal, $x[-1] = x[-2] = 0$. Taking z-transform on both sides of the above equation we get

$$\begin{aligned}
&Y[z] - 9[z^{-1}Y[z] + y[-1]] + 20[z^{-2}Y[z] + z^{-1}y[-1] + y[-2]] \\
&= 4[z^{-1}X[z] + x[-1] + 2[z^{-2}X[z] + z^{-1}x[-1] + z^{-2}x[-2]]] \\
&= [4z^{-1} + 2z^{-2}]X[z] \\
&= [1 - 9z^{-1} + 20z^{-2}]Y[z] - 18 + 40z^{-1} + 20 = (4z^{-1} + 2z^{-2})X[z] \\
&\frac{[z^2 - 9z + 20]}{z^2} Y[z] = -(2 + 40z^{-1}) + (4z^{-1} + 2z^{-2})X[z]
\end{aligned}$$

Substitute $(z^2 - 9z + 20) = (z-4)(z-5)$ and $X[z] = \frac{z}{(z-0.5)}$

$$\begin{aligned}\frac{Y[z]}{z} &= \frac{(-2z^2 - 35z + 22)}{(z-0.5)(z-4)(z-5)} \\ &= \frac{A_1}{(z-0.5)} + \frac{A_2}{(z-4)} + \frac{A_3}{(z-5)} \\ (-2z^{-2} - 35z + 22) &= A_1(z-4)(z-5) + A_2(z-0.5)(z-5) \\ &\quad + A_3(z-0.5)(z-4)\end{aligned}$$

Substitute $z = 0.5$

$$-0.5 - 17.5 + 22 = A_1(-3.5)(-4.5); \qquad A_1 = 0.254$$

Substitute $z = 4$

$$-32 - 140 + 22 = A_2(3.5)(-1); \qquad A_2 = 42.86$$

Substitute $z = 5$

$$-50 - 175 + 22 = A_3(4.5); \qquad A_3 = -45.1$$

$$Y[z] = \frac{0.254z}{(z-0.5)} + \frac{42.86z}{(z-4)} - \frac{45.1z}{(z-5)}$$

$$y[n] = \left[0.254(0.5)^n + 42.86(4)^n - 45.1(5)^n\right] u[n]$$

5.14 Zero-Input and Zero State Response

The total solution of the difference equation is separated into zero input and zero state components. The response due to the initial conditions alone (in the absence of the input) is called zero input response. The response due to the input alone (assuming that the initial conditions are zero) is called zero state response. The total response is the sum of zero input response and zero state response. This is illustrated in the following examples.

Example 5.41

$$y[n] + 5y[n-1] + 6y[n-2] = x[n-1] + 2x[n]$$

where $x[n] = u[n]$. The initial conditions are $y[-1] = 1$ and $y[-2] = 0$. Find (a) Zero input response, (b) Zero state response, and (c) Total response.

Solution (a) **Zero-Input Response**

$$y[n] \overset{Z}{\longleftrightarrow} Y[z]$$
$$y[n-1] \overset{Z}{\longleftrightarrow} z^{-1}Y[z] + y[-1]$$
$$y[n-2] \overset{Z}{\longleftrightarrow} z^{-2}Y[z] + z^{-1}y[-1] + y[-2]$$

Assuming the input is zero, taking z-transform on both sides of the given equation we get

$$\begin{aligned}
Y[z] + 5(z^{-1}Y[z] + y[-1]) + 6(z^{-2}Y[z] & \\
+z^{-1}y[-1] + y[-2]) &= 0 \\
(1 + 5z^{-1} + 6z^{-2})Y[z] + 5 + 6z^{-1} &= 0 \\
\frac{(z+2)(z+3)}{z^2}Y[z] &= -\frac{(5z+6)}{z} \\
\frac{Y[z]}{z} &= -\frac{(5z+6)}{(z+2)(z+3)} \\
&= \frac{A_1}{(z+2)} + \frac{A_2}{(z+3)} \\
-(5z+6) &= A_1(z+3) + A_2(z+2)
\end{aligned}$$

Substitute $z = -2$

$$10 - 6 = A_1; \qquad A_1 = 4$$

Substitute $z = -3$

$$15 - 6 = A_2(-1); \qquad A_2 = -9$$

$$Y[z] = \frac{4z}{(z+2)} - \frac{9z}{(z+3)}$$

$$y[n] = \left[4(-2)^n - 9(-3)^n\right]u[n]$$

The initial condition can be easily checked as explained below. Substitute $n = -1$

$$y[-1] = 4\frac{1}{(-2)} - 9\left(\frac{1}{-3}\right)$$
$$= -2 + 3 = 1$$

Substitute $n = -2$

$$y[-2] = 4\frac{1}{(-2)^2} - 9\frac{1}{(-3)^2}$$
$$= 1 - 1 = 0$$

(b) **Zero State Response** Assuming the zero initial conditions and noting $x[-1] = 0$, we get

$$[1 + 5z^{-1} + 6z^{-2}]Y[z] = z^{-1}X[z] - x[-1] + 2X[z]$$
$$\frac{[z^2 + 5z + 6]}{z^2}Y[z] = [z^{-1} + 2]X[z]$$
$$= \frac{(2z+1)}{z}\frac{z}{(z-1)}$$
$$\frac{Y[z]}{z} = \frac{z(2z+1)}{(z-1)(z+2)(z+3)}$$
$$= \frac{A_1}{(z-1)} + \frac{A_2}{(z+2)} + \frac{A_3}{(z+3)}$$
$$2z^2 + 3 = A_1(z+2)(z+3) + A_2(z-1)(z+3)$$
$$+A_3(z-1)(z+2)$$

Substitute $z = 1$

$$2 + 1 = A_1(3)(4); \qquad A_1 = \frac{1}{4}$$

Substitute $z = -2$

$$8 - 2 = A_2(-3); \qquad A_2 = -2$$

Substitute $z = -3$

$$18 - 3 = A_3(-4)(-1); \qquad A_3 = \frac{15}{4}$$

$$Y[z] = \frac{1}{4}\frac{z}{(z-1)} - \frac{2z}{(z+2)} + \frac{15}{4}\frac{z}{(z+3)}$$

$$y[n] = \left[\frac{1}{4} - 2(-2)^n + \frac{15}{4}(-3)^n\right]u[n]$$

(c) **Total Response**

$$\text{Total response} = \text{Zero input response} + \text{Zero state response}$$
$$y[n] = 4(-2)^n - 9(-3)^n + \frac{1}{4} - 2(-2)^n + \frac{15}{4}(-3)^n$$

$$y[n] = \left[\frac{1}{4} + 2(-2)^n - \frac{21}{5}(-3)^n\right]u[n]$$

5.15 Natural and Forced Response

In the total response, the response due to the characteristic modes are called forced response. The terms which include characteristic modes (Eigen values) are called natural response. In Example 5.41 the Eigen values are $\lambda_1 = -2$ and $\lambda_2 = -3$. In the total response $y[n] = \frac{1}{4}u[n]$ is free from characteristic modes. Hence, it is the forced response. This is illustrated in the following example.

Example 5.42 Consider the following difference equation

$$y[n+2] - 6y[n+1] + 8y[n] = x[n]$$

where $x[n] = (\frac{1}{4})^n u[n]$. The initial conditions are $y[0] = 1$ and $y[1] = 2$. Find (a) Zero state response; (b) Zero input response; (c) Natural response; (d) Forced response and (e) Total response.

Solution (a) Taking z-transform on both sides we get

$$X[z] = z^2Y[z] - z^2y[0] - zy[1] - 6\{zY[z] - zy[0]\} + 8Y[z]$$
$$X[z] = [z^2 - 6z + 8]Y[z] - z^2 - 2z + 6z$$

Substituting $X[z] = \frac{z}{(z-0.25)}$ and $z^2 - 6z + 8 = (z-2)(z-4)$ we get

$$(z-2)(z-4)Y[z] = z^2 - 4z + \frac{z}{(z-0.25)}$$

$z = 2$ and $z = 4$ are the Eigen values. If the initial conditions are zero we get

$$\frac{Y[z]}{z} = \frac{1}{(z-2)(z-4)(z-0.25)}$$
$$= \frac{A_1}{(z-2)} + \frac{A_2}{(z-4)} + \frac{A_3}{(z-0.25)}$$
$$1 = A_1(z-4)(z-0.25) + A_2(z-2)(z-0.25) + A_3(z-2)(z-4)$$

Substitute $z = 2$

$$1 = A_1(-2)(1.75); \qquad A_1 = -\frac{2}{7}$$

Substitute $z = 4$

$$1 = A_2(2)(3.75); \qquad A_2 = \frac{2}{15}$$

Substitute $z = 0.25$

$$1 = A_3(-1.75)(-3.75); \qquad A_3 = \frac{16}{105}$$

Let $y_{0s}[n]$ denote zero state response and $y_{0i}[n]$ denote zero input response.

$$Y_{0s}[z] = -\frac{2}{7}\frac{z}{(z-2)} + \frac{2}{15}\frac{z}{(z-4)} + \frac{16}{105}\frac{z}{(z-0.25)}$$

$$y_{0s}[n] = \left[-\frac{2}{7}(2)^n + \frac{2}{15}(4)^n + \frac{16}{105}(0.25)^n\right]u[n]$$

(b) If we assume the input is zero, $X[z] = 0$

$$\frac{Y_{0i}[z]}{z} = \frac{(z-4)}{(z-2)(z-4)}$$
$$Y_{0i}[z] = \frac{z}{(z-2)}$$

$$y_{0i}[n] = (2)^n u[n]$$

(c) The total response $y[n]$ is given by

$$y[n] = y_{0s}[n] + y_{0i}[n] = \left[-\frac{2}{7}(2)^n + \frac{2}{15}(4)^n + \frac{16}{105}(0.25)^n + (2)^n\right]u[n]$$
$$= \Big[\underbrace{\frac{5}{7}(2)^n + \frac{2}{15}(4)^n}_{\text{Natural response}} + \underbrace{\frac{16}{105}(0.25)^n}_{\text{Forced response}}\Big]u[n]$$

Let us denote $y_n[n]$ and $y_f[n]$ as the natural and forced responses respectively. The natural response is the response which is due to the characteristic roots $z = 2$ and $z = 4$. The remaining portion of $y[n]$ is the forced response.

$$y_f[n] = \frac{16}{105}(0.25)^n u[n]$$

(d) The natural response is

$$y_n[n] = \left[\frac{5}{7}(2)^n + \frac{2}{15}(4)^n\right]u[n]$$

The forced response is

$$y_f[n] = \frac{16}{105}(0.25)^n u[n]$$

(e) The total response is

$$y[n] = \left[\frac{5}{7}(2)^n + \frac{2}{15}(4)^n + \frac{16}{105}(0.25)^n\right]u[n]$$

5.16 Difference Equation from System Function

Let the system function $H[z]$ be expressed as

$$\frac{Y[z]}{X[z]} = H[z] = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1}z + b_N}{z^N + a_1 z^{N-1} + \cdots + a_{N-1}z + a_N}$$

Cross multiplying and operating z on $Y[z]$ and $X[z]$ we get

$$y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1}y[n+1] + a_N y[n]$$
$$= b_0 x[n+N] + b_1 x[n+N-1] + \cdots + b_{N-1}x[n+1] + b_N x[n] \quad (5.58)$$

Similar procedure has to be followed if the system frequency response $H(e^{j\omega})$ is given. Here $e^{j\omega}$ has to be treated as z. The following examples demonstrate the above methods.

Example 5.43 For the system functions given below determine the difference equation

(a) $$H[z] = \frac{(1 - z^{-1})}{\left(1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right)}$$

(b) $$H[z] = \frac{(z - 1)}{(z + 1)(z - 2)}$$

(c) $$H[z] = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)}$$

(*Anna University, December, 2006*)

(d) Consider the system consisting of the cascade of two LTI system with frequency responses

$$H_1(e^{j\omega}) = \frac{2 - e^{j\omega}}{\left(1 + \frac{1}{2}e^{-j\omega}\right)}$$

$$H_2(e^{j\omega}) = \frac{1}{\left(1 - \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega}\right)}$$

Find the difference equation describing the overall system.

(*Anna University, April, 2008*)

(e) Write a difference equation that characterizes a system whose frequency response is

$$H(e^{j\omega}) = \frac{\left(1 - \frac{1}{2}e^{-j\omega} + e^{-3j\omega}\right)}{\left(1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-2j\omega}\right)}$$

(*Anna University, May, 2007*)

Solution (a) $\boldsymbol{H[z] = \dfrac{(1 - z^{-1})}{\left(1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right)}}$

$$\frac{Y[z]}{X[z]} = \frac{\left(1 - z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right)}$$

$$Y[z] - \frac{1}{2}z^{-1}Y[z] + \frac{1}{4}z^{-2}Y[z] = X[z] - z^{-1}X[z]$$

$$y[n] - \frac{1}{2}y[n-1] + \frac{1}{4}y[n-2] = x[n] - x[n-1]$$

(b) $\boldsymbol{H[z] = \frac{(z-1)}{(z+1)(z-2)}}$

$$\frac{Y[z]}{X[z]} = \frac{(z-1)}{(z+1)(z-2)}$$
$$= \frac{(z-1)}{(z^2-z-2)}$$
$$z^2Y[z] - zY[z] - 2Y[z] = zX[z] - X[z]$$

$$y[n+2] - y[n+1] - 2y[n] = x[n+1] - x[n]$$

(c) $\boldsymbol{H[z] = \frac{Y[z]}{X[z]} = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)}}$

$$\left[1 - \frac{1}{4}z^{-1}\right]Y[z] = X[z]$$

$$y[n] - \frac{1}{4}y[n-1] = x[n]$$

(d) $\boldsymbol{H_1(e^{j\omega}) = \frac{2-e^{j\omega}}{(1+\frac{1}{2}e^{-j\omega})}}$ **and** $\boldsymbol{H_2(e^{j\omega}) = \frac{1}{(1-\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-j2\omega})}}$

$$H_1H_2(e^{j\omega}) = \frac{Y(j\omega)}{X(j\omega)} = \frac{(2-e^{-j\omega})}{\left(1+\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-j2\omega}\right)}$$
$$= \frac{(2-e^{-j\omega})}{\left(1-\frac{1}{2}e^{-j\omega}+\frac{1}{4}e^{-j2\omega}+\frac{1}{2}e^{-j\omega}-\frac{1}{4}e^{-j2\omega}+\frac{1}{8}e^{-j3\omega}\right)}$$
$$Y[e^{j\omega}]\left[1+\frac{1}{8}e^{-j3\omega}\right] = [2-e^{-j\omega}]X[e^{j\omega}]$$

$$y[n] + \frac{1}{8}y[n-3] = 2x[n] - x[n-1]$$

(e) $\boldsymbol{\frac{Y[e^{j\omega}]}{X[e^{j\omega}]} = H(e^{j\omega}) = \frac{(1-e^{-j\omega}+e^{-3j\omega})}{(1+\frac{1}{2}e^{-j\omega}+\frac{3}{4}e^{-2j\omega})}}$

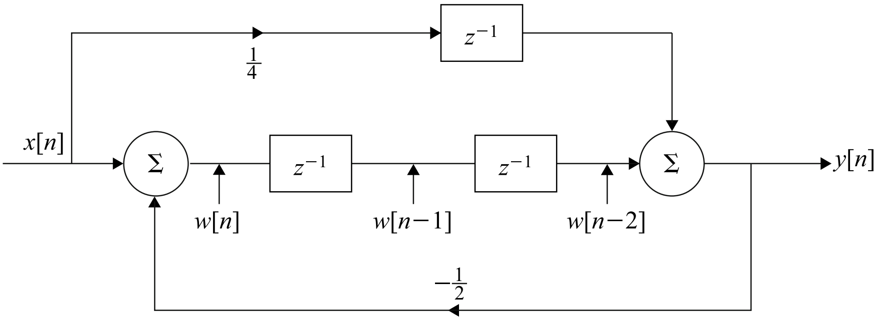


Fig. 5.18 Block diagram of Example 5.44

$$\left[1+\frac{1}{2}e^{-j\omega}+\frac{3}{4}e^{-2j\omega}\right]Y[e^{j\omega}]=\left[1-e^{-j\omega}+e^{-3j\omega}\right]X[e^{j\omega}]$$

$$y[n]+\frac{1}{2}y[n-1]+\frac{3}{4}y[n-2]=x[n]-x[n-1]+x[n-3]$$

Example 5.44 Obtain the difference equation for the block diagram shown in Fig. 5.18.

Solution From Fig. 5.18, the following equations are written:

$$w[n]=x[n]-\frac{1}{2}y[n]$$

Replace n by $(n-2)$

$$\begin{aligned} w[n-2] &= x[n-2]-\frac{1}{2}y[n-2] \\ y[n] &= \frac{1}{4}x[n-1]+w[n-2] \\ &= \frac{1}{4}x[n-1]+x[n-2]-\frac{1}{2}y[n-2] \end{aligned}$$

$$y[n]+\frac{1}{2}y[n-2]=\frac{1}{4}x[n-1]+x[n-2]$$

Summary

1. The z-transform for discrete-time signals and systems has been developed. This resembles corresponding treatment of Laplace transform for continuous time system.
2. A definite connection exists between Laplace transform, Fourier transform and z-transform. The Laplace transform reduces to Fourier transform on the imaginary axis in the s-plane. Then z-transform reduces to Fourier transform on the unit circle in the complex z-plane.
3. For the causal signals system (right sided), the z-transform exists if the ROC is exterior of the circle which passes through the outermost pole of the system function. For the anti-causal signal and system (left sided) the z-transform exists if the ROC is interior of the circle which passes through the innermost pole of the system. For the right- and left-sided signals, the ROC is a ring which does not include any pole of system function.
4. The application of the properties of z-transform very much simplifies the procedure to determine z-transform and inverse z-transform.
5. For an LTID system to be causal, the system function should be rational and the ROC is the exterior of the circle which passes through the outermost pole of the system function $H[z]$.
6. An LTID system is said to be stable if the ROC of the system function $H[z]$ includes the unit circle.
7. An LTID system is said to be causal and stable if all the poles of the system function $H[z]$ lie inside the unit circle in the z-plane.
8. Using the properties of z-transform, LTID systems described by constant coefficient difference equation can be converted into algebraic equations and easily analyzed. The solution obtained is classified as zero state response, zero input response, natural response, and forced response.
9. An LTID system structure is realized using adders, multipliers, and unit delay. System is realized in direct form-I, direct form-II, parallel form, cascade form, and transposed form.

Exercise

I. Short Answer Type Questions

1. **Define z-transform**.
   The z-transform of a discrete-time signal $x[n]$ is defined as

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

   where z is a complex variable.

2. **Define z -transform pair**.
When the discrete-time signal $x[n]$ is z-transformed it is expressed as

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

If we want to recover $x[n]$ form $X[z]$, it is obtained using the following integration

$$x[n] = \frac{1}{2\pi j} \oint X[z]z^{n-1} dz$$

This equation is called as inverse z-transform. The above two equations for z-transform and inverse z-transform are called z-transform pair.
3. **What do you understand by ROC of z -transform?**
The range of values of z for which the function $X[z]$ converges is called region of convergence which is expressed in abbreviated form as ROC.
4. **Mention the properties of ROC**.
    1. The ROC of $X[z]$ is in the form of a ring in the z-plane which is centered about the origin.
    2. The ROC does not include any poles.
    3. For the right-sided sequence $x[n]$, the ROC is the exterior of the outermost pole.
    4. For the left-sided sequence $x[n]$, the ROC is the interior of the innermost pole.
    5. If the sequence $x[n]$ is two sided, then the ROC consist of a ring in the z-plane.
5. **What is the scaling property of z -transform?**
If

$$x[n] \overset{Z}{\longleftrightarrow} X[z] \qquad \text{ROC: } R$$

then

$$a^n x[n] \longleftrightarrow X\left[\frac{z}{a}\right] \qquad \text{ROC: } aR$$

By using multiplication property, the z-transform is obtained by replacing z by $\frac{z}{a}$ with ROC R replaced by aR.
6. **What is the convolution property of z -transform?**
If

$$x_1[n] \overset{Z}{\longleftrightarrow} X_1[z] \qquad \text{ROC: } R_1$$
$$x_2[n] \overset{Z}{\longleftrightarrow} X_2[z] \qquad \text{ROC: } R_2$$

then

$$x_1[n] * x_2[n] \overset{Z}{\longleftrightarrow} X_1[z]X_2[z] \qquad \text{ROC: } R_1 \cap R_2$$

7. **What is difference property in the z -transform?**
   If

$$x[n] \overset{Z}{\longleftrightarrow} X[z] \qquad \text{ROC: } R$$

   then

$$nx[n] \overset{Z}{\longleftrightarrow} -z\frac{dx[z]}{dz} \qquad \text{ROC: } R$$

8. **What are initial and final value theorems?**
   If $x[n] = 0$ for $n < 0$, then

$$x[0] = \underset{z\to\infty}{Lt}\ X[z]$$

   is called initial value theorem. According to the finial value theorem if $X[z]$ is the z-transform $x[n]$ and if all the poles of $X[z]$ are inside the unit circle, then the final value of $x[n] = x[\infty]$ is obtained form

$$x[\infty] = \underset{z\to 1}{Lt}\,(z-1)X[z]$$

9. **What do you understand by the time reversal property of z -transform?**

   If

$$x[n] \overset{Z}{\longleftrightarrow} X[z] \qquad \text{ROC: } R$$

   then

$$x[-n] \overset{Z}{\longleftrightarrow} X\left[\frac{1}{z}\right] \qquad \text{ROC: } \frac{1}{R}$$

   Thus, the z-transform of the time reversal signal is obtained by replacing z by its reciprocal and also its ROC by its reciprocal.
10. **What do you understand by the causality of an LTID system?**
    An linear time invariant discrete-time system is said to be causal if the ROC of the system function $H[z]$ is the exterior of the circle containing all the poles of $H[z]$.
11. **What do you understand by stability of an LTID system?**
    An LTID system is said to be stable if the ROC of the system function $H[z]$ includes the unit circle in the z-plane.

12. **When the system is said to be both causal and stable?**
An LTID system is said to be both causal and stable if all the poles of the system function $H[z]$ are inside the unit circle in the z-plane.
13. **Define system function**.
System function or transfer function $H[z]$ is defined as the ratio of the z-transform of output sequence $y[n]$ and the input sequence $x[n]$

$$H[z] = \frac{Y[z]}{X[z]}$$

14. **What is the z-transform of $\delta[n-2]$**?

$$\delta[n-2] \overset{Z}{\longleftrightarrow} z^{-2}$$

15. **What is the z-transform of $u[n]$ and $\delta[n]$**?

$$u[n] \overset{Z}{\longleftrightarrow} \frac{z}{(z-1)}$$
$$\delta[n] \overset{Z}{\longleftrightarrow} 1$$

16. **Find the z-transform of $x[n] = u[n] - u[n-5]$**.

$$X[z] = \frac{z}{(z-1)}[1 - z^{-5}]$$

17. **Write the relationship between z-transform and Fourier transform**. The z-transform reduces to Fourier transform on the unit circle in the complex z-plane.
18. **Write the relationship between z-transform and Laplace transform**. The Laplace transform and z-transform are related as

$$e^s = z$$
$$X[s] = X[z]\Big|_{z=e^s}$$

19. **What is the inverse z-transform of $X[\frac{z}{a}]$**?

$$X\left[\frac{z}{a}\right] \overset{Z^{-1}}{\longleftrightarrow} a^n x[n]$$

20. **Find the system function of the following first-order difference equation $y[n] - 2y[n-1] = x[n] + x[n-1]$**?

$$H[z] = \frac{Y[z]}{X[z]} = \frac{[1 + z^{-1}]}{[1 - 2z^{-1}]}$$
$$= \frac{[z + 1]}{[z - 2]}$$

II. Long Answer Type Questions

1. **Find the z-transform of the following sequence.**

$$\boldsymbol{x[n] = [3^{n-1} - (-3)^{n-1}]u[n]}$$

$$X[z] = \frac{2z^2}{3(z^2 - 9)} \qquad \text{ROC: } |z| > 3.$$

2. **Find the z-transform of**

$$\boldsymbol{x[n] = \sum_{n=0}^{\infty} \frac{1}{3} z^{-n} + \frac{1}{4}(-2)^n z^{-n}}$$

$$X[z] = \frac{1}{3}\frac{1}{(1 - z^{-1})} + \frac{1}{3}\frac{1}{(1 + 2z^{-1})} \qquad \text{ROC: } |z| > 2.$$

3. **Find the z-transform of**

$$\boldsymbol{x[n] = \sum_{n=-1}^{\infty} \left(\frac{1}{4}\right)^{n+1} z^{-n}}$$

$$X[z] = z + \frac{1}{4}\frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)} \qquad \text{ROC: } |z| > \frac{1}{4}.$$

4. **Find the z-transform of**

$$\boldsymbol{x[n] = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{-n+1} z^{-n}}$$

$$X[z] = \frac{1}{4} + \frac{4}{(1 - 4z)} \qquad \text{ROC: } |z| < \frac{1}{4}.$$

5. **Find the z-transform of**

$$\boldsymbol{x[n] = \left(\frac{1}{10}\right)^n u[n-4]}$$

$$X[z] = 10^{-4}\frac{z^{-3}}{\left(z-\frac{1}{10}\right)} \qquad \text{ROC: } |z| > \frac{1}{10}.$$

6. **Find the z-transform of**

$$\begin{aligned} (a) \quad x[n] &= 1\, 0 \le n \le 9 \\ &= 0 \quad otherwise \\ (b) \quad y[n] &= x[n] - x[n-1] \end{aligned}$$

$$\begin{aligned} &\text{(a)} \quad X[z] = \frac{(1-z^{-10})}{(1-z^{-1})} \qquad \text{ROC: } |z| > 0 \\ &\text{(b)} \quad Y[z] = 1 - z^{-10} \qquad \text{ROC: } |z| > 0 \end{aligned}$$

7. **Find the unilateral z-transform and the ROC for the following sequences:**

$$\begin{aligned} \textbf{(a)} \quad x[n] &= \left(\frac{1}{6}\right)^n u[n+6] \\ \textbf{(b)} \quad x[n] &= 3\delta[n+4] + \delta[n] + (3)^n u[-n] \\ \textbf{(c)} \quad x[n] &= \left(\frac{1}{4}\right)^{|n|} \end{aligned}$$

$$\begin{aligned} &\text{(a)} \quad X[z] = \frac{1}{(1-6z^{-1})} \qquad \text{ROC: } |z| > 6 \\ &\text{(b)} \quad X[z] = 4 \qquad \text{ROC: all } z \\ &\text{(c)} \quad X[z] = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)} \qquad \text{ROC: } |z| > \frac{1}{4} \end{aligned}$$

8. **By applying properties of z-transform find the z-transform of the following sequences given $x[n] \stackrel{Z}{\longleftrightarrow} \frac{z}{(z^2+2)}$**

$$
\begin{aligned}
&\textbf{(a)} \quad y[n] = x[n-3] \\
&\textbf{(b)} \quad y[n] = nx[n] \\
&\textbf{(c)} \quad y[n] = x[n+1] + x[n-1] \\
&\textbf{(d)} \quad x[n] = 2^n x[n] \\
&\textbf{(e)} \quad x[n] = (n-2)x[n-1] \\
&\textbf{(f)} \quad x[n] = x[-n]
\end{aligned}
$$

(a) $Y[z] = \dfrac{z^{-2}}{(z^2+2)}$ ROC: $|z| < 2$ (Time shifting property)

(b) $Y[z] = \dfrac{z[z^2-2]}{(z^2+2)^2}$ (Differentiation property)

(c) $Y[z] = \dfrac{(z^2+1)}{(z^2+2)}$ (Time advancing and time delaying)

(d) $Y[z] = \dfrac{2z}{(z^2+8)}$ (Multiplying property)

(e) $Y[z] = \dfrac{-4}{(z^2+2)^2}$ (Time differentiation and time shifting)

(f) $Y[z] = \dfrac{z}{(1+2z^2)}$ (Time reversal)

9. **Find the z-transform of**

$$
\begin{aligned}
&\textbf{(a)} \quad x[n] = 2^n u[n-2] \\
&\textbf{(b)} \quad x[n] = \left(\frac{1}{4}\right)^n u[-n]
\end{aligned}
$$

(a) $X[z] = \dfrac{4z^{-2}}{(1-2z^{-1})}$

(b) $X[z] = \dfrac{1}{(1-4z)}$

10. **Find the z-transform of**

$$
\begin{aligned}
&\textbf{(a)} \quad x[n] = (n-4)u[n-4] \\
&\textbf{(b)} \quad x[n] = u[n] - u[n-4] \\
&\textbf{(c)} \quad x[n] = (n-4)u[n] \\
&\textbf{(d)} \quad x[n] = n[u[n] - u[n-4]]
\end{aligned}
$$

(a) $$X[z] = \frac{z^{-3}}{(z-1)^2} \qquad \text{ROC: } |z| > 1$$

(b) $$X[z] = \frac{z}{(z-1)}[1 - z^{-4}] \qquad \text{ROC: } |z| > 1$$

(c) $$X[z] = \frac{(5z - 4z^2)}{(z-1)^2} \qquad \text{ROC: } |z| > 1$$

(d) $$X[z] = \frac{(z - 3z^{-2} + 2z^{-3})}{(z-1)^2} \qquad \text{ROC: } |z| > 1$$

11. **Find the z-transform of the following sequence?**

$$\begin{aligned} \boldsymbol{x[n]} &= \left(\frac{1}{4}\right)^n u[n] \\ &= \left(-\frac{1}{2}\right)^n u[-n-1] \end{aligned}$$

$$X[z] = \frac{3}{4}\left[\frac{z}{\left(z - \frac{1}{4}\right)\left(z + \frac{1}{2}\right)}\right] \qquad \text{ROC: } \frac{1}{4} < |z| < \frac{1}{2}$$

12. **Using convolution find $y[n]$ given**

$$\begin{aligned} \boldsymbol{x[n]} &= \left(\frac{1}{2}\right)^n u[n] \\ \boldsymbol{h[n]} &= \left(\frac{1}{3}\right)^n u[n] \\ \boldsymbol{y[n]} &= x[n] * h[n] \end{aligned}$$

$$y[n] = \left[3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n\right] u[n] \qquad \text{ROC: } |z| < \frac{1}{2}$$

13. **Using partial function find the inverse z-transform**

$$\boldsymbol{H[z] = \frac{(1 - z^{-1} + z^{-2})}{(1 - z^{-1})(1 - 2z^{-1})(1 - 4z^{-1})} \qquad \text{ROC: } 2 < |z| < 4}$$

$$h[n] = \left[\frac{1}{3} - \frac{3}{2}(2)^n\right] u[n] + \frac{3}{16}(4)^n u[-n-1]$$

14. **Find the inverse z-transform of**

$$H[z] = \frac{4z+1}{z-\frac{1}{4}}$$

using power series expansion?

$$(a) \quad ROC: |z| > \frac{1}{4}$$
$$(b) \quad ROC: |z| < \frac{1}{4}$$

(a) $x[n] = \left\{ \underset{\uparrow}{4}, 2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots \right\}$

(b) $x[n] = \{\ldots, 2048, -512, -128, -32, \underset{\uparrow}{-4}\}$

15. **Consider the algebraic expression for the z-transform of $x[n]$**

$$x[n] = \frac{\left(1-\frac{1}{4}z^{-2}\right)}{\left(1+\frac{1}{4}z^{-1}\right)\left(1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}\right)}$$

How many different ROCs could correspond the $X[z]$?

(a) ROC: $|z| > \frac{1}{2}$

(b) ROC: $0 < |z| < \frac{1}{4}$

(c) ROC: $\frac{1}{3} < |z| < \frac{1}{2}$

16. **Consider the algebraic expression for the z-transform of $x[n]$**

$$x[n] = \frac{\left(1+z^{-1}+4z^{-2}\right)}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{7}{24}z^{-1}+\frac{1}{48}z^{-2}\right)}$$

ROC: $|z| > \frac{1}{4}$. Find whether the system is causal and stable. $X[z]$ is rational and the poles are at $z = \frac{1}{4}$, $z = \frac{1}{6}$, and $z = \frac{1}{8}$. Since the ROC is exterior of the outermost pole the system is causal. The ROC includes unit circle and the poles are inside the unit circle. The system is stable. Therefore the system is causal and stable.

17. **A system with impulse response $h[n] = 5(3)^n u[n-1]$ produces on output $y[n] = (-4)^n u[n-1]$. Determine the input $x[n]$.**

$$x[n] = \frac{1}{5}\left[(-4)^n u[n] - 3(-4)^{n-1} u[n-1]\right] \qquad \text{ROC: } |z| > 4$$

18. **Consider the following difference equation.**

$$y[n] - y[n-1] - 2y[n-2] = x[n] + 2x[n-1]$$

The initial conditions are $y[-1] = 1$ and $y[-2] = 2$. The input $x[n] = u[n]$. Find (a) Zero input response, (b) Zero state response, (c) Natural response, (d) Forced response, and (e) Total response.

(a) $y_{0i}[n] = [(-1)^n + 4(2)^n]u[n]$

(b) $y_{0s}[n] = \left[-\frac{1}{6}(-1)^n + \frac{8}{3}(2)^n - \frac{3}{2}\right]u[n]$

(c) $y_n[n] = -\frac{3}{2}u[n]$

(d) $y_f[n] = \left[\frac{5}{6}(-1)^n + \frac{20}{3}(2)^n\right]u[n]$

(e) $y_{\text{total}}[n] = \left[-\frac{3}{2} + \frac{5}{6}(-1)^n + \frac{20}{3}(2)^n\right]u[n]$

19. Consider the causal LTID system represented in block diagram shown in Fig. 5.19. (a) Determine the difference equation relating the output $y[n]$ and input $x[n]$ and (b) Is the system stable?

(a)

$$y[n] - \frac{1}{4}y[n-1] + \frac{1}{64}y[n-2] = x[n] - 5x[n-1] + 7x[n-2]$$

(b) The ROC includes unit circle and the poles of system function are within unit circle. Hence, the system is stable.

20. **For each of the following difference equations determine the output response $y[n]$?**

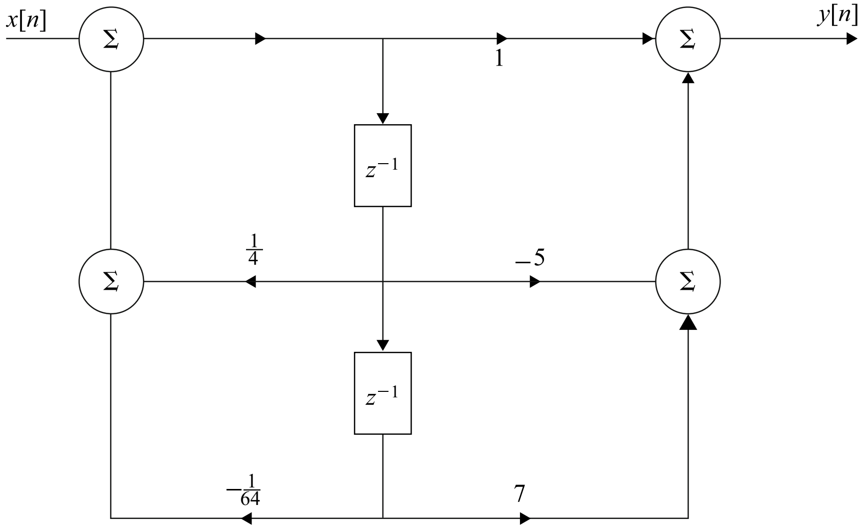


Fig. 5.19 Block diagram of Problem 19

$$\textbf{(a)}\quad \boldsymbol{y[n]-4y[n-1]=x[n]\ with\ y[-1]=2\ and\ x[n]=\left(\frac{1}{3}\right)^n u[n]}$$

$$\textbf{(b)}\quad \boldsymbol{y[n]+\frac{1}{3}y[n-1]=x[n]-\frac{1}{3}x[n-1]}$$
$$\boldsymbol{with\ y[-1]=1\ and\ x[n]=u[n]}$$

$$\text{(a)}\qquad y[n]=\left[\frac{100}{11}(4)^n-\frac{1}{11}\left(\frac{1}{3}\right)^n\right]u[n]$$

$$\text{(b)}\qquad y[n]=\left[\frac{3}{2}-\frac{5}{6}\left(-\frac{1}{3}\right)^n\right]u[n]$$

Chapter 6
State Space Modeling and Analysis

Chapter Objectives

- To define the state of a system.
- To represent the mechanical systems and electrical networks by state equations.
- To convert transfer function model to state space model of continuous-time system.
- To find the solution of the state equation of continuous-time system.
- To represent the discrete-time system by state equations.

6.1 Introduction

The transfer function (T.F.) model, for quite a long time, was used for the analysis and design of linear time invariant continuous time systems. However, this model has many limitations in that it is expressed as a ratio of output to input variables and thus the internal behavior of the system is hidden. Further, the T.F. method is valid only for linear systems with initial conditions being zero. It is powerless for non-linear, time varying and multi-input and multi-output (MIMO) systems. It is also difficult to handle large-scale complex systems with transfer function model. Furthermore, system design modeling by T.F. is based on trial and error procedure which in general will not lead to optimal control systems. All these limitations are overcome by representing the system in state space model. This is a differential (or difference) equation model which is expressed in n first-order equations which are written in a specific format. The model is valid for linear, non-linear, and time

S. Palani, *Basic System Analysis*,
https://doi.org/10.1007/978-3-031-28280-5_6

varying systems and also when initial conditions are not zero. Unlike the T.F. model, it gives a complete description of the internal behavior of all physical variables in the system. In this chapter, we first develop state space model of mechanical systems and electrical networks followed by the conversion of T.F. model to state space model. The solution of state equation is derived and illustrated by simple examples. Finally, discrete-time system represented by the difference equation is converted into state equations.

6.2 The State of a System and State Equation of Continuous-Time System

For a linear continuous-time system, **the state of a system is defined as the minimum number of initial conditions that must be specified at any initial time t_0 so that the complete dynamic behavior of the system at any time $t > t_0$ is determined when the input $x(t)$ is known.**

When the input $x(t)$ is applied, the future states of the system, for $t > t_0$ also change and we can uniquely determine these states. Since the states of the system vary with respect to time we call these variables as state variables.

The number of state variables depends on the dynamic model selected to describe the physical systems. For a system described by nth-order differential equation, there will be n state variables. If these n state variables form the coordinates of a n-dimensional vector space, it is known as state space. For a continuous-time system described by nth-order differential equation, if the variables which represent the states are chosen less than n, then the system is not fully represented and information about the missing states will not be known. Similarly, if the states are chosen more than n, then some of the states chosen are redundant and they can always be expressed in terms of other known states. Hence, for an nth-order model of a system, it is necessary strictly to choose only n appropriate states and they are represented by n first-order equations together with the input.

6.3 Vector-Matrix Differential Equation of Continuous-Time System

State variable equations, whether linear or non-linear, are expressed in the time domain by using compact vector–matrix notations. **These equations are called vector–matrix differential equations**. The standard form of representing these state equations for a continuous-time system is

$$\dot{q}(t) = Aq(t) + Bx(t) \tag{6.1}$$

$$y(t) = Cq(t) + Dx(t) \tag{6.2}$$

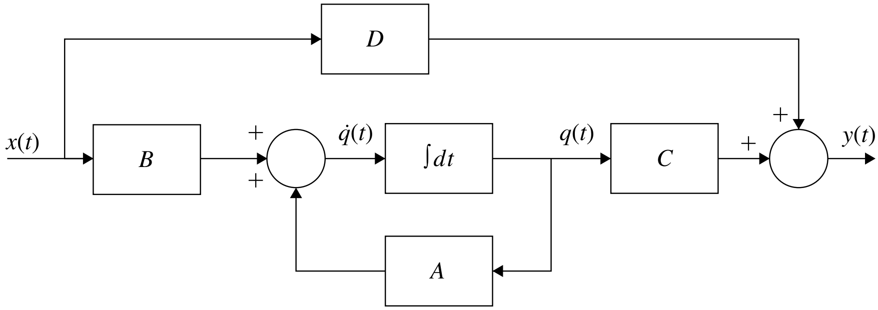


Fig. 6.1 Block diagram representation of state Eqs. (6.1) and (6.2)

Equation (6.1) is called the state equation and Eq. (6.2) is called output equation. Equations (6.1) and (6.2) together describe the system dynamics and they are called vector–matrix differential equations.

In Eqs. (6.1) and (6.2),

$q(t)$ = State vector ($n \times 1$ dimension)
$y(t)$ = Output vector ($p \times 1$ dimension)
$x(t)$ = Input vector ($r \times 1$ dimension)
A = State matrix ($n \times n$ dimension)
B = Input matrix ($n \times r$ dimension)
C = Output matrix ($p \times n$ dimension)
D = Direct transmission matrix ($p \times r$ dimension)

A block diagram representation of Eqs. (6.1) and (6.2) is shown in Fig. 6.1.

Depending upon the dimensions of the vector q, x, and y, the appropriate dimensions of the matrices, A, B, and C are chosen. In most of the practical applications, the direct transmission of input $x(t)$ to the output $y(t)$ is not done, and hence $D = 0$. In forming the state Eq. (6.1), it is to be observed that only the first derivative of $q(t)$ appears on the left side of the equation and no derivative of $q(t)$ appears on the right side. The right side of the equation contains only the states and input. The following examples illustrate the method of forming state equations.

6.3.1 State Equations for Mechanical Systems

The dynamic equations of mechanical systems are written from the free body diagram. The physical variables such as displacement, velocity, etc., are chosen as the states and for each state variable the equation for its first derivative is obtained and converted in the format of Eq. (6.1).

Example 6.1 For the mechanical system shown in Fig. 6.2a, form the state equation.

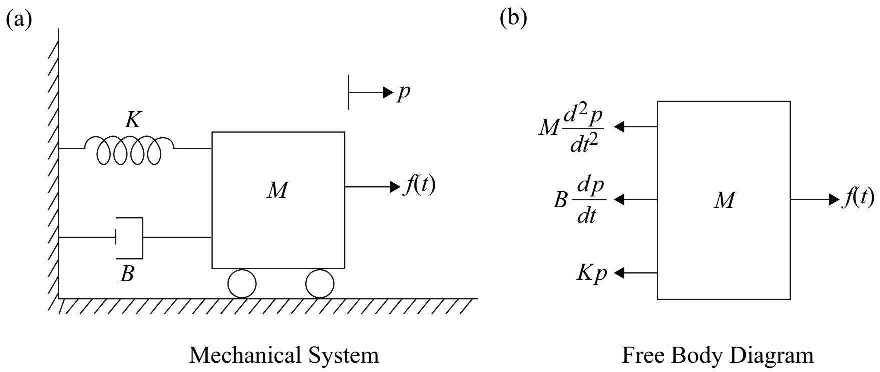


Fig. 6.2 Mechanical system and its free body diagram

Solution

1. The mechanical system is shown in Fig. 6.2a. Its free body diagram is shown in Fig. 6.2b.
2. p is the displacement of mass M and $f(t)$ is the force applied. In the free body diagram, the opposing forces act in the direction opposite to the direction of motion.
3. From Fig. 6.2b, the following dynamic equations for the given mechanical system are written:
$$M\frac{d^2p}{dt^2} + B\frac{dp}{dt} + Kp = f(t) \tag{6.3}$$
4. Let us choose the displacement p, which is the physical variable in the mechanical system as one state variable. Thus,
$$p = q_1(t) \tag{6.4}$$
5. $$\frac{dp}{dt} = \dot{q}_1(t)$$
Let us choose the velocity $\frac{dp}{dt}$ as the second state variable. Thus,
$$\frac{dp}{dt} = \dot{q}_1(t) = q_2(t) \tag{6.5}$$
6. Equation (6.3), gives a complete description of the given mechanical system and it is a second-order system. Therefore, there should be two states and two-state equations. Equation (6.5) represents one state equation. Similar to that, we should obtain an equation for $\dot{q}_2(t)$.

7. Now consider Eq. (6.3). By substituting $p = q_1(t)$, $\frac{dp}{dt} = q_2(t)$ and $f(t) = x(t)$, we get

$$M\dot{q}_2(t) + Bq_2(t) + Kq_1(t) = x(t)$$

Solving for $\dot{q}_2(t)$, we get

$$\dot{q}_2(t) = -\frac{K}{M}q_1(t) - \frac{B}{M}q_2(t) + \frac{1}{M}x(t) \tag{6.6}$$

8. From Eqs. (6.5) and (6.6), the following vector–matrix differential equation is formed.

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = [\dot{q}(t)] = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix}}_{A} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}_{B} x(t) \tag{6.7}$$

In Eq. (6.7),

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}$$

Note that A matrix is a square matrix. In the examples to follow, for convenience, the state variables are denoted as q instead of $q(t)$.

Example 6.2 Consider the mechanical system shown in Fig. 6.3a. From the state equation, the displacement plus velocity of mass M is taken as the output.

Solution

1. The mechanical system is represented in Fig. 6.3a and its free body diagram in Fig. 6.3b. The mass M is given a displacement of p_2. The point A moves with a displacement p_1.
2. From free body diagram, the following equations are written:

$$M\ddot{p}_2 + B(\dot{p}_2 - \dot{p}_1) + K_2 p_2 = f(t) \tag{6.8}$$

$$B(\dot{p}_1 - \dot{p}_2) + K_1 p_1 = 0 \tag{6.9}$$

The following state variables are chosen

$$p_2 = q_1$$
$$\dot{p}_2 = q_2$$
$$p_1 = q_3$$

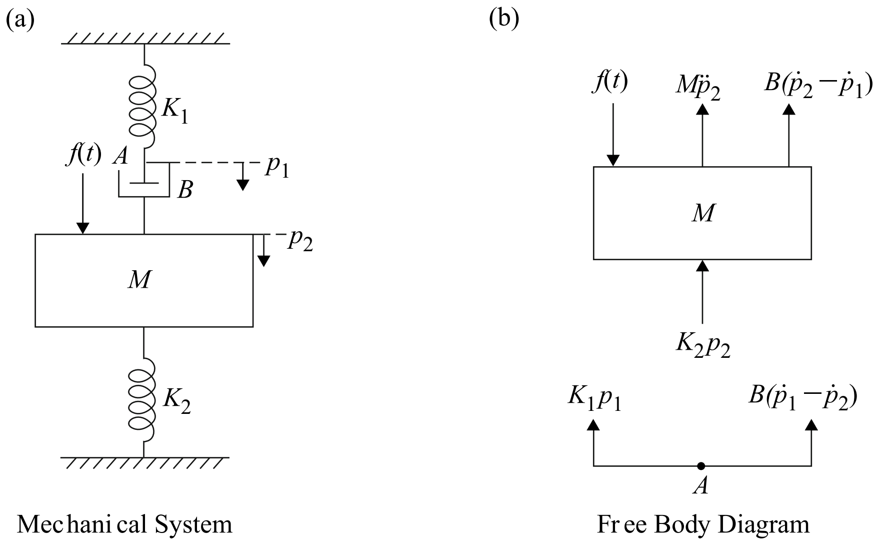


Fig. 6.3 Mechanical system and its free body diagram

The equations for the first derivatives of q_1, q_2, and q_3 are to be obtained.

$$\dot{q}_1 = \dot{p}_2 = q_2$$

$$M\dot{q}_2 + K_1q_3 + K_2q_1 = x(t)$$

$$\dot{q}_2 = -\frac{K_2}{M}q_1 - \frac{K_1}{M}q_3 + \frac{1}{M}x(t)$$

$$B\dot{q}_3 - Bq_2 + K_1q_3 = 0$$

$$\dot{q}_3 = q_2 - \frac{K_1}{B}q_3$$

Thus,

$$\dot{q}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -\frac{K_2}{M} & 0 & -\frac{K_1}{M} \\ 0 & 1 & -\frac{K_1}{B} \end{bmatrix}}_{A} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \end{bmatrix}}_{B} x(t)$$

The output y is given by

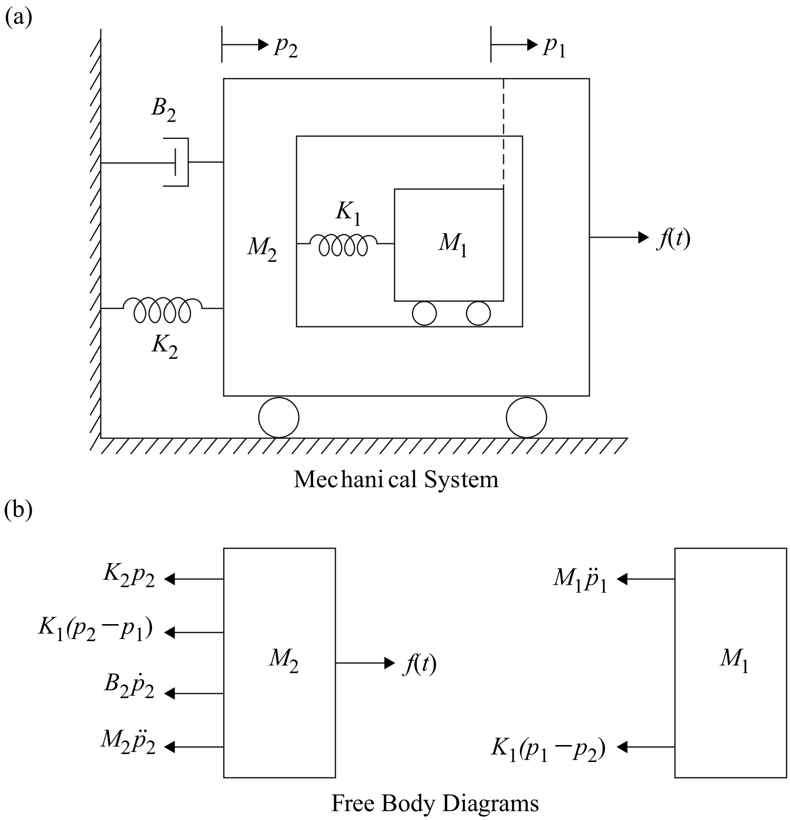


Fig. 6.4 Mechanical system and its free body diagrams

$$y = \underbrace{[1 \quad 1 \quad 0]}_{C} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Example 6.3 Consider the mechanical system shown in Fig. 6.4a. Form the state equation.

Solution

1. The mechanical system is shown in Fig. 6.4a and its free body diagram in Fig. 6.4b.
2. From Fig. 6.4b, the following dynamic equations are written:

$$M_2 \ddot{p}_2 + B_2 \dot{p}_2 + K_2 p_2 + K_1(p_2 - p_1) = f(t) \tag{6.10}$$

$$M_1 \ddot{p}_1 + K_1(p_1 - p_2) = 0 \tag{6.11}$$

3. The following state variables are chosen

$$\begin{aligned} q_1 &= p_1 \\ q_2 &= \dot{q}_1 = \dot{p}_1 \\ q_3 &= p_2 \\ q_4 &= \dot{q}_3 = \dot{p}_2 \end{aligned}$$

4. The first derivatives $\dot{q}_1$ and $\dot{q}_3$ are known. The first derivatives $\dot{q}_2$ and $\dot{q}_4$ are obtained from Eqs. (6.11) and (6.10) respectively and they are given below:

$$\dot{q}_2 = -\frac{K_1}{M_1}q_1 + \frac{K_1}{M_1}q_3 \tag{6.12}$$

$$\dot{q}_4 = \frac{K_1}{M_2}q_1 - \frac{(K_1 + K_2)}{M_2}q_3 - \frac{B_2}{M_2}q_4 + \frac{1}{M_2}f(t) \tag{6.13}$$

5. The vector–matrix differential equation is thus obtained as

$$\dot{q}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_1}{M_1} & 0 & -\frac{K_1}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K_1}{M_2} & 0 & -\frac{(K_1 + K_2)}{M_2} & -\frac{B_2}{M_2} \end{bmatrix} q(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_2} \end{bmatrix} x(t)$$

Example 6.4 For the mechanical system shown in Fig. 6.5, obtain the state space model.

Solution

1. A hybrid mechanical system which is a combination of translational and rotational systems is shown in Fig. 6.5. Just by inspection, the following dynamic equations are written:
2.

$$T(t) = B_1(\dot{\theta}_1 - \dot{\theta}_2) + K_1(\theta_1 - \theta_2) \tag{6.14}$$

Also

$$T(t) = J\ddot{\theta}_2 + B_2\dot{\theta}_2 + r(M\ddot{p} + B_3\dot{p} + K_2 p) \tag{6.15}$$

Substituting $p = r\theta_2$ in Eq. (6.15), we get

$$T(t) = (J + r^2 M)\ddot{\theta}_2 + (B_2 + r^2 B_3)\dot{\theta}_2 + r^2 K_3\theta_2 \tag{6.16}$$

3. The following state variables are chosen

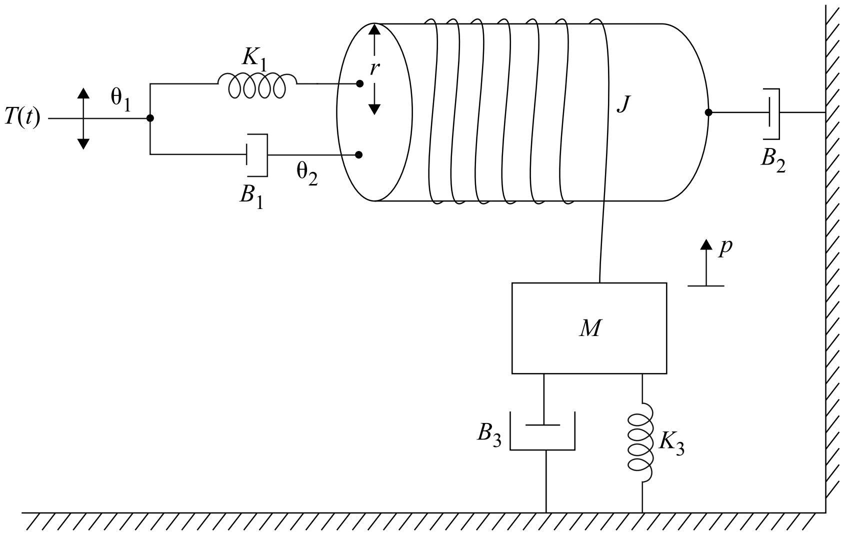


Fig. 6.5 A hybrid mechanical system

$$q_1 = \theta_1$$
$$q_2 = \theta_2$$
$$q_3 = \dot{\theta}_2 = \dot{q}_2$$

4. The derivatives of q_1 and q_2 are obtained from Eqs. (6.14) and (6.16) and are given as

$$\dot{q}_1 = -\frac{K_1}{B_1}q_1 + \frac{K_1}{B_1}q_2 + q_3 + \frac{1}{B_1}T(t)$$
$$\dot{q}_2 = q_3$$
$$\dot{q}_3 = -\frac{K_{eq}}{J_{eq}}q_2 - \frac{B_{eq}}{J_{eq}}q_3 + \frac{1}{J_{eq}}T(t)$$

where $J_{eq} = (J + r^2 M)$; $B_{eq} = (B_2 + r^2 B_3)$ and $K_{eq} = r^2 K_3$

$$\dot{q}(t) = \begin{bmatrix} -\frac{K_1}{B_1} & \frac{K_1}{B_1} & 1 \\ 0 & 0 & 1 \\ 0 & -\frac{K_{eq}}{J_{eq}} & -\frac{B_{eq}}{J_{eq}} \end{bmatrix} q(t) + \begin{bmatrix} \frac{1}{B_1} \\ 0 \\ \frac{1}{J_{eq}} \end{bmatrix} x(t)$$

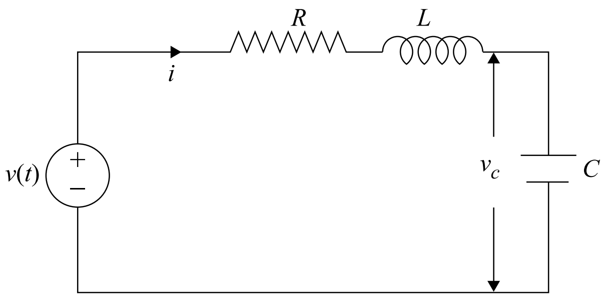


Fig. 6.6 *R-L-C* series circuit

6.3.2 State Equations for Electrical Circuits

The number of independent energy-storing elements in the electrical circuit determine the number of state variables. The capacitor and inductor are the two energy-storing elements in electrical circuits. The physical variables namely the current through the inductor and voltages across the capacitor are chosen as the states variables. The following steps are followed while forming the state space equations for electrical circuits.

1. Choose all independent inductor currents and the capacitor voltages as the state variables.
2. The state variables and their first derivatives are expressed in terms of a set of loop circuits.
3. Write loop equations and eliminate all variables other than the state variables and their first derivatives.

The following examples illustrate the method of forming state equations for electrical networks.

Example 6.5 Write the state equations for the network shown in Fig. 6.6.

Solution

1. Let i be the current passing through the inductor L and v_C be the voltage across the capacitor C. These variables are chosen as state variables.

$$q_1 = i$$
$$q_2 = v_c$$

2. The following loop equation is written

$$L\frac{di}{dt} + Ri + v_c = v(t)$$
$$\dot{q}_1 = -\frac{R}{L}q_1 - \frac{1}{L}q_2 + \frac{1}{L}v(t) \quad (6.17)$$

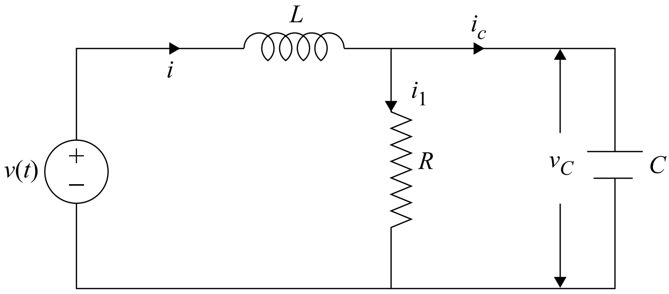


Fig. 6.7 Electrical circuit of Example 6.6

Also

$$v_c = \frac{1}{C}\int i\,dt$$

Differentiating both sides, we get

$$\dot{v}_c = \frac{1}{C}i$$
$$\dot{q}_2 = \frac{1}{C}q_1 \tag{6.18}$$

3. Equations (6.17) and (6.18) represent the first derivatives of the chosen states. Hence,

$$\dot{q}(t) = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} q(t) + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v(t)$$

Example 6.6 For the electrical network shown in Fig. 6.7, form the state equation.

Solution

1. Let i be the current passing through the inductor L and v_c be the voltage across the capacitor C. These variables are taken as the state variables. Thus,

$$q_1 = i$$
$$q_2 = v_c$$

2. The following loop equation connecting L and C is written

$$L\frac{di}{dt} + v_c = v(t)$$
$$\dot{q}_1 = -\frac{1}{L}q_2 + \frac{1}{L}v(t) \tag{6.19}$$

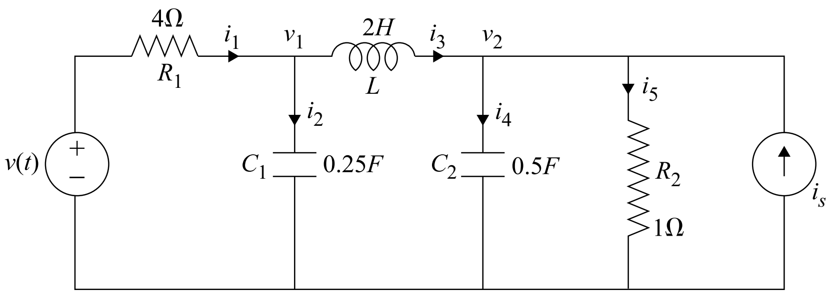


Fig. 6.8 Electrical network of Example 6.7

3. Also

$$
\begin{aligned}
i_c &= i - i_1 \\
C\frac{dv_c}{dt} &= i - \frac{v_c}{R} \\
\dot{q}_2 &= \frac{1}{C}q_1 - \frac{1}{RC}q_2
\end{aligned}
\tag{6.20}
$$

4. Combining Eqs. (6.19) and (6.20), we get

$$
\dot{q}(t) = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix} q(t) + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} v(t)
$$

Example 6.7 For the electrical network shown in Fig. 6.8, form the state equations.

Solution

1. There are three energy-storing elements, namely L, C_1, and C_2, and they are independently connected and there should be three state variables. The following state variables are chosen.

$$
\begin{aligned}
q_1 &= v_1 \\
q_2 &= v_2 \\
q_3 &= i_3
\end{aligned}
$$

2. The first derivative of these variables is obtained as follows. At v_1 node,

$$
\begin{aligned}
i_1 &= i_2 + i_3 \\
\frac{(v - v_1)}{R_1} &= C_1\frac{dv_1}{dt} + i_3
\end{aligned}
$$

$$\frac{dv_1}{dt} = -\frac{1}{R_1C_1}v_1 - \frac{1}{C_1}i_3 + \frac{1}{R_1C_1}v$$

$$\dot{q}_1 = -\frac{1}{R_1C_1}q_1 - \frac{1}{C_1}q_3 + \frac{1}{R_1C_1}v \tag{6.21}$$

3. At note v_2, the following equation is written

$$i_3 + i_s = i_4 + i_5$$
$$= C_2\frac{dv_2}{dt} + \frac{v_2}{R_2}$$

Substituting the state variables, we get

$$q_3 + i_s = C_2\dot{q}_2 + \frac{q_2}{R_2}$$

$$\dot{q}_2 = -\frac{1}{R_2C_2}q_2 + \frac{1}{C_2}q_3 + \frac{1}{C_2}i_s \tag{6.22}$$

4. The voltage drop across the inductor L is

$$L\frac{di_3}{dt} = v_1 - v_2$$

$$\dot{q}_3 = \frac{1}{L}q_1 - \frac{1}{L}q_2 \tag{6.23}$$

5. Combining Eqs. (6.21), (6.22) and (6.23), we get

$$\dot{q}(t) = \begin{bmatrix} -\frac{1}{R_1C_1} & 0 & -\frac{1}{C_1} \\ 0 & -\frac{1}{R_2C_2} & \frac{1}{C_2} \\ \frac{1}{L} & -\frac{1}{L} & 0 \end{bmatrix} q(t) + \begin{bmatrix} \frac{1}{R_1C_1} & 0 \\ 0 & \frac{1}{C_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ i_s(t) \end{bmatrix}$$

Substituting the numerical values, we get

$$\dot{q}(t) = \begin{bmatrix} -1 & 0 & -4 \\ 0 & -2 & 2 \\ 0.5 & -0.5 & 0 \end{bmatrix} q(t) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v(t) \\ i_s(t) \end{bmatrix}$$

Example 6.8 For the electrical network shown in Fig. 6.9, form the vector–matrix differential equation.

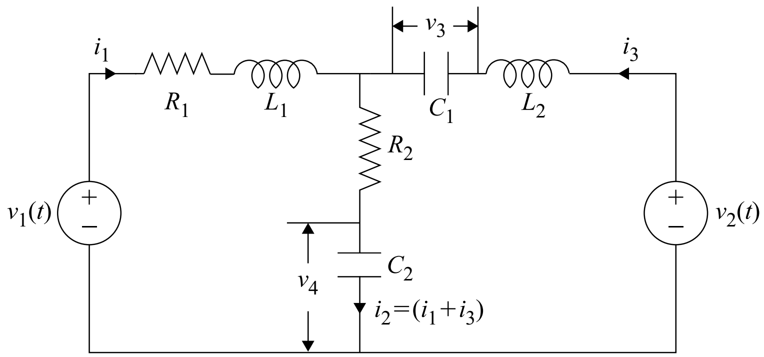


Fig. 6.9 Electrical network of Example 6.8

Solution

1. There are four energy-storing elements, namely $L_1,\ L_2,\ C_1$, and C_2. Hence, four state variables, the current through the inductors and the voltages across the capacitors are chosen. Thus,

$$q_1 = i_1$$
$$q_2 = i_3$$
$$q_3 = v_3$$
$$q_4 = v_4$$

   The dimension of A matrix is 4×4. Since there are two inputs namely $v_1(t)$ and $v_2(t)$, the dimension of B matrix is 3×2.

2. The following loop equation is written connecting $v_1(t),\ R_1,\ L_1,\ R_2$, and C_2.

$$v_1(t) = L_1\frac{di_1}{dt} + i_1R_1 + (i_1 + i_3)R_2 + v_4$$
$$\dot{q}_1 = -\frac{(R_1 + R_2)}{L_1}q_1 - \frac{R_2}{L_1}q_2 - \frac{1}{L_1}q_4 + \frac{1}{L_1}v_1(t) \tag{6.24}$$

3. The following loop equation is written connecting $v_2(t),\ L_2,\ C_1,\ R_2$, and C_2.

$$v_2(t) = L_2\frac{di_3}{dt} + v_3 + (i_1 + i_3)R_2 + v_4$$
$$\dot{q}_2 = -\frac{R_2}{L_2}q_1 - \frac{R_2}{L_2}q_2 - \frac{1}{L_2}q_3 - \frac{1}{L_2}q_4 + \frac{1}{L_2}v(t) \tag{6.25}$$

4. For the capacitance C_1, the following equation is written

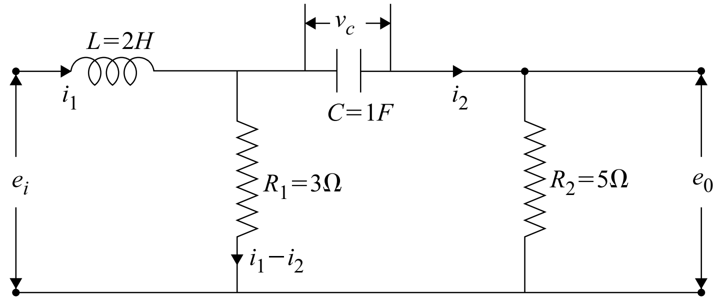


Fig. 6.10 Electrical network of Example 6.9

$$C_1\frac{dv_3}{dt} = i_3$$
$$\dot{q}_3 = \frac{1}{C_1}q_2 \tag{6.26}$$

5. For the capacitance C_2, the following equation is written

$$C_2\frac{dv_4}{dt} = i_1 + i_3$$
$$\dot{q}_4 = \frac{1}{C_2}q_1 + \frac{1}{C_2}q_2 \tag{6.27}$$

6. Combination Eqs. (6.24)–(6.26) and (6.27) the following vector–matrix differential equation is obtained

$$\dot{q}(t) = \begin{bmatrix} -\frac{(R_1+R_2)}{L_1} & -\frac{R_2}{L_1} & 0 & -\frac{1}{L_1} \\ -\frac{R_2}{L_2} & -\frac{R_2}{L_2} & -\frac{1}{L_2} & -\frac{1}{L_2} \\ 0 & \frac{1}{C_1} & 0 & 0 \\ \frac{1}{C_2} & \frac{1}{C_2} & 0 & 0 \end{bmatrix} q(t) + \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ \\ v_2(t) \end{bmatrix}$$

Example 6.9 Develop the state model for the electrical network shown in Fig. 6.10.

(*Anna University, April, 2004*)

Solution

1. There are two energy-storing elements, L and C. The current i_1 through L_1 and voltage v_c across C are chosen as the state variables. Thus,

$$q_1 = i_1$$
$$q_2 = v_c$$

2. The following equation is written connecting R_1, R_2, and C.

$$v_c + i_2 R_2 = (i_1 - i_2) R_1$$
$$i_2 = i_1 \frac{R_1}{(R_1 + R_2)} - \frac{v_c}{(R_1 + R_2)} \tag{6.28}$$

The following equation is written connecting e_i, L, C, and R_2.

$$e_i = L\frac{di_1}{dt} + v_c + i_2 R_2$$
$$= L\frac{di_1}{dt} + v_c + i_1 \frac{R_1 R_2}{R_1 + R_2} - \frac{v_c R_2}{(R_1 + R_2)}$$
$$\dot{q}_1 = -\frac{R_1 R_2}{L(R_1 + R_2)} q_1 + \frac{1}{L}\left(\frac{R_2}{(R_1 + R_2)} - 1\right) q_2 + \frac{1}{L} e_i \tag{6.29}$$

3. For the capacitor C, the following equation is written

$$C\frac{dv_c}{dt} = i_2 = \frac{i_1 R_1}{(R_1 + R_2)} - \frac{v_c}{(R_1 + R_2)}$$
$$\dot{q}_2 = -\frac{R_1}{C(R_1 + R_2)} q_1 - \frac{1}{C(R_1 + R_2)} q_2 \tag{6.30}$$

4. Combining Eqs. (6.29) and (6.30), we get

$$\dot{q}(t) = \begin{bmatrix} -\frac{(R_1 R_2)}{L(R_1+R_2)} & \frac{1}{L}\left(\frac{R_2}{R_1+R_2} - 1\right) \\ \frac{R_1}{C(R_1+R_2)} & -\frac{1}{C(R_1+R_2)} \end{bmatrix} q(t) + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} e_i$$

Substituting the numerical value, we get

$$\dot{q}(t) = \begin{bmatrix} \frac{15}{16} & -\frac{3}{16} \\ \frac{3}{8} & \frac{1}{8} \end{bmatrix} q(t) + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} e_i$$

5. The output

$$e_0 = i_2 R_2$$
$$= \frac{R_1 R_2}{R_1 + R_2} i_1 - \frac{v_c}{R_1 + R_2}$$
$$= \frac{1}{8}(15 q_1 - q_2)$$
$$y = e_0 = \frac{1}{8}[15 - 1] \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

6.4 State Equations from Transfer Function

State equations can be easily obtained from the transfer function of the system. Consider the first-order system with the following transfer function:

$$H(s) = \frac{10}{(s+a)}$$

The system realization is shown in Fig. 6.11.

From Fig. 6.11, the following equations are derived.

$$\begin{aligned} \dot{q} &= -aq + x \\ y &= 10q \end{aligned} \tag{6.31}$$

The output of each integrator ($\frac{1}{s}$) is chosen as one state variable. Thus, for an nth-order system, n integrators are required. The following methods of realization are used to determine the state equation. They are

1. The direct form-II.
2. The cascade form.
3. The parallel form.

Example 6.10 Determine the state space model of a continuous-time system whose transfer function is given by

$$H(s) = \frac{3s^2 + 24s + 44}{(s^3 + 12s^2 + 44s + 48)}$$

Use the following methods:

(a) The direct form-II.
(b) The cascade form.
(c) The parallel form.

Show that the A matrix is not unique for the given system.

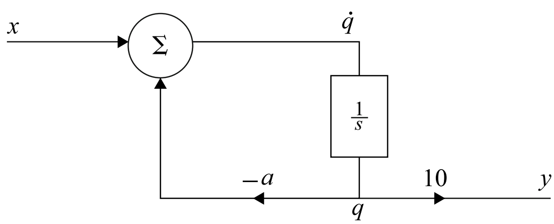


Fig. 6.11 First-order T.F. realization

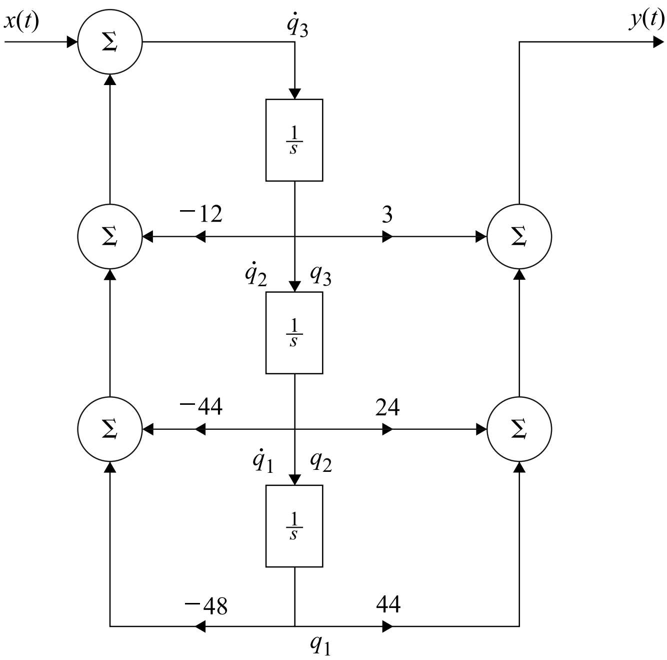


Fig. 6.12 Direct form-II realization of $H(s) = \dfrac{3s^2 + 24s + 44}{(s^3 + 12s^2 + 44s + 48)}$

Solution

(a) **The Direct Form-II**

$$H(s) = \frac{3s^2 + 24s + 44}{(s^3 + 12s^2 + 44s + 48)}$$

The above equation can be written as

$$H(s) = \frac{\frac{3}{s} + \frac{24}{s^2} + \frac{44}{s^3}}{1 + 12\frac{1}{s} + 44\frac{1}{s^2} + \frac{48}{s^3}}$$

Here, $b_0 = 0; b_1 = 3; b_2 = 24; b_3 = 44; a_1 = 12; a_2 = 44; a_3 = 48$. The direct form-II realization of $H(s)$ is shown in Fig. 6.12. From Fig. 6.12, the following equations are written:

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= q_3 \\ \dot{q}_3 &= -48q_1 - 44q_2 - 12q_3 + x(t) \end{aligned}$$

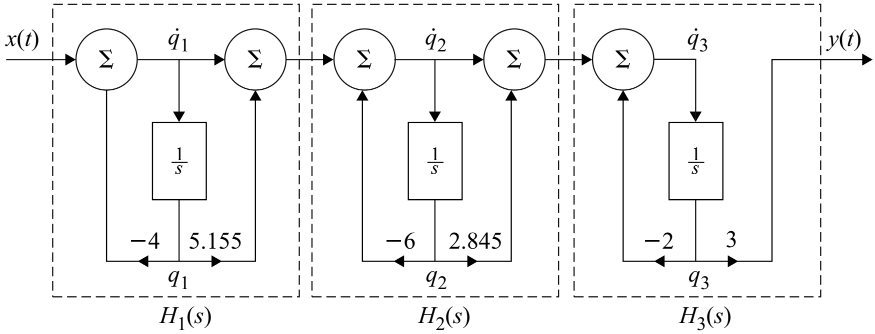


Fig. 6.13 The cascade realization of $H(s) = \dfrac{3(s+5.155)(s+2.845)}{(s+4)(s+6)(s+2)}$

The state equation for $H(s)$ is therefore written as

$$\dot{q}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -48 & -44 & -12 \end{bmatrix}}_{A} q(t) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} x(t) \tag{6.32}$$

Also from Fig. 6.12, the output $y(t)$ is obtained as

$$y(t) = 44q_1 + 24q_2 + 3q_3$$
$$y(t) = [\underbrace{44 \quad 24 \quad 3}_{C}]q(t)$$

(b) **The Cascade Form**

$$H(s) = \frac{3s^2 + 24s + 44}{(s^3 + 12s^2 + 44s + 48)}$$

The above equation can be written as

$$\begin{aligned} H(s) &= \frac{(s+5.155)}{(s+4)} \frac{(s+2.845)}{(s+6)} 3 \frac{1}{(s+2)} \\ &= H_1(s) H_2(s) H_3(s) \end{aligned}$$

The cascade form realization is shown in Fig. 6.13.
From Fig. 6.13, the following equations are for the first derivatives of the states

$$\begin{aligned}
\dot{q}_1 &= -4q_1 + x(t) \\
\dot{q}_2 &= -6q_2 + \dot{q}_1 + 5.155q_1 \\
&= -6q_2 - 4q_1 + x(t) + 5.155q_1 \\
&= 1.155q_1 - 6q_2 + x(t) \\
\dot{q}_3 &= -2q_3 + \dot{q}_2 + 2.8455q_2 \\
&= -2q_3 + 1.155\dot{q}_1 - 6q_2 + x(t) + 2.845q_2
\end{aligned}$$

$$\begin{aligned}
\dot{q}_3 &= 1.155q_1 - 3.155q_2 - 2q_3 + x(t) \\
y(t) &= 3q_3
\end{aligned}$$

The state equations are given below:

$$\begin{aligned}
\dot{q}(t) &= \begin{bmatrix} -4 & 0 & 0 \\ 1.155 & -6 & 0 \\ 1.155 & -3.155 & -2 \end{bmatrix} q(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x(t) \\
y(t) &= [0 \quad 0 \quad 3]\, q(t)
\end{aligned} \tag{6.33}$$

(c) **The Parallel Form Realization**

$$\begin{aligned}
H(s) &= \frac{3s^2 + 24s + 44}{(s^3 + 12s^2 + 44s + 48)} \\
&= \frac{3s^2 + 24s + 44}{(s+2)(s+4)(s+6)} \\
&= \frac{1}{(s+2)} + \frac{1}{(s+4)} + \frac{1}{(s+6)} \\
&= H_1(s) + H_2(s) + H_3(s)
\end{aligned}$$

The parallel form realization of $H(s)$ is shown in Fig. 6.14. From Fig. 6.14, the following equations are written for the first derivatives of the states

$$\begin{aligned}
\dot{q}_1 &= -2q_1 + x(t) \\
\dot{q}_2 &= -4q_2 + x(t) \\
\dot{q}_3 &= -6q_3 + x(t) \\
y(t) &= q_1 + q_2 + q_3
\end{aligned}$$

$$\begin{aligned}
\dot{q}(t) &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix} q(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x(t) \\
y(t) &= [1 \quad 1 \quad 1]q(t)
\end{aligned} \tag{6.34}$$

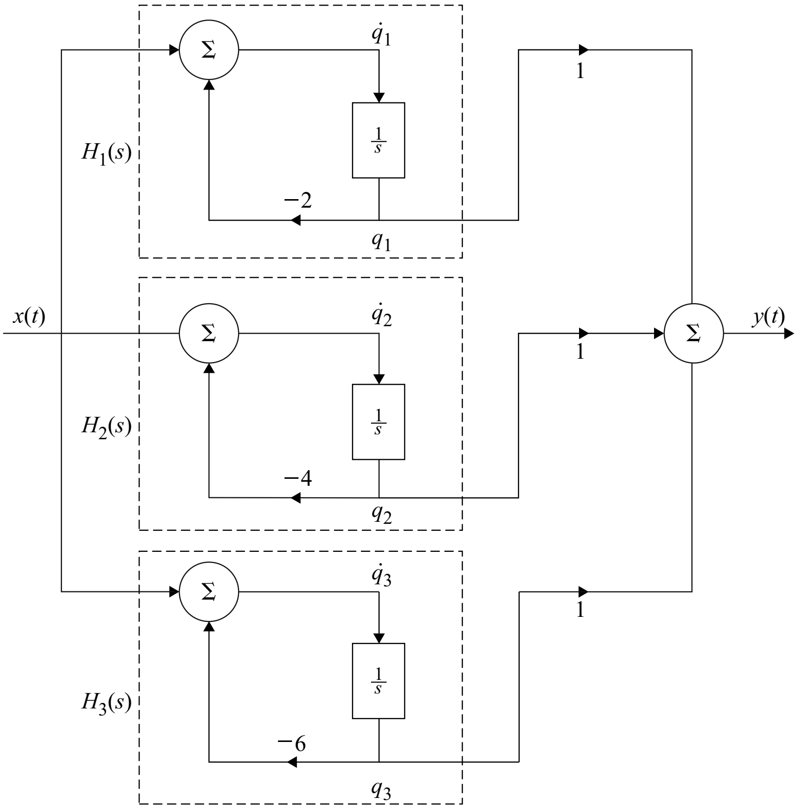


Fig. 6.14 The parallel form realization of $H(s)$

Equations (6.32), (6.33) and (6.34) give the state space description of the system T.F. and **the system A matrices are not unique**. But all of them will give the same characteristics of the system. In parallel form representation, the eigen values of the system T.F. form the diagonal elements of the A matrix.

Example 6.11 Consider the following T.F. of a certain system

$$H(s) = \frac{10(s+2)}{s^2(s+1)^2(s+4)}$$

Determine A, B, and C matrices using parallel form realization.

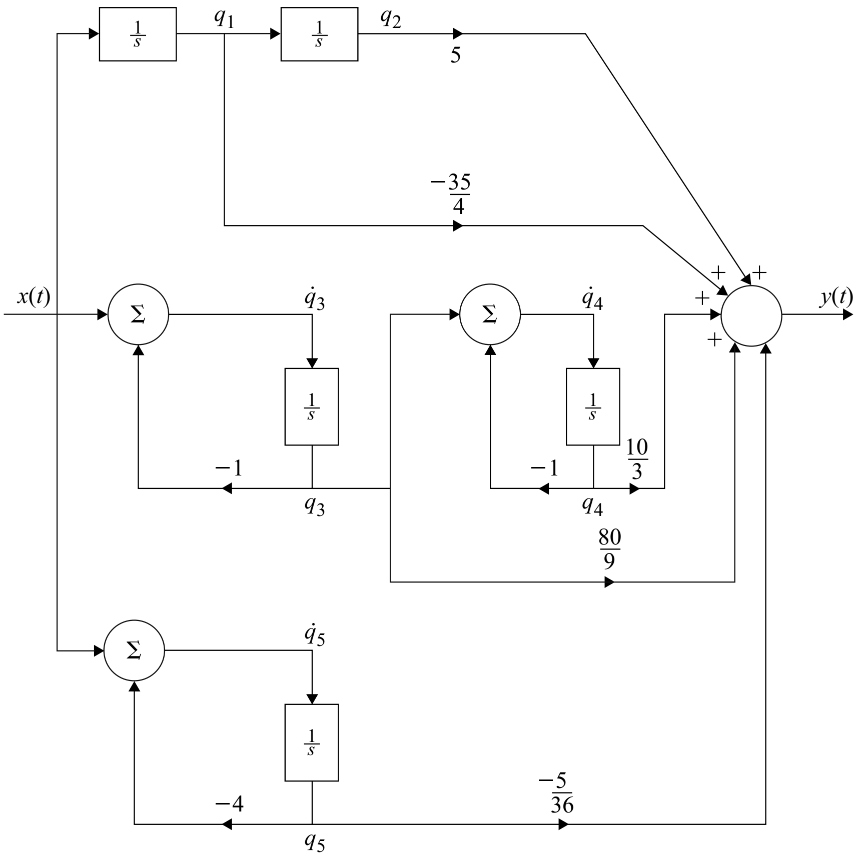


Fig. 6.15 The parallel form realization of $H(s)$ for Example 6.11

Solution

$$
\begin{aligned}
H(s) &= \frac{10(s+2)}{s^2(s+1)^2(s+4)} \\
&= \frac{A_1}{s^2} + \frac{A_2}{s} + \frac{A_3}{(s+1)^2} + \frac{A_4}{(s+1)} + \frac{A_5}{(s+4)}
\end{aligned}
\tag{6.35}
$$

The residues A_1, A_2, A_3, A_4, and A_5 are determined by any one method discussed in previous chapters and are given below:

$$
A_1 = 5; \quad A_2 = -\frac{35}{4}; \quad A_3 = \frac{10}{3}; \quad A_4 = \frac{80}{9}; \quad \text{and} \quad A_5 = -\frac{5}{36}
$$

$$H(s) = -\frac{35}{4}\frac{1}{s} + \frac{5}{s^2} + \frac{80}{9}\frac{1}{(s+1)} + \frac{10}{3}\frac{1}{(s+1)^2} - \frac{5}{36}\frac{1}{(s+1)} \tag{6.36}$$

Equation (6.36) is represented in Fig. 6.15. From Fig. 6.15, the following equations in terms of state variables

$$\begin{aligned}
\dot{q}_1 &= x(t) \\
\dot{q}_2 &= q_1 \\
\dot{q}_3 &= -q_3 + x(t) \\
\dot{q}_4 &= q_3 - q_4 \\
\dot{q}_5 &= -4q_5 + x(t) \\
y(t) &= -\frac{35}{4}q_1 + 5q_2 + \frac{80}{9}q_3 + \frac{10}{3}q_4 - \frac{5}{36}q_5
\end{aligned}$$

The above equations are written in vector–matrix differential equation form as

$$\dot{q}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix} q(t) + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} x(t)$$

$$y(t) = \left[-\frac{35}{4},\ 5,\ \frac{80}{9},\ \frac{10}{3},\ -\frac{5}{36}\right] q(t)$$

6.4.1 General Case of Representation

The state space description can be done in several ways. However, the state variables obtained from direct form II are quite convenient since the state equations can be immediately written just by inspection of the transfer function. Consider the general Nth-order transfer function given below:

$$\begin{aligned}
H(s) &= \frac{b_0 s^N + b_1 s^{N-1} + b_2 s^{N-2} + \cdots + b_N}{s^N + a_1 s^{N-1} + a_2 s^{N-2} + \cdots + a_N} \\
&= \frac{\left(b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \cdots + \frac{b_N}{s^N}\right)}{\left(1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \cdots + \frac{a_N}{s^N}\right)}
\end{aligned} \tag{6.37}$$

Equation (6.37) is realized in direct form II structure and is shown in Fig. 6.16. From Fig. 6.16, the following equation for the state variable is written:

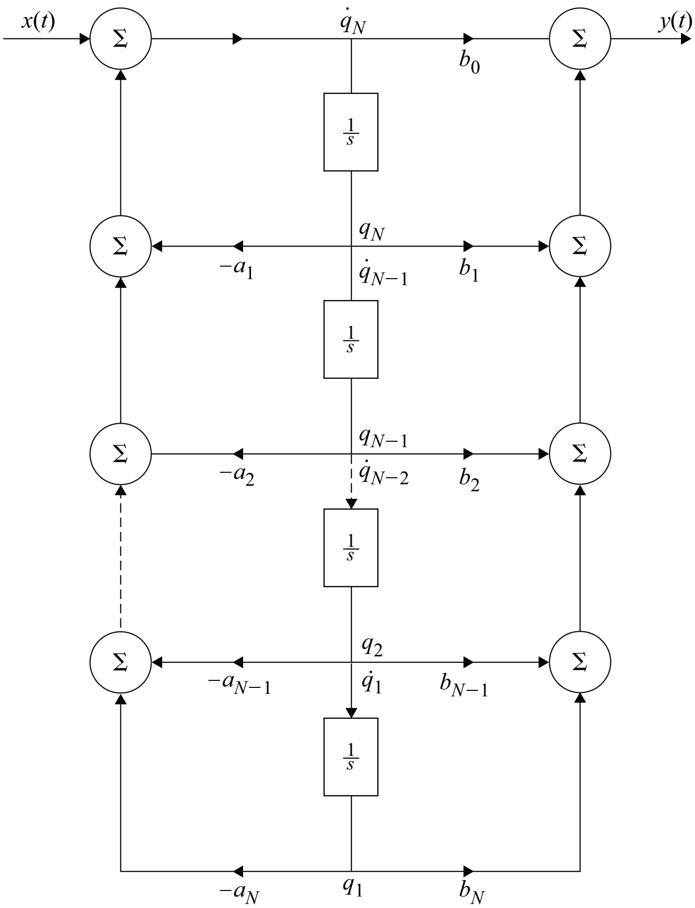


Fig. 6.16 Direct form II realization of *N*th-order LTIC system

$$\begin{aligned}
\dot{q}_1 &= q_2 \\
\dot{q}_2 &= q_3 \\
\dot{q}_3 &= q_4 \\
&\vdots \\
\dot{q}_{N-1} &= q_N \\
\dot{q}_N &= -a_N q_1 - a_{N-1} q_2, \ldots, -a_2 q_{N-1} - a_1 q_N + x(t)
\end{aligned} \tag{6.38}$$

$$\begin{aligned} y(t) &= b_N q_1 + b_{N-1} q_2 + \cdots + b_1 q_N + b_0 \dot{q}_N \\ &= (b_N - b_0 a_N) q_1 + (b_{N-1} - b_0 a_{N-1}) q_2 + \cdots \\ &\quad + (b_1 - b_0 a_1) q_N + b_0 x(t) \end{aligned} \tag{6.39}$$

Equations (6.38) and (6.39) can be represented in matrix form as given below:

$$\begin{aligned} \dot{q}(t) &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{N-1} \\ q_N \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} x(t) \\ y(t) &= [(b_N - b_0 a_N)(b_{N-1} - b_0 a_{N-1}) \ldots (b_1 - b_0 a_1)] q + b_0 x(t) \\ &= [\bar{b}_N \quad \bar{b}_{N-1} \ldots \bar{b}_1] q + b_0 x(t) \end{aligned} \tag{6.40}$$

where $\bar{b}_N = (b_N - b_0 a_N)$. The A matrix given in Eq. (6.40) is said to be in phase variable canonical form.

6.4.2 Step by Step Procedure to Determine A, B and C Matrices

1. If the system is described by linear differential equation, convert that into T.F. form. Find the coefficients of numerator polynomial $b_0, b_1, \ldots, b_N$ and the coefficients of the denominator polynomials $a_0, a_1, \ldots, a_N$.
2. The elements of A matrix are written in phase variable canonical form. The elements of the last row are written in the reverse order with a negative sign as $-a_N, -a_{N-1}, \ldots, -a_2, a_1$. The elements of B matrix and identified with 0s in all the rows and 1 in the last row.
3. The elements of C matrix are identified as given in Eq. (6.40). To remember this in an easier way, the elements of the first column are obtained from the product $b_0 a_N$ being subtracted from b_N, the second column from the product $b_0 a_{N-1}$ being subtracted from b_{N-1} and so on. This can be easily viewed from Fig. 6.16. For the state q_1, the right side branch gain is b_N. The left side branch gain after being multiplied by b_0 is $-b_0 a_N$. The sum of these two is $(b_N - b_0 a_N)$. Similarly, the second column of C is obtained which corresponds to the state q_2. This is nothing but $(b_{N-1} - b_0 a_{N-1})$.

Example 6.12 Consider the following differential equation which describes the dynamics of a continuous-time system

$$5\frac{d^4 y}{dt^4} + 2\frac{d^3 y}{dt^3} + 4\frac{d^2 y}{dt^2} + 7\frac{dy}{dt} + 8y = 8\frac{dx(t)}{dt} + 7x(t)$$

Form the state space equation.

Solution

$$5\frac{d^4y}{dt^4} + 2\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 8y = 8\frac{dx(t)}{dt} + 7x(t)$$

Taking Laplace transform on both sides of the above equation, we get

$$H(s) = \frac{Y(s)}{X(s)} = \frac{(8s+7)}{5\left(s^4 + \frac{2}{5}s^3 + \frac{4}{5}s^2 + \frac{7}{5}s + \frac{8}{5}\right)}$$

where

$$b_0 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = \frac{8}{5},\ b_4 = \frac{7}{5},\ a_1 = \frac{2}{5},\ a_2 = \frac{4}{5},\ a_3 = \frac{7}{5},\ a_4 = \frac{8}{5}.$$

The state equation is written in phase variable canonical form as given below:

$$\dot{q}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{8}{5} & -\frac{7}{5} & -\frac{4}{5} & -\frac{2}{5} \end{bmatrix} q(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x(t)$$

$$\bar{b}_4 = b_4 - b_0a_4 = \frac{7}{5}$$

$$\bar{b}_3 = b_3 - b_0a_3 = \frac{8}{5}$$

$$\bar{b}_2 = b_2 - b_0a_2 = 0$$

$$\bar{b}_1 = b_1 - b_0a_1 = 0$$

$$y = \frac{1}{5}\underbrace{[7 \quad 8 \quad 0 \quad 0]}_{C}q$$

Example 6.13 Consider the following T.F. of a certain continuous-time system

$$H(s) = \frac{7s^3 + 11s^2 + 14s + 10}{s^3 + 8s^2 + 5s + 4}$$

Form the state equations.

Solution

$$H(s) = \frac{7s^3 + 11s^2 + 14s + 10}{s^3 + 8s^2 + 5s + 4}$$

where

$$b_0 = 7,\ b_1 = 11,\ b_2 = 14,\ b_3 = 10,\ a_1 = 8,\ a_2 = 5,\ a_3 = 4.$$

$$\dot{q}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -5 & -8 \end{bmatrix} q(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x(t)$$

$$\bar{b}_3 = (b_3 - b_0 a_3) = (10 - 28) = -18$$
$$\bar{b}_2 = (b_2 - b_0 a_2) = (14 - 35) = -21$$
$$\bar{b}_1 = (b_1 - b_0 a_1) = (11 - 56) = -45$$
$$y = [-18 \quad -21 \quad -45]q + 7x(t)$$

6.5 Transfer Function of Continuous-Time System from State Equations

Consider the state Eqs. (6.1) and (6.2)

$$\dot{q}(t) = Aq(t) + Bx(t)$$
$$y(t) = Cq(t) + Dx(t)$$

Let the initial conditions be zero. Taking Laplace transform on both sides of the above equations, we get

$$[sI - A]Q(s) = BX(s)$$

Pre-multiplying both sides by $[sI - A]^{-1}$, we get

$$Q(s) = [sI - A]^{-1}BX(s) \tag{6.41}$$

Similarly,

$$Y(s) = CQ(s) + DX(s) = C[sI - A]^{-1}BX(s) + DX(s)$$

$$\frac{Y(s)}{X(s)} = H(s) = C[sI - A]^{-1}B + D \tag{6.42}$$

$[sI - A]^{-1}$ is called the state transition matrix (STM). In Eq. (6.42),

$$[sI - A]^{-1} = \frac{\text{Adjoint } [sI - A]}{\text{Determinant } [sI - A]}$$

Example 6.14 Consider the following state equations

$$\dot{q} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} q + \begin{bmatrix} 1 \\ 0 \end{bmatrix} q(t)$$
$$y = [0 \quad 1] q(t)$$

Determine the system function.

Solution

$$\dot{q} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} q + \begin{bmatrix} 1 \\ 0 \end{bmatrix} q$$
$$y = [0 \quad 1] q$$
$$(sI - A) = \begin{bmatrix} (s+3) & -1 \\ 2 & s \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s & 1 \\ -2 & (s+3) \end{bmatrix}$$
$$(sI - A)^{-1} B = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s & 1 \\ -2 & (s+3) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \frac{1}{(s+1)(s+2)} \begin{bmatrix} s \\ -2 \end{bmatrix}$$
$$H(s) = \frac{Y(s)}{X(s)} = C[sI - A]^{-1} B$$
$$= \frac{1}{(s+1)(s+2)} [0 \quad 1] \begin{bmatrix} s \\ -2 \end{bmatrix}$$

$$H(s) = \frac{-2}{(s+1)(s+2)}$$

6.6 Solution of State Equations

The state equations of a linear time invariant system are solved in both time and frequency domains. In the frequency domain, the Laplace transform method is used. These two methods are discussed below.

6.6.1 Laplace Transform Solution of State Equations

Consider the vector–matrix differential equation of (6.1)

$$\dot{q} = Aq + Bx(t)$$

Taking Laplace transform on both sides of the above equation, we get

$$\begin{aligned} sQ(s) - q(0) &= AQ(s) + BX(s) \\ (sI - A)Q(s) &= q(0) + BX(s) \end{aligned} \tag{6.43}$$

Pre-multiplying both sides of Eq. (6.43) by $[sI - A]^{-1}$, we get

$$\begin{aligned} Q(s) &= [sI - A]^{-1}[q(0) + BX(s)] \\ &= \phi(s)[q(0) + BX(s)] \end{aligned} \tag{6.44}$$

where

$$\phi(s) = [sI - A]^{-1} \tag{6.45}$$

$$Q(s) = \phi(s)q(0) + \phi(s)BX(s)$$

Taking the inverse Laplace transform, we get

$$q(t) = L^{-1}[\phi(s)q(0)] + L^{-1}[\phi(s)BX(s)] \tag{6.46}$$

$\phi(s)$ defined in Eq. (6.45) is the STM. In Eq. (6.46) $L^{-1}[\phi(s)q(0)]$ gives the zero input response and $L^{-1}[\phi(s)BX(s)]$ gives zero state response.

Example 6.15 A certain system is described by the following state equation:

$$\dot{q} = Aq + Bx(t)$$

where

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad x(t) = u(t)$$

The initial conditions are $q_1(0) = 1$ and $q_2(0) = -1$. Find STM and hence $q(t)$. Also find $y(t)$ if $C = [0 \quad 1]$.

Solution

$$\dot{q} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} q + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t)$$

$$(sI - A) = \begin{bmatrix} (s+3) & 1 \\ 2 & s \end{bmatrix}$$

The STM is

$$\phi(s) = [sI - A]^{-1} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}$$

$$Q(s) = [sI - A]^{-1} q(0) + [sI - A]^{-1} BX(s)$$

$$(sI - A)^{-1} q(0) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{(s+1)(s+2)} \begin{bmatrix} (s-1) \\ -(s+5) \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{(s-1)}{(s+1)(s+2)} \\ -\dfrac{(s+5)}{(s+1)(s+2)} \end{bmatrix}$$

Given $x(t) = u(t)$ and $X(s) = \frac{1}{s}$

$$[sI - A]^{-1} BX(s) = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{s}$$

$$= \frac{1}{s(s+1)(s+2)} \begin{bmatrix} s \\ -2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{(s+1)(s+2)} \\ -\dfrac{2}{s(s+1)(s+2)} \end{bmatrix}$$

$$q(t) = L^{-1}\{[sI - A]^{-1} q(0) + [sI - A]^{-1} BX(s)\}$$

$$= L^{-1} \begin{bmatrix} \dfrac{(s-1)}{(s+1)(s+2)} \\ \dfrac{(s+5)}{(s+1)(s+2)} \end{bmatrix} + L^{-1} \begin{bmatrix} \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{s(s+1)(s+2)} \end{bmatrix}$$

$$= L^{-1} \begin{bmatrix} -\dfrac{2}{(s+1)} + \dfrac{3}{(s+2)} \\ -\dfrac{4}{(s+1)} + \dfrac{3}{(s+2)} \end{bmatrix} + L^{-1} \begin{bmatrix} \dfrac{1}{(s+1)} - \dfrac{1}{(s+2)} \\ -\dfrac{1}{s} + \dfrac{2}{(s+1)} - \dfrac{1}{(s+2)} \end{bmatrix}$$

$$q(t) = \begin{bmatrix} 2e^{-2t} - e^{-t} \\ 2e^{-2t} - 2e^{-t} - 1 \end{bmatrix}$$

$$q_1(t) = (2e^{-2t} - e^{-t})u(t)$$
$$q_2(t) = (2e^{-2t} - 2e^{-t} - 1)u(t)$$
$$y(t) = Cq(t) = 2e^{-2t} - 2e^{t} - 1$$

6.6.2 Time Domain Solution to State Equations

Consider the state equation

$$\dot{q} = Aq + Bx(t) \tag{6.47}$$

Pre-multiplying Eq. (6.47) by e^{-At} on both sides, we get

$$e^{-At}\dot{q} = e^{-At}Aq + e^{-At}Bx(t) \tag{6.48}$$
$$(e^{-At}\dot{q} - e^{-At}Aq) = e^{-At}Bx(t)$$
$$\frac{d}{dt}[e^{-At}q] = e^{-At}Bx(t)$$

By integrating both sides of the above equation from 0 to t, we get

$$e^{-At}q - q(0) = \int_0^t e^{-A\tau}Bx(\tau)d\tau$$

$$q(t) = e^{At}q(0) + \int_0^t e^{A(t-\tau)}Bx(\tau)d\tau \tag{6.49}$$

In Eq. (6.49), e^{At} is the STM, Eq. (6.49) can be generalized to any initial value t_0 and hence it can be modified as

$$\begin{aligned} q(t) &= e^{A(t-t_0)}q(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bx(\tau)d\tau \\ &= \text{Free response} + \text{Forced response} \end{aligned} \tag{6.50}$$

6.6.3 Determination of e^{At}—The Cayley–Hamilton Theorem

To determine $q(t)$ which is the solution of the vector–matrix differential Eq. (6.49), it is necessary to determine the STM e^{At}. This can be obtained using the Cayley–Hamilton theorem. According to this theorem, an $n \times n$ square matrix A satisfies its own characteristic equation $|\lambda I - A| = 0$ where λ_s are the eigen values of system matrix A. Let

$$F(A) = \sum_{k=0}^{n-1} \alpha^k A^k$$

$$F(\lambda) = \sum_{k=0}^{n-1} \alpha^k \lambda^k$$

$$e^{At} = \sum_{k=0}^{n-1} \alpha^k A^k \tag{6.51}$$

Equation (6.51) should satisfy for all the eigen values of A.

6.6.3.1 Determination of e^{At} for Distinct Eigen Values of A

If the eigen values of A are distinct, then the following procedure is followed to evaluate e^{At}.

1. Determine the eigen values of the matrix A from $|\lambda I - A| = 0$.
2. For distinct eigen values the following equations are written:

$$\begin{aligned} e^{\lambda_1 t} &= \alpha_0 + \alpha_1\lambda_1 + \alpha_2\lambda_1^2 + \cdots + \alpha_{n-1}\lambda_1^{n-1} \\ e^{\lambda_2 t} &= \alpha_0 + \alpha_1\lambda_2 + \alpha_2\lambda_2^2 + \cdots + \alpha_{n-1}\lambda_2^{n-1} \\ &\vdots \\ e^{\lambda_n t} &= \alpha_0 + \alpha_1\lambda_n + \alpha_2\lambda_n^2 + \cdots + \alpha_{n-1}\lambda_n^{n-1} \end{aligned} \tag{6.52}$$

   Thus, we will have n simultaneous equations if there are n distinct eigen values. By solving these simultaneous equations, $\alpha_0,\ \alpha_1,\ \alpha_2, \ldots, \alpha_n$ can be determined.
3. Using the following equation, e^{At} can be evaluated.

$$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \cdots + \alpha_{n-1} A^{n-1} \tag{6.53}$$

   where $I = n \times n$ identity matrix.

6.6.3.2 Determination of e^{At} for Multiple Eigen Values of A

Let us assume that $\lambda = \lambda_i$ has multiplicity of m. If all the other eigen values are distinct, then the number of distinct eigen values are $(n - m + 1)$. Corresponding to these eigen values, we will have $(n - m + 1)$ independent equations. For the rest $(m - 1)$ we use Cauchy's residue theorem.

$$\left.\frac{d^i f(\lambda)}{d\lambda^i}\right|_{\lambda=\lambda_i} = \frac{d^i}{d\lambda_i}\left[\sum_{k=0}^{n-1} \alpha_k \lambda^k\right]_{\lambda=\lambda_i} \tag{6.54}$$

where $i = 1, 2, 3, \ldots, (m-1)$. The following examples illustrate the method of evaluating STM and the solution for $q(t)$.

Example 6.16 Consider the following vector–matrix differential equation

$$\dot{q} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} q + \begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t)$$

with the initial conditions

$$q(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Find (a) e^{At} and (b) $q(t)$ for unit step input signal.

Solution

$$\begin{aligned} A &= \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \\ |\lambda I - A| &= \lambda^2 + 3\lambda + 2 \\ &= (\lambda + 1)(\lambda + 2) \\ \lambda_1 &= -1 \\ \lambda_2 &= -2 \end{aligned}$$

using Eq. (6.52), we get

$$\begin{aligned} e^{-t} &= \alpha_0 - \alpha_1 \\ e^{-2t} &= \alpha_0 - 2\alpha_1 \end{aligned}$$

solving the above simultaneous equations for α_0 and α_1, we get

$$\begin{aligned} \alpha_0 &= 2e^{-t} - e^{-2t} \\ \alpha_1 &= e^{-t} - e^{-2t} \\ e^{At} &= \alpha_0 I + \alpha_1 A \\ &= \begin{bmatrix} (2e^{-t} - e^{-2t}) & 0 \\ 0 & (2e^{-t} - e^{-2t}) \end{bmatrix} + (e^{-t} - e^{-2t}) \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (-e^{-t} + 2e^{-2t}) & (e^{-t} - e^{-2t}) \\ 2(-e^{-t} + e^{-2t}) & (2e^{-t} - e^{-2t}) \end{bmatrix} \\ q(t) &= e^{At} q(0) + \int_0^t e^{A(t-\tau)} B x(\tau) d\tau \end{aligned}$$

Free Response $q_{FR}(t)$

$$q_{FR}(t) = e^{At}q(0) = \begin{bmatrix} (-e^{-t} + 2e^{-2t}) & (e^{-t} - e^{-2t}) \\ 2(-e^{-t} + e^{-2t}) & (2e^{-t} - e^{-2t}) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} (3e^{-2t} - 2e^{-t}) \\ (3e^{-2t} - 4e^{-t}) \end{bmatrix}$$

Forced Response $q_{FO}(t)$

$$q_{FO}(t) = \int_0^t e^{A(t-\tau)} Bx(\tau) d\tau$$
$$= \int_0^t \begin{bmatrix} -e^{-(t-\tau)} + 2e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ 2(-e^{-(t-\tau)} + e^{-2(t-\tau)}) & 2e^{-(t-\tau)} - e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau$$
$$= \int_0^t \left[\begin{Bmatrix} -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \\ 2(-e^{-(t-\tau)} + e^{-2(t-\tau)}) \end{Bmatrix} \right] d\tau$$
$$= \left[\begin{Bmatrix} -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \\ 2(-e^{-(t-\tau)} + e^{-2(t-\tau)}) \end{Bmatrix}_0^t \right]$$
$$= \begin{bmatrix} e^{-t} - e^{-2t} \\ -1 + 2e^{-t} - e^{-2t} \end{bmatrix}$$

Hence, the total response is

$$q(t) = q_{FR}(t) + q_{FO}(t)$$
$$= \begin{bmatrix} 3e^{-2t} - 2e^{-t} \\ 3e^{-2t} - 4e^{-t} \end{bmatrix} + \begin{bmatrix} e^{-t} - e^{-2t} \\ 2e^{-t} - e^{-2t} - 1 \end{bmatrix}$$

$$q(t) = \begin{bmatrix} (2e^{-2t} - e^{-t}) \\ (2e^{-2t} - 2e^{-t} - 1) \end{bmatrix}$$

The result is same as derived in Example 6.15.

Example 6.17 Determine e^{At} for the following A matrix

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

$$\begin{aligned} F(\lambda) &= |\lambda I - A| \\ &= \lambda^3 - 7\lambda^2 - 9 + 15\lambda \\ &= (\lambda - 1)(\lambda - 3)^2 \end{aligned}$$

This is the case of repeated eigen values.

$$\begin{aligned} \lambda_1 &= 1 \\ \lambda_2 &= 3 \\ \lambda_3 &= 3 \\ e^t &= \alpha_0 + \alpha_1 + \alpha_2 \\ e^{3t} &= \alpha_0 + 3\alpha_1 + 9\alpha_2 \\ \frac{d}{d\lambda} e^{\lambda t} &= \frac{d}{d\lambda}(\alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2) \end{aligned}$$

For $\lambda = 3$

$$te^{3t} = \alpha_1 + 6\alpha_2$$

Solving the above three simultaneous equations, we get

$$\begin{aligned} \alpha_0 &= \frac{1}{4}(9e^t + 6te^{3t} - 5e^{3t}) \\ \alpha_1 &= \frac{1}{4}(-6e^t - 8te^{3t} + 6e^{3t}) \\ \alpha_2 &= \frac{1}{4}(e^t + 2te^{3t} - e^{3t}) \end{aligned}$$

$$\begin{aligned} e^{At} &= \alpha_0 I + \alpha_1 A + \alpha_2 A^2 \\ &= \alpha_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 15 & 6 & -12 \\ 6 & -1 & 4 \\ 6 & -2 & 5 \end{bmatrix} \end{aligned}$$

$$e^{At} = \begin{bmatrix} (-te^{3t} + e^{3t}) & (te^{3t}) & (-2te^{3t}) \\ (te^{3t}) & (2e^t + te^{3t} - e^{3t}) & -2(e^t + te^{3t} - e^{3t}) \\ (te^{3t}) & (e^t + te^{3t} - e^{3t}) & (-e^t - 2te^{3t} + 2e^{3t}) \end{bmatrix}$$

6.7 State Equations of A Discrete-Time System

The discrete-time system is described by difference equation. For a continuous-time system which is described by an nth-order differential equation, it is converted into n first-order equations and the descriptions for n state variables are given. Analogous to continuous-time system, the discrete system described by Nth-order difference equation is converted into N first-order difference equations and the description for N state variables are given in the form of vector matrix difference equation. (In difference equation, the order of the equation is denoted by N instead of n to avoid confusion between order and sequence number).

6.7.1 Canonical Form II Model

Let $H(s)$ be the discrete-time system transfer function which can be expressed as

$$H(s) = \frac{b_0 z^N + b_1 z^{N-1} + b_2 z^{N-2} + \cdots + b_{N-1} z + b_N}{z^N + a_1 z^{N-1} + a_2 s^{N-2} + \cdots + a_{N-1} + a_N} \tag{6.55}$$

The input $x[n]$ and the output $y[n]$ are related by the following difference equation

$$\begin{aligned}&(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) y[n] \\ &= (b_0 E^N + b_1 E^{N-1} + \cdots + b_{N-1} E + b_N) x[n]\end{aligned} \tag{6.56}$$

where

$$\begin{aligned}E^N y[n] &= y[n] \\ E^{N-1} y[n] &= y[n-1] \\ E^{N+1} y[n] &= y[n+1]\end{aligned}$$

The direct form II realization of Eqs. (6.55) and (6.56) is represented in Fig. 6.17.

The output of N delay elements are denoted by $q_1[n],\ q_2[n], \ldots, q_N[n]$. The output of the first delay is $q_N[n+1]$. Since there are N delays, N equations can be written one each at the input point.

$$\begin{aligned}q_1[n+1] &= q_2[n] \\ q_2[n+1] &= q_1[n] \\ &\vdots \\ q_{N-1}[n+1] &= q_N[n] \\ q_N[n+1] &= -a_N q_1[n] - a_{N-1} q_2[n] - \ldots - a_1 q_N[n] + x[n] \\ y[n] &= b_N q_1[n] - b_{N-1} q_2[n] + \ldots + b_1 q_N[n] + b_0 q_{N+1}[n]\end{aligned} \tag{6.57}$$

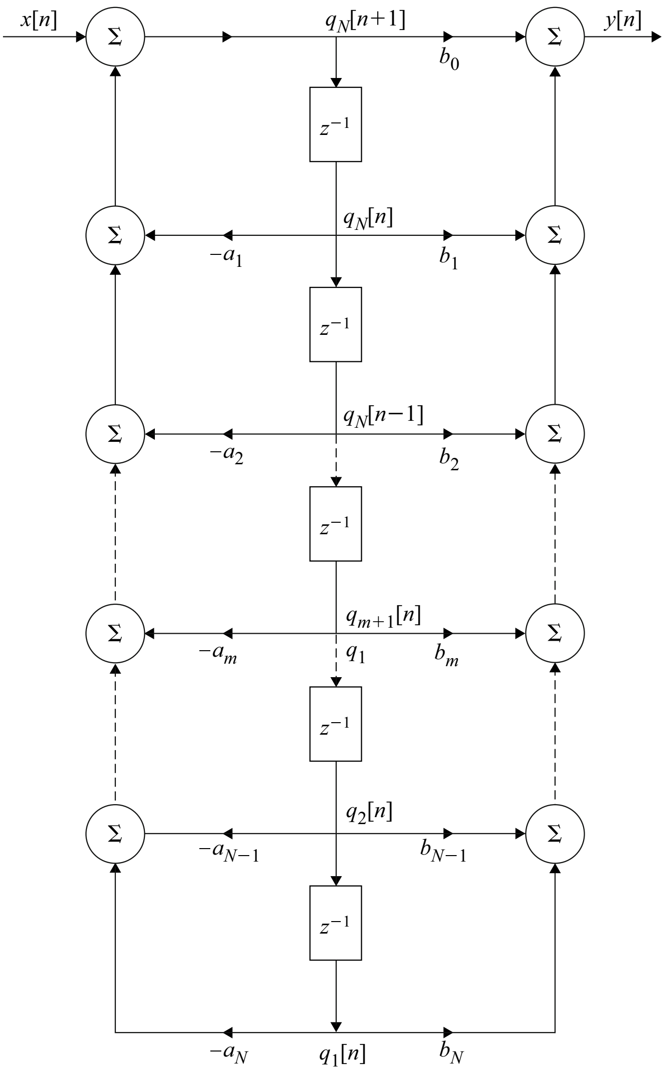


Fig. 6.17 Direct form II realization of Nth-order discrete-time system

Substituting for $q_{N+1}[n]$ from Eq. (6.57), we get

$$y[n]=(b_N-b_0a_N)q_1[n]+(b_{N-1}-b_0a_{N-1})q_2[n]+\cdots+(b_1-b_0a_1)q_N[n]+b_0x[n]$$
$$=\bar{b}_Nq_1[n]+\bar{b}_{N-1}q_2[n]+\cdots+\bar{b}_1q_N+b_0x[n] \quad (6.58)$$

where $\bar{b}_i=(b_i-b_0a_i)$. Equation (6.57) represents the state equations and Eq. (6.58) represents the output equation. The above equations are written in matrix form as

$$\underbrace{\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \\ q_3[n+1] \\ \vdots \\ q_{N-1}[n] \\ q_N[n] \end{bmatrix}}_{q[n+1]} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & -a_2 & -a_1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} q_1[n] \\ q_2[n] \\ q_3[n] \\ \vdots \\ q_{N-1}[n] \\ q_N[n] \end{bmatrix}}_{q[n]} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B} \quad (6.59)$$

$$y[n] = \underbrace{[\bar{b}_N \quad \bar{b}_{N-1} \ldots \bar{b}_1]}_{C} + \underbrace{b_0}_{D} \; x[n] \quad (6.60)$$

The general form of state and output equation is therefore written as

$$\begin{aligned} q[n+1] &= Aq[n] + Bx[n] \\ y[n] &= Cq[n] + Dx[n] \end{aligned} \quad (6.61)$$

The state equations for a discrete system can be obtained by several methods. However, we represented here by direct forms I, II and parallel form. The following analogy between continuous- and discrete-time system are to be noted.

1. In the continuous-time system, the output of each integrator is identified as a state. In the discrete-time system, the output of each delay element is identified as the state.
2. In the continuous-time system, the input of each integrator is identified as the first derivative from which the first-order differential equation is formed. In the discrete-time system, the input to each delayed element is identified to form the first-order difference equation.

The following examples, illustrate the method of forming state equations.

Example 6.18 Form the state equations of canonical form II for the following T.F. of a discrete-time system,

$$H[z] = \frac{5z^4 + 7z^3 + 8z^2 + 2z + 10}{z^4 + 6z^3 + 7z^2 + 4z + 9}$$

Solution

$$\begin{aligned} H[z] &= \frac{5z^4 + 7z^3 + 8z^2 + 2z + 10}{z^4 + 6z^3 + 7z^2 + 4z + 9} \\ &= \frac{5 + 7z^{-1} + 8z^{-2} + 2z^{-3} + 10z^{-4}}{1 + 6z^{-1} + 7z^{-2} + 4z^{-3} + 9z^{-4}} \end{aligned}$$

where $b_0 = 5;\ b_1 = 7;\ b_2 = 8;\ b_3 = 2;\ b_4 = 10;\ a_1 = 6;\ a_2 = 7;\ a_3 = 4;\ a_4 = 9$

$$q[n+1] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -9 & -4 & -7 & -6 \end{bmatrix} q[n] + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} x[n]$$

$$\bar{b}_4 = b_4 - b_0 a_4 = 10 - 5 \times 9 = -35$$
$$\bar{b}_3 = b_3 - b_0 a_3 = 2 - 5 \times 4 = -18$$
$$\bar{b}_2 = b_2 - b_0 a_2 = 8 - 5 \times 7 = -27$$
$$\bar{b}_1 = b_1 - b_0 a_1 = 7 - 5 \times 6 = -23$$

$$y[n] = [-35 \quad -18 \quad -27 \quad -23] q[n] + 5x[n]$$

Example 6.19 Consider the following difference equation

$$4y[n-3] + 6y[n-2] - 5y[n-1] + y[n] = x[n] + 5x[n-1]$$

Form state equations.

Solution

$$\frac{Y[z]}{X[z]} = H[z] = \frac{(1 + 5z^{-1})}{(1 - 5z^{-1} + 6z^{-2} + 4z^{-3})}$$

where $b_0 = 1;\ b_1 = 5;\ b_2 = 0;\ b_3 = 0;\ a_1 = -5;\ a_2 = 6;\ a_3 = 4$

$$q[n+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -6 & 5 \end{bmatrix} q[n] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x[n]$$

$$\bar{b}_3 = b_3 - b_0 a_3 = -4$$
$$\bar{b}_2 = b_2 - b_0 a_2 = -6$$
$$\bar{b}_1 = b_1 - b_0 a_1 = 5 + 5 = 10$$

$$y[n] = [-4 \quad -6 \quad 10] q[n] + x[n].$$

6.7.2 Canonical Form I Model

Consider the following system function

$$\frac{Y[z]}{X[z]} = H[z] = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

By cross-multiplying the above equation, we get

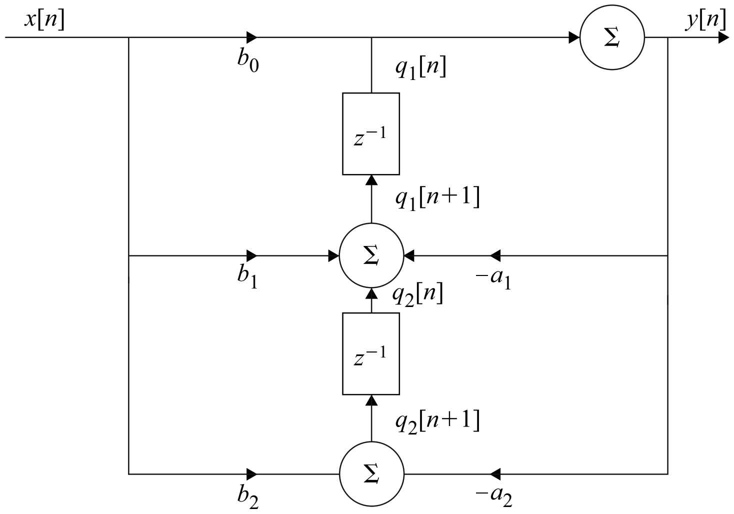


Fig. 6.18 Canonical form I structure

$$\begin{aligned}[1 + a_1 z^{-1} + a_2 z^{-2}]Y[z] &= [b_0 + b_1 z^{-1} + b_2 z^{-2}]X[z] \\ Y[z] &= -a_1 z^{-1} Y[z] - a_2 z^{-2} Y[z] + b_0 X[z] \\ &\quad + b_1 z^{-1} X[z] + b_2 z^{-2} X[z] \end{aligned} \tag{6.62}$$

Equation (6.62) is represented in Fig. 6.18, where the states and the delay elements are shown.

From Fig. 6.18, the following equations are written in terms of states

$$\begin{aligned} y[n] &= q_1[n] + b_0 x[n] \\ q_1[n+1] &= -a_1 y[n] + q_2 y[n] + b_1 x[n] \\ &= -a_1 q_1[n] + q_2[n] + (b_1 - b_0 a_1) x[n] \end{aligned} \tag{6.63}$$

$$\begin{aligned} q_2[n+1] &= -a_2 y[n] + b_2 x[n] \\ &= -a_2 q_1[n] + (b_2 - b_0 a_1) x[n] \end{aligned} \tag{6.64}$$

Equations (6.63) and (6.64) can be combined and expressed in matrix form as

$$\begin{aligned} q[n+1] &= \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} q[n] + \begin{bmatrix} (b_1 - b_0 a_1) \\ (b_2 - b_0 a_2) \end{bmatrix} x[n] \\ y[n] &= [1 \quad 0] q[n] + b_0 x[n] \end{aligned}$$

In general, for an Nth-order difference equation,

$$q[n+1] = \begin{bmatrix} -a_1 & 1 & 0 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & 0 & \cdots & 0 \\ -a_3 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_{N-1} & 0 & 0 & 0 & \cdots & 1 \\ -a_N & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} q[n] + \begin{bmatrix} (b_1 - b_0a_1) \\ (b_2 - b_0a_2) \\ (b_3 - b_0a_3) \\ \vdots \\ (b_{N-1} - b_0a_{N-1}) \\ (b_N - b_0a_N) \end{bmatrix} x[n] \quad (6.65)$$

where $\bar{b}_i$ is defined as $\bar{b}_i = (b_i - b_0a_i)$

$$y[n] = [1 \quad 0 \quad 0 \cdots 0]q[n] + b_0x[n] \quad (6.66)$$

Example 6.20 Form the state equations of canonical form I for the following T.F. of a discrete-time system

$$H[z] = \frac{5z^4 + 7z^3 + 8z^2 + 2z + 10}{z^4 + 6z^3 + 7z^2 + 4z + 9}$$

Solution

$$H[z] = \frac{5z^4 + 7z^3 + 8z^2 + 2z + 10}{z^4 + 6z^3 + 7z^2 + 4z + 9}$$
$$= \frac{5 + 7z^{-1} + 8z^{-2} + 2z^{-3} + 10z^{-4}}{1 + 6z^{-1} + 7z^{-2} + 4z^{-3} + 9z^{-4}}$$

where $b_0 = 5;\ b_1 = 7;\ b_2 = 8;\ b_3 = 2;\ b_4 = 10;\ a_1 = 6;\ a_2 = 7;\ a_3 = 4;\ a_4 = 9$

$$\bar{b}_1 = b_1 - b_0a_1 = 7 - 5 \times 6 = -23$$
$$\bar{b}_2 = b_2 - b_0a_2 = 8 - 5 \times 7 = -27$$
$$\bar{b}_3 = b_3 - b_0a_3 = 2 - 5 \times 4 = -18$$
$$\bar{b}_4 = b_4 - b_0a_4 = 10 - 5 \times 9 = -35$$

$$q[n+1] = \begin{bmatrix} -6 & 1 & 0 & 0 \\ -7 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \\ -9 & 0 & 0 & 0 \end{bmatrix} q[n] + \begin{bmatrix} -23 \\ -27 \\ -18 \\ -35 \end{bmatrix} x[n]$$

$$y[n] = [1 \quad 0 \quad 0 \quad 0]q[n] + 5x[n]$$

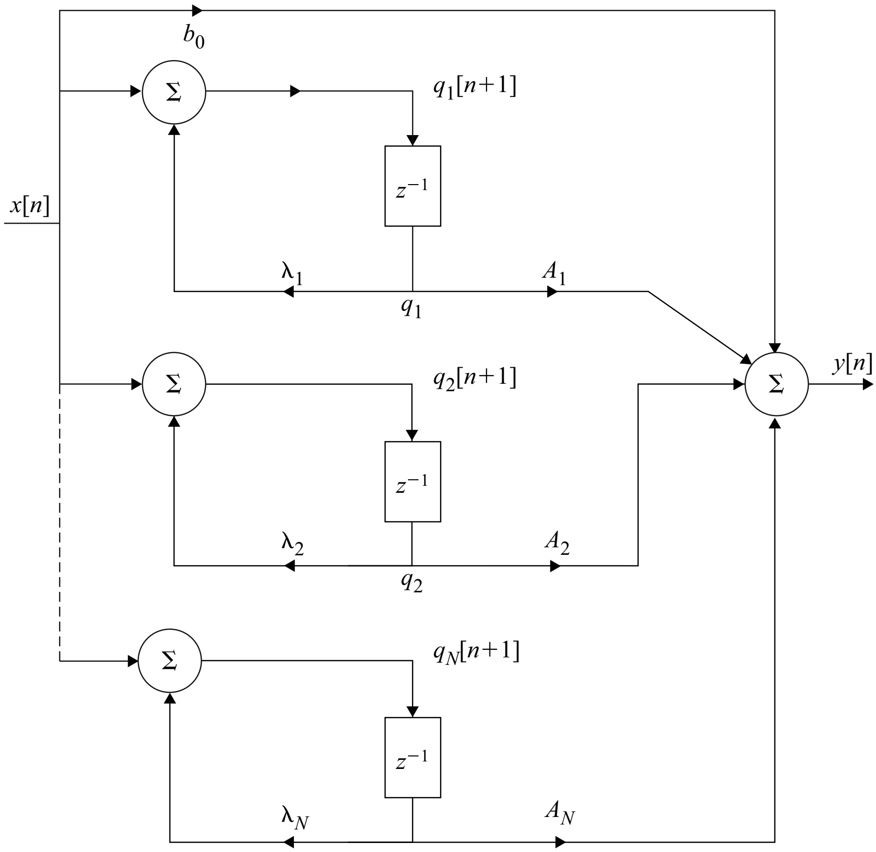


Fig. 6.19 Diagonal form model

6.7.3 Diagonal Form (Parallel Form) Model

Consider the system transfer function given below:

$$H[z] = \frac{b_0 z^N + b_1 z^{N+1} + \cdots + b_{N-1} z + b_N}{z^N + a_1 z^{N+1} + \cdots + a_{N-1} z + a_N} \tag{6.67}$$

$$= \frac{b_0 + b_1 z^{-1} + \cdots + b_{N-1} z^{-(N-1)} + b_N z^{-N}}{1 + a_1 z^{-1} + \cdots + a_{N-1} z^{-(N-1)} + a_N z^{-N}}$$

$$= b_0 + \frac{A_1}{(z - \lambda_1)} + \cdots + \frac{A_N}{(z - \lambda_N)} \tag{6.68}$$

where $\lambda_1,\ \lambda_2, \ldots, \lambda_n$ are the distinct eigen values of $H[z]$. Equation (6.68) is represented in Fig. 6.19.

From Fig. 6.19, the following state equations are written

$$q_1[n+1] = \lambda_1 q_1[n] + x[n]$$

$$q_2[n+1] = \lambda_2 q_2[n] + x[n]$$
$$\vdots$$
$$q_N[n+1] = \lambda_N q_N[n] + x[n] \tag{6.69}$$

$$y[n] = A_1 q_1[n] + A_2 q_2[n] + \cdots + A_N q_N[n] + b_0 x[n] \tag{6.70}$$

Equations (6.69) and (6.70) can be represented in matrix form as

$$q[n+1] = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_N \end{bmatrix} q[n] + \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} x[n] \tag{6.71}$$

$$y[n] = [A_1 \quad A_2 \quad A_3 \cdots A_N] q[n] + b_0 x[n] \tag{6.72}$$

Example 6.21 A certain discrete-time system has the following T.F.

$$H[z] = \frac{z^3 + 10z^2 + 32z + 29}{z^3 + 9z^2 + 26z + 24}$$

Form the state equation with its A matrix in diagonal form.

Solution Dividing the numerator polynomial with denominator polynomial, we get

$$\begin{array}{r} 1 \\ z^3 + 9z^2 + 26z + 24 \overline{\big) z^3 + 10z^2 + 32z + 29} \\ \underline{z^3 + 9z^2 + 26z + 24} \\ \underline{z^2 + 6z + 5} \end{array}$$

$$H[z] = 1 + \frac{(z^2 + 6z + 5)}{(z^3 + 9z^2 + 26z + 24)}$$

$$(z^2 + 6z + 5) = (z+1)(z+5)$$

$$(z^3 + 9z^2 + 26z + 24) = (z+2)(z+3)(z+4)$$

$$H[z] = 1 + \frac{(z+1)(z+5)}{(z+2)(z+3)(z+4)}$$

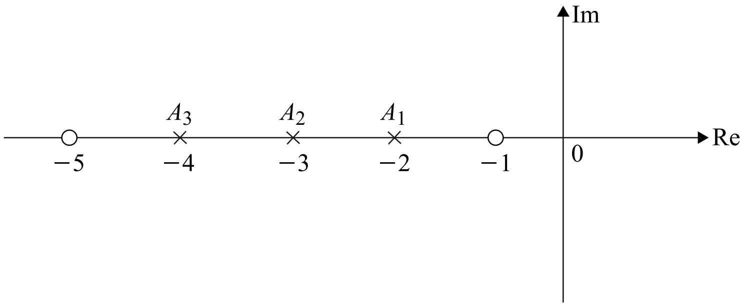


Fig. 6.20 Pole-zero diagram

$$\frac{(z+1)(z+5)}{(z+2)(z+3)(z+4)} = \frac{A_1}{(z+2)} + \frac{A_2}{(z+3)} + \frac{A_3}{(z+4)}$$

The pole-zero locations are shown in Fig. 6.20.

$$A_1 = \frac{-3 \times 1}{2 \times 1} = -\frac{3}{2}$$
$$A_2 = \frac{2 \times 2}{1 \times 1} = 4$$
$$A_3 = \frac{-3 \times 1}{2 \times 1} = -\frac{3}{2}$$

$$H[z] = 1 - \frac{3}{2}\frac{1}{(z+2)} + \frac{4}{(z+3)} - \frac{3}{2}\frac{1}{(z+4)}$$

The eigen values are $\lambda_1 = -2;\ \lambda_2 = -3$; and $\lambda_3 = -4$

$$q[n+1] = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} q[n] + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x[n]$$

$$y[n] = \begin{bmatrix} -\frac{3}{2} & 4 & -\frac{3}{2} \end{bmatrix} q[n] + x[n]$$

Example 6.22 Find the state variable matrices $A,\ B,\ C$ and D for the equation

$$y[n] - 3y[n-1] - 2y[n-2] = x[n] + 5x[n-1] + 6x[n-2]$$

(*Anna University, November, 2007*)

Solution Taking z-transform on both sides of the equation, we get

$$\frac{Y[z]}{X[z]} = H[z] = \frac{(1 + 5z^{-1} + 6z^{-2})}{(1 - 3z^{-1} - 2z^{-2})}$$

where $b_0 = 1;\ b_1 = 5;\ b_2 = 6;\ a_1 = -3;\ a_2 = -2$

$$\begin{aligned} \bar{b}_2 &= b_2 - b_0 a_2 = 6 + 2 = 8 \\ \bar{b}_1 &= b_1 - b_0 a_1 = 5 + 3 = 8 \\ D &= b_0 = 1 \end{aligned}$$

$$\begin{aligned} q[n+1] &= \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}}_{A} q[n] + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} x[n] \\ y[n] &= \underbrace{[8 \quad 8]}_{C} q[n] + \underbrace{1}_{D} x[n] \end{aligned}$$

Example 6.23 A continuous-time system has the state variable description

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad C = [3 \quad 1]; \quad D = [2]$$

Determine the transfer function.

Solution

$$\begin{aligned} (sI - A) &= \begin{bmatrix} (s-2) & 1 \\ -1 & s \end{bmatrix} \\ (sI - A)^{-1} &= \frac{\text{cofactor}}{\text{determinant}} \\ &= \frac{1}{(s^2 - 2s + 1)} \begin{bmatrix} s & 1 \\ 1 & (2-s) \end{bmatrix} \\ (sI - A)^{-1}B &= \frac{1}{(s^2 - 2s + 1)} \begin{bmatrix} s & 1 \\ 1 & (2-s) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{(s^2 - 2s + 1)} \begin{bmatrix} s \\ 1 \end{bmatrix} \\ C(sI - A)^{-1}B &= \frac{1}{(s^2 - 2s + 1)} [3 \quad 1] \begin{bmatrix} s \\ 1 \end{bmatrix} = \frac{(3s+1)}{(s^2 - 2s + 1)} \\ C(sI - A)^{-1}B + D &= \frac{(3s+1)}{(s^2 - 2s + 1)} + 2 \end{aligned}$$

$$H(s) = \frac{(2s^2 - s + 3)}{(s^2 - 2s + 1)}$$

Summary

1. An Nth-order systems (continuous as well as discrete) can be described in terms of N variables with N first-order equations. These variable are called state variables.
2. State variables representation is not unique. However, the solution of state equation is unique.
3. The state equations are written in specific format. Such an equation is called vector–matrix differential/difference equation.
4. The state variables give internal and external description of the system. Thus, physical variables can be chosen as state variables and their behavior can be readily studied.
5. State equations are written from system structure or from block diagrams.
6. State equations are solved by time domain or frequency domain methods.

Exercise

I. Short Answer Type Questions

1. **Define the state of a system?**
   The state of a system is defined as the minimum number of initial conditions that must be specified at any initial time t_0 so that the complete behavior of the system at any time $t > t_0$ is determined if the input $x(t)$ is known.
2. **What do you understand by state vector?**
   If N state variables are required to completely describe the behavior of the system then these N variables are the N components of a vector q. Such a vector is called state vector.
3. **What is state space?**
   The N dimensional space whose coordinate axes consist of q_1 axis, q_2 axis, $\ldots$, q_N axis, is called state space.
4. **What is state-space equations?**
   Input variables, output variables and state variables are involved in the modeling of dynamic systems. The dynamic equations involving these three variables are called state space equations.
5. **What is vector–matrix differential/difference equation?**
   State variables equations are expressed in the time domain by using compact vector–matrix notations. These equations when written for a CT systems are

called vector–matrix differential equations and when written to DT systems are called vector–matrix difference equation.

6. **What is state transition matrix STM?**
The matrix which is unique for a given system that transforms any initial state $q(t_0)$ to any final state $q(t_f)$ is called state transition matrix. It contains all the informations about the system dynamics at all time. For a continuous-time system the STM $\phi(t) = e^{At}$.
7. **What is Cayley–Hamilton theorem?**
According to Cayley–Hamilton theorem, the matrix A satisfies its own characteristic equation. This property is used to evaluate the STM e^{At}.
8. **What physical variables are chosen as state variables in electrical circuit and mechanical systems?**
The current through the inductor and the voltage across the capacitor are chosen as state variables in electrical circuits. The displacement and velocity of energy-storing elements such as mass (inertia) and spring are chosen as state variables in mechanical systems.
9. **What are the advantages of state space model over that of transfer function model?**

   (a) State space model is applicable to linear, non-linear and time varying systems whereas T.F. model is applicable only to linear system.
   (b) T.F. model requires initial conditions to be zero whereas for state space model, the initial conditions need not be zero.
   (c) System design modeled by T.F. is based on trial and error and in will not in general lead to optimal control. The famous optimal control theory which follows a systematic design procedure uses state space model of the system.

10. **For a particular system, the A matrix is represented in more than one form. What is the nature of characteristic equation?**
In sate space model, even though the system A matrix is not unique, the characteristic equation is same and is unique.

II. Long Answer Type Questions

1. **Consider the mechanical system shown in Fig. 6.21. Form the state space equation.**

$$\dot{q}(t) = \begin{bmatrix} -\frac{R}{L} & 0 & 0 \\ 0 & 0 & 1 \\ \dfrac{K_s}{M_{eq}} & -\dfrac{K_{eq}}{M_{eq}} & -\dfrac{B_{eq}}{M_{eq}} \end{bmatrix} q(t) + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix} e(t)$$

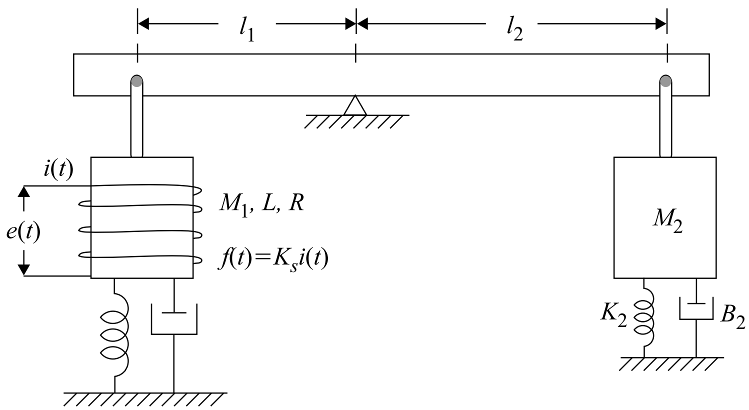


Fig. 6.21 Mechanical system with lower arrangement

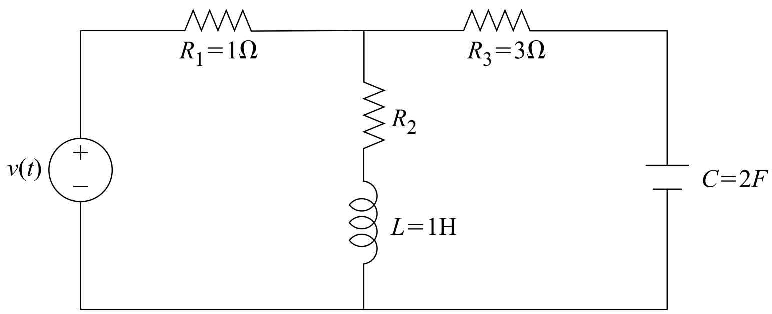


Fig. 6.22 Electrical circuit for Example 6.2

where

$$
\begin{aligned}
M_{eq} &= \frac{l_1}{l_2} M_1 + \frac{l_2}{l_1} M_2 \\
B_{eq} &= \frac{l_1}{l_2} B_1 + \frac{l_2}{l_1} B_2 \\
K_{eq} &= \frac{l_1}{l_2} K_1 + \frac{l_2}{l_1} K_2
\end{aligned}
$$

2. **Consider the electrical circuit shown in Fig. 6.22. Form the state space equation.**

$$\dot{q}(t) = \begin{bmatrix} -\frac{11}{4} & \frac{1}{4} \\ \frac{3}{8} & -\frac{1}{8} \end{bmatrix} q(t) + \begin{bmatrix} \frac{5}{4} \\ -\frac{1}{8} \end{bmatrix} v(t)$$

3. **Consider the following T.F. of a certain continuous-time system. Form the state space equations in canonical form I model.**

$$\boldsymbol{H(z) = \frac{7s^2 + 8s + 4}{s^3 + 5s^2 + 9s + 10}}$$

$$\dot{q}(t) = \begin{bmatrix} -5 & 1 & 0 \\ -9 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix} q(t) + \begin{bmatrix} 7 \\ 8 \\ 4 \end{bmatrix} x(t)$$

$$y(t) = [1 \quad 0 \quad 0] q(t)$$

4. **Consider the following T.F. of continuous-time system. Form the state space equation in canonical form II model.**

$$\boldsymbol{H(z) = \frac{5s^3 + 6s^2 + 2s + 10}{s^3 + 7s^2 + 4s + 5}}$$

$$\dot{q}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -4 & -7 \end{bmatrix} q(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x(t)$$

$$y(t) = [-15 \quad -18 \quad -29] q(t) + 5x(t)$$

5. **Consider the following T.F. of continuous-time system. Form the state space equation in diagonal form model.**

$$\boldsymbol{H(z) = \frac{(s^2 + 6s + 8)}{(s^3 + 27s^2 + 230s + 600)}}$$

$$\dot{q}(t) = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -12 \end{bmatrix} q(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x(t)$$

$$y(t) = \begin{bmatrix} \frac{3}{35} & -\frac{24}{5} & \frac{40}{7} \end{bmatrix} q(t)$$

6. **Consider the following differential equation**

$$\frac{d^3y}{dt^3} - 5\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 7y = 2\frac{dx}{dt} + 5x(t)$$

Form the state space equation in canonical form II model.

$$\dot{q}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -6 & 5 \end{bmatrix} q(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x(t)$$

$$y(t) = [5 \quad 2 \quad 0]\, q(t)$$

7. **Consider the following T.F. of a certain discrete-time system**

$$H(z) = \frac{4z^2 - 5z + 10}{z^3 + 2z^2 - 7z + 9}$$

Form the state variable equation in canonical form II model.

$$q[n+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & 7 & -2 \end{bmatrix} q[n] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x[n]$$

$$y[n] = [10 \quad -5 \quad 4]\, q[n]$$

8. **Consider the following T.F. of a certain discrete system given below. Form the state space equation of canonical form I model.**

$$H(z) = \frac{2z^2 + 6z + 9}{z^3 + 8z^2 + 7z + 16}$$

$$q[n-1] = \begin{bmatrix} -8 & 1 & 0 \\ -7 & 0 & 1 \\ -16 & 0 & 0 \end{bmatrix} q[n] + \begin{bmatrix} 2 \\ 6 \\ 9 \end{bmatrix} x[n]$$

$$y[n] = [1 \quad 0 \quad 0]\, q[n]$$

9. **A certain discrete-time system is described by the following difference equation.**

$$\boldsymbol{y[n] - \tfrac{1}{12}y[n-1] - \tfrac{1}{6}y[n-2] + \tfrac{1}{48}y[n-3] = x[n-1] - \tfrac{3}{8}x[n-2] + \tfrac{1}{32}x[n-3]}$$

Determine the system T.F. Form the state space equation in the diagonal form. What are the eigen values of the system?

$$q[n+1] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} q[n] + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x[n]$$

$$y[n] = \begin{bmatrix} \dfrac{3}{16} & -\dfrac{35}{16} & 3 \end{bmatrix} q[n]$$

$$H[z] = \frac{z^{-1} - \frac{3}{8}z^{-2} + \frac{1}{32}z^{-3}}{1 - \frac{1}{12}z^{-1} - \frac{1}{6}z^{-2} - \frac{1}{48}z^{-3}}$$

The eigen values are $\lambda_1 = \frac{1}{2},\ \lambda_2 = -\frac{1}{6},\ \lambda_3 = -\frac{1}{4}$.

10. **Find the state equation of a continuous-time LTI system described by**

$$\boldsymbol{\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)}$$

(*Anna University, May, 2007*)

$$\dot{q}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} q(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(t).$$

Chapter 7
Application of MATLAB and Python Programs to Solve Problems

7.1 Application of MATLAB Program

Example 7.1 Write a MATLAB program for a signal $x(t)$ shown in Fig. 1.23a, and sketch the output waveforms. (Refer Example 1.2)

(a) $x(3t + 2)$ (b) $x(\frac{-t}{2} + 2)$

Program (a)

```
clc;
clf;
clear all;
start_time=-1;
end_time=1;
time=start_time:0.5:end_time;
Amplitude=[ 1 2 2 1 0 ];
subplot(3,1,1)
y=stairs(time,Amplitude);
xticks(time);
yticks(0:1:2);
ylim([0 2]);
xlabel('time (t)');
ylabel('x(t)');
title('plot of x(t)');

shift=2;
time=start_time-shift:0.5:end_time-shift;
subplot(3,1,2)
y=stairs(time,Amplitude);
xticks(time);
yticks(0:1:2);
```


S. Palani, *Basic System Analysis*,
https://doi.org/10.1007/978-3-031-28280-5_7

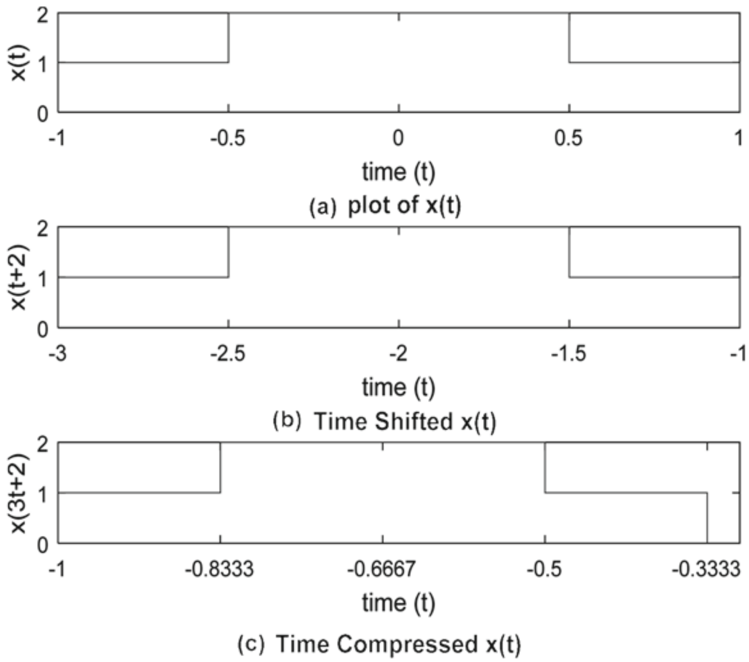


Fig. 7.1 Plot of response of the signal $x(t)$, $x(t+2)$ and $x(3t+2)$ of Example 7.1

```
ylim([0 2]);
xlabel('time (t)');
ylabel('x(t+2)');
title('Time Shifted x(t)');

scale=3;
time=time./scale;
subplot(3,1,3)
y=stairs(time,Amplitude);
xticks(time);
yticks(0:1:2);
ylim([0 2]);
xlabel('time (t)');
ylabel('x(3t+2)');
title('Time Compressed x(t)');
```

Figure 7.1 represents the response of the signal $x(t)$, $x(t+2)$ and $x(3t+2)$. The input signal $x(t)$, time shifted signal $x(t+2)$ and time compressed signal $x(3t+2)$ are plotted in Fig. 7.1a, b, c, respectively.

Program (b)

$$x\left(-\frac{t}{2}+2\right)$$

```
clf;
clear all;
start_time=-1;
end_time=1;
time=start_time:0.5:end_time;
Amplitude=[ 1 2 2 1 0 ];
subplot(3,1,1)
y=stairs(time,Amplitude);
xticks(time);
yticks(0:1:2);
ylim([0 2]);
xlabel('time (t)');
ylabel('x(-t)');
title('Folded x(-t)');

shift=1;
time=start_time-shift:0.5:end_time-shift;
subplot(3,1,2)
y=stairs(time,Amplitude);
xticks(time);
yticks(0:1:2);
ylim([0 2]);
xlabel('time (-t-1)');
ylabel('x(t+2)');
title('Time Shifted x(-t)');

scale=0.5;
time=time./scale;
subplot(3,1,3)
y=stairs(time,Amplitude);
xticks(time);
yticks(0:1:2);
ylim([0 2]);
xlabel('time (t)');
ylabel('x(-t/2-1)');
title('Time expansion of x(-t-1) to get x(-t/2-1)');
```

Figure 7.2 represents the response of the signal $x(t)$ shown in Fig. 1.23b. The input signal $x(-t)$, time shifted signal $x(-t)$ and time expanded signal $x((-t/2)-1))$ are plotted in Fig. 7.2a, b, c respectively.

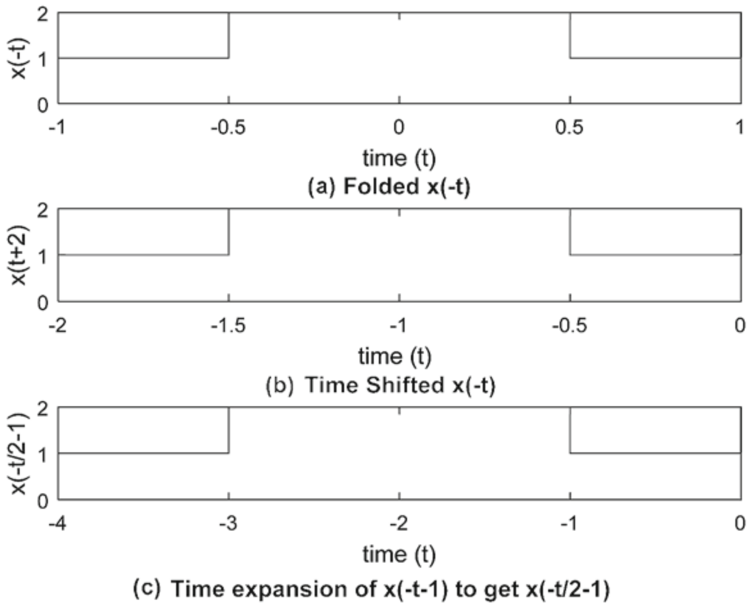


Fig. 7.2 Response of the signal $x(t)$ of Example 7.1b

Example 7.2 Write a MATLAB program to represent the signal $x(t) = 5u(4 - t)$ shown in Fig. 1.27 (Refer Example 1.6).

```
clc;
clearall;
start_time=0;
end_time=-10;
time=end_time:1:start_time;
Amplitude=5*ones(1,length(time));
subplot(2,1,1)
y=stairs(time,Amplitude);
xticks(time);
xlabel('time (-t)');
ylabel('5u(-t))');
title('Original Signal 5u(-t))');

shift=4;
time=end_time:1:start_time+shift;
Amplitude=5*ones(1,length(time+shift));
subplot(2,1,2)
y=stairs(time,Amplitude);
xticks(time);
```

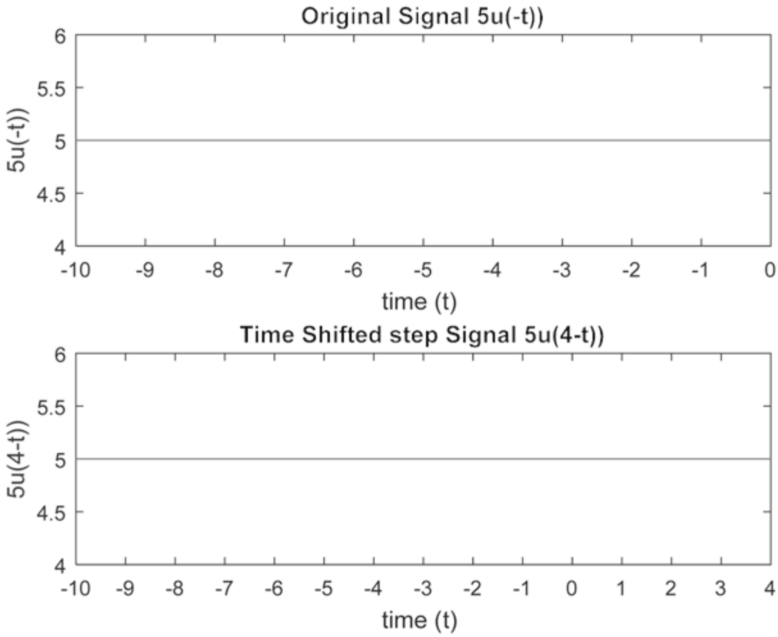


Fig. 7.3 The original and time shifted response of the signal $y(t) = 5u(4-t)$ of Example 7.2

```
xlabel('time (-t)');
ylabel('5u(4-t))');
title('Time Shifted step Signal 5u(4-t))');% %
```

The signals $5u(-t)$ and $5u(4-t)$ are shown in Fig. 7.3a and b respectively.

Example 7.3 Write a MATLAB program to check the system $y(t) = 5x(t)\sin 10t$ is linear. (Refer Example 1.45a)

```
n=0:5;
x1=(5.*n).*sin(10*n);
x2=(5.*n).*sin(10*n);
a1=1;
a2=1;
z=a1*x1+a2*x2;
y1=n.*z
z1=n.*x1;
z2=n.*x2;
y2=a1*z1+a2*z2
if y1==y2
    fprintf('The System y(t)=5x(t) sin 10t is Linear \n');
else
```

```
    fprintf('The System y(t)=5x(t) sin 10t is non-linear \n');
end;
```

Output:

y1 = 0 −5.4402 36.5178 −88.9228 119.2181 −65.5937

y2 = 0 −5.4402 36.5178 −88.9228 119.2181 −65.5937

y1 = y2

The System $y(t) = 5x(t)\sin 10t$ is Linear.

Example 7.4 Write a MATLAB program to check the system $y(t) = 3x(t) + 5$ is linear or not (Refer Example 1.45b)

```
n=0:5;
x1=(3.*n)+5;
x2=(3.*n)+5;
a1=1;
a2=1;
z=a1*x1+a2*x2;
y1=n.*z
z1=n.*x1;
z2=n.*x2;
y2=a1*z1+a2*z2+5
if y1==y2
    fprintf('The System y(t)=3x(t)+5 is Linear \n');
else
     fprintf('The System y(t)=3x(t)+5 is non-linear \n');
end;
```

Output:

y1 = 0 16 44 84 136 200

y2 = 5 21 49 89 141 205

y1 $\neq$ y2

The System y(t)=3x(t)+5 is non-linear

Example 7.5 Write a MATLAB program to check the system $y(t) = t^2 x(t + 1)$ is linear or not (Refer Example 1.45c)

```
n=0:5;
x1=(n.^2).*(n+1);
x2=(n.^2).*(n+1);
a1=1;
a2=1;
z=a1*x1+a2*x2;
y1=n.*z
z1=n.*x1;
z2=n.*x2;
y2=a1*z1+a2*z2
if y1==y2
    fprintf('The System y(t)=t^2*x(t+1) is Linear \n')
else
     fprintf('The System y(t)=t^2*x(t+1) is non-linear \n')
end;
```

Output:

y1 = 0 4 48 216 640 1500

y2 = 0 4 48 216 640 1500

The System y(t)=t^2*x(t+1) is Linear

Example 7.6 Write a MATLAB program to check the system $y(t) = x(t^2)$ is linear or not (Refer Example 1.45e)

```
n=0:5;
x1=n.^2;
x2=n.^2;
a1=1;
a2=1;
z=a1*x1+a2*x2;
y1=n.*z
z1=n.*x1;
z2=n.*x2;
y2=a1*z1+a2*z2
if y1==y2
    fprintf('The System y(t)=x(t^2) is Linear \n');
else
    fprintf('The System y(t)=x(t^2) is non-linear \n');
end;
```

Output:

y1 = 0 2 16 54 128 250

y2 = 0 2 16 54 128 250

y1 = y2

The System y(t)=x(t^2) is Linear

Example 7.7 Write a MATLAB program to check the system $y(t) = x(t + 1) + 5$ is static or dynamic (Refer Example 1.48a)

```
clc;
clear all;
t=0;
if t < t+1
    fprintf(' The system y(t)=x(t+1)+5 is Dynamic \n');
else
   fprintf(' The system y(t)=x(t+1)+5 is Static \n');
end
```

Output:

The system y(t)=x(t+1)+5 is dynamic

Example 7.8 Write a MATLAB program to check the system $y(t) = x(t^2)$ is static or dynamic (Refer Example 1.48b)

```
clc;
clear all;
t=2;
if t < t^2
    fprintf(' The system y(t)=x(t^2) is Dynamic \n');
else
    fprintf(' The system y(t)=x(t^2) is Static \n');
end
```

Output:

The signal y(t)=x(t^2) is dynamic

Example 7.9 Write a MATLAB program to check the system $y(t) = x(t^2)$ is causal or non-causal (Refer Example 1.49c)

```
t=0:5;
for i=0 : length(t-1)
if i >= i^2
   fprintf(' The signal y(t)=x(t^2) is causal \n');
```

```
else
    fprintf(' The signal y(t)=x(t^2) is non-causal \n');
end
end;
```

Output:

The system y(t)=x(t^2) is non-causal

Example 7.10 Write a MATLAB program to check the system $y(t) = x(t + 1)$ is causal or non-causal (Refer Example 1.49e)

```
t=0:5;
for i=0 : length(t-1)
if i >= i+1
     fprintf(' The signal y(t)=x(t+1) is causal \n');
else
     fprintf(' The signal y(t)=x(t+1) is non-causal \n');
end
end;
```

Output:

The system y(t)=x(t+1) is non-causal

Example 7.11 Write a MATLAB program to check the system $y(t) = x(t - 1)$ is causal or non-causal (Refer Example 1.49f)

```
t=0:5;
for i=0 : length(t-1)
if i >= i-1
     fprintf(' The signal y(t)=x(t+1) is causal \n');
else
     fprintf(' The signal y(t)=x(t+1) is non-causal \n');
end
end;
```

Output:

The system y(t)=x(t-1) is causal

Example 7.12 Write a MATLAB program to check the system $y(t) = tx(t)$ is stable or not (Refer Example 1.50a)

```
clc;
clear all;
clf;
t=0:.1:10;
```

```
x=cos(2*pi*t);
plot(t,x);
xlabel('Time (t)');
ylabel('Magnitude of the Signal');
legend('x(t)')
ylim([-2 2]);
figure
y2=t.*x;
plot(t,y2);
legend('y(t)')
xlabel('Time (t)');
ylabel('Magnitude of the Signal');
if max(x)>max(y2)
    fprintf('The System y(t)=t(x(t) is stable \n');
else
    fprintf('The System y(t)=t(x(t) is unstable \n');
end
```

Output:

The System y(t)=t(x(t)) is unstable

Figure 7.4a represents $x(t)$ and Fig. 7.4b represents the output $t\ x(t)$.

Example 7.13 Write a MATLAB program to check the system $y(t) = x(t) \sin t$ is stable or not (Refer Example 1.50c)

```
clc;
clear all;
clf;
t=0:.1:10;
x=cos(2*pi*t);
plot(t,x);
xlabel('Time (t)');
ylabel('Magnitude of the Signal');
legend('x(t)')
ylim([-2 2]);
figure
y2=sin(t).*x;
plot(t,y2);
legend('y(t)')
xlabel('Time (t)');
ylabel('Magnitude of the Signal');
if max(x)>max(y2)
    fprintf('The System y(t)=(x(t) sin t is stable \n');
else
```

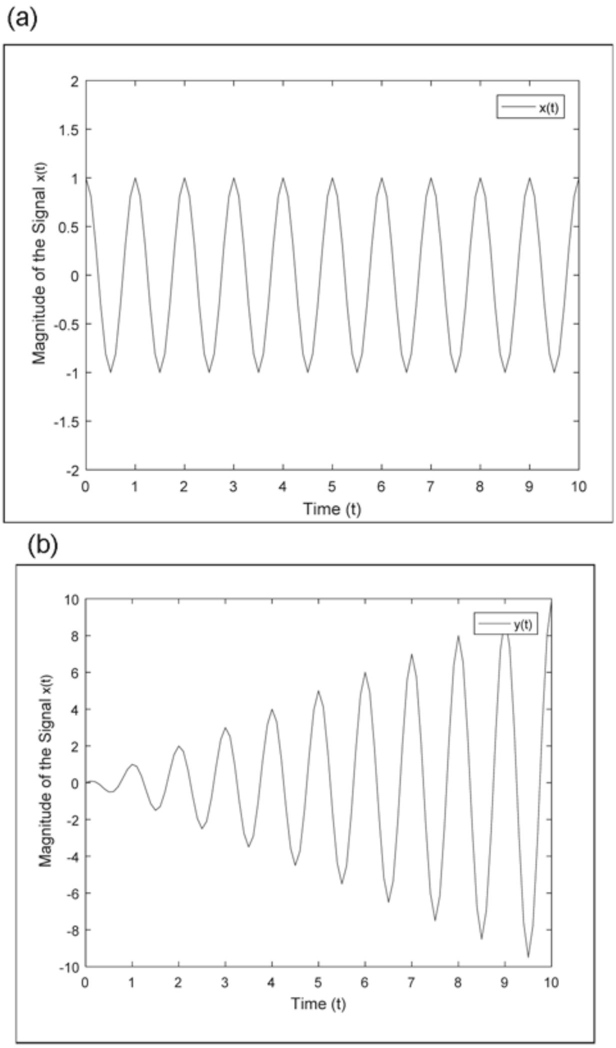


Fig. 7.4 The output response of the signal $y(t) = t(x(t))$

```
    fprintf('The System y(t)=(x(t) sin t is unstable \n');
end
```

Output:

The System y(t)=(x(t) sin t is stable

Figure 7.5a represents $x(t)$ and Fig. 7.5b represents $y(t) = x(t)\sin t$ respectively.

Example 7.14 Write a MATLAB program to determine the trigonometric Fourier series of Example 2.1.

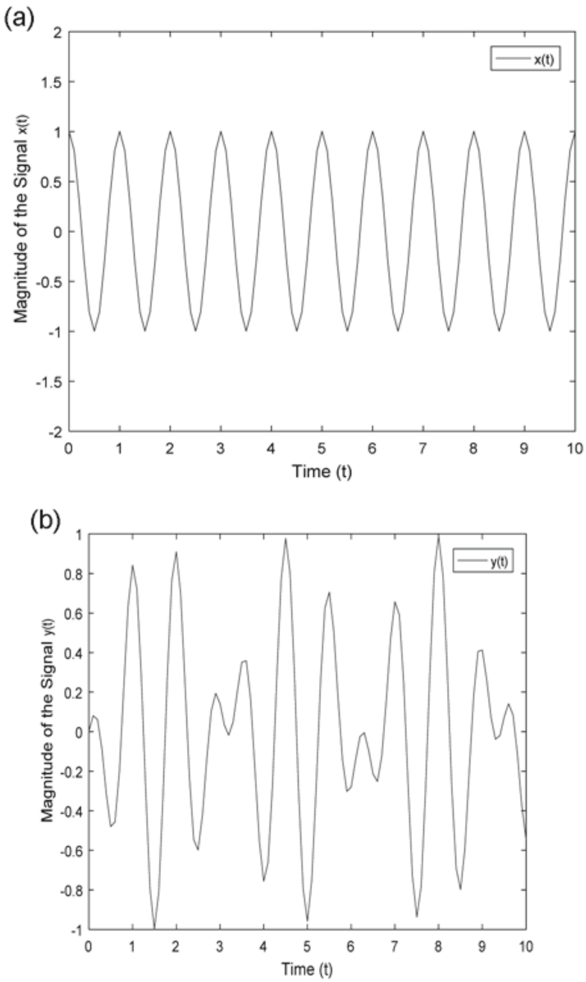


Fig. 7.5 The output response of the signal $y(t) = (x(t) \sin t$ of Example 7.13

```
clear all;
syms t n A pi
n = [1:3];
A=1;
T0=4;
wo=pi/2;
up_limit1=1;
low_limit1=-1;
up_limit2=3;
low_limit2=1;
```

```
half_a0 = (1/T0)*(int(A,t,low_limit1, up_limit1)+int(-A,t,low_limit2,up_limit2))
ai = (2/T0)*(int(A*cos(n*wo*t),t,low_limit1, up_limit1)
        +int((-A)*cos(n*wo*t),t,low_limit2, up_limit2))
bi = (2/T0)*(int(A*sin(n*wo*t),t,low_limit1, up_limit1 )
        +int((-A)*sin(n*wo*t),t,low_limit2, up_limit2))
ft = half_a0;
for k=1:length(n)
ft = ft + ai(k)*cos(k*wo*t) + bi(k)*sin(k*wo*t);
end;
ezplot(ft),grid
xlabel('Time (t) ');
ylabel('x(t) ');
title('Output response of x(t)');
ft
```

Output:

```
half_a0 =0

ai =[ 4/pi, 0, -4/(3*pi)],...

bi = [ 0, 0, 0]

ft =(4*cos((pi*t)/2))/pi - (4*cos((3*pi*t)/2))/(3*pi)+...
```

The plot of $x(t)$ for $n = 3$ is shown in Fig. 7.6. By increasing n to 20, $x(t)$ may be plotted and the original signal can be obtained in the form of periodic square wave as represented in Fig. 2.1.

Example 7.15 Write a MATLAB program to determine the trigonometric Fourier series of Example 2.2.

```
clc;
clear all;
syms t n A pi
n = [1:3];
A=t;
T0=2;
wo=pi;
up_limit=1;
low_limit=-1;
half_a0 = (1/T0)*(int(A,t,low_limit, up_limit))
ai = (2/T0)*(int(A*cos(n*wo*t),t,low_limit, up_limit))
bi = (2/T0)*(int(A*sin(n*wo*t),t,low_limit, up_limit))
ft = half_a0;
for k=1:length(n)
```

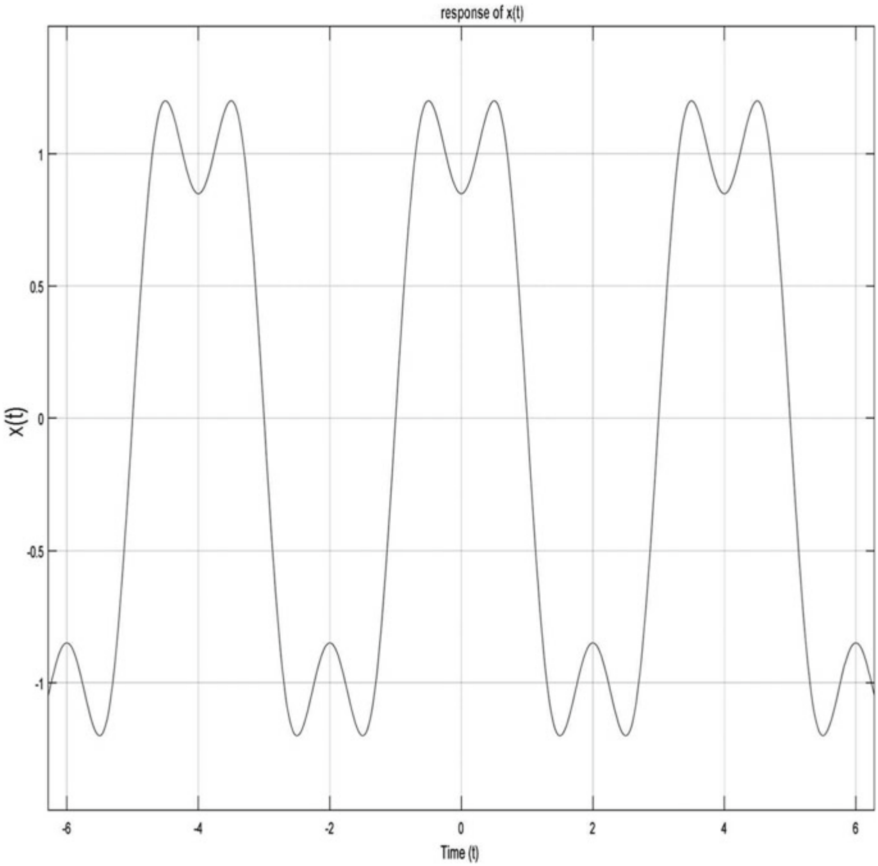


Fig. 7.6 $x(t)$ response of Example 7.14 for $n = 3$

```
ft = ft + ai(k)*cos(k*wo*t) + bi(k)*sin(k*wo*t);
end;
ezplot(ft),grid
xlabel('Time (t) ');
ylabel('x(t) ');
title('Output response of x(t)');
ft
```

Output:

```
a0 = 0.0

ai = [ 0, 0, 0]

bi = [ 2/pi, -1/pi, 2/(3*pi)],. . .
```

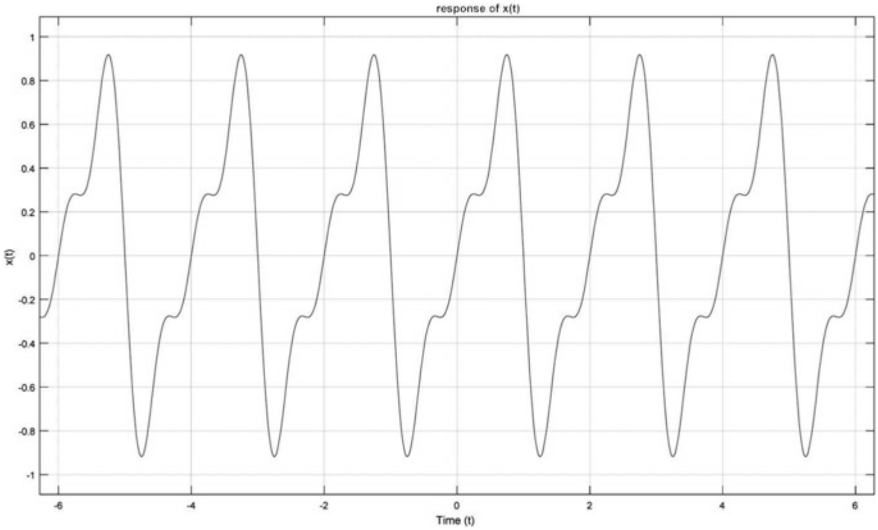


Fig. 7.7 $x(t)$ response of Example 7.15 for $n = 3$

```
ft = (2*sin(pi*t))/pi - sin(2*pi*t)/pi + (2*sin(3*pi*t))/(3*pi) + . . .
```

The output response is shown in Fig. 7.7 for $n = 3$.

Example 7.16 Write a MATLAB program to determine the trigonometric Fourier series of Example 2.3.

```
clc;
clear all;
syms t n A pi
n = [1:3];
A=t/(2*pi);
T0=2*pi;
wo=1;
up_limit=0;
low_limit=2*pi;
half_a0 = (1/T0)*(int(A,t,low_limit, up_limit))
ai = (2/T0)*(int(A*cos(n*wo*t),t,low_limit, up_limit))
bi = (2/T0)*(int(A*sin(n*wo*t),t,low_limit, up_limit))
ft = half_a0;
for k=1:length(n)
ft = ft + ai(k)*cos(k*wo*t) + bi(k)*sin(k*wo*t);
end;
ezplot(ft),grid
```

```
xlabel('Time (t) ');
ylabel('x(t) ');
title('Output response of x(t)');
ft
```

Output:

```
half_a0 = 1/2

ai = [ 0, 0, 0]

bi = [ 1/pi, 1/(2*pi), 1/(3*pi)]+ . . .

ft = -sin(2*t)/(2*pi) + sin(3*t)/(3*pi) - sin(t)/pi + 1/2 + . . .
```

Example 7.17 Write a MATLAB program to determine the trigonometric Fourier series of Example 2.4

```
clc;
clear all;
syms t n A pi
n = [1:3];
A=sin(t);
T0=pi;
wo=2;
up_limit=pi;
low_limit=0;
half_a0 = (1/T0)*(int(A,t,low_limit, up_limit))
ai = (2/T0)*(int(A*cos(n*wo*t),t,low_limit, up_limit))
bi = (2/T0)*(int(A*sin(n*wo*t),t,low_limit, up_limit))
ft = half_a0;
for k=1:length(n)
ft = ft + ai(k)*cos(k*wo*t) + bi(k)*sin(k*wo*t);
end;
ezplot(ft),grid
xlabel('Time (t) ');
ylabel('x(t) ');
title('Output response of x(t)');
ft
```

Output:

```
half_a0 = 2/pi

ai = [ -4/(3*pi), -4/(15*pi), -4/(35*pi)] + . . .
```

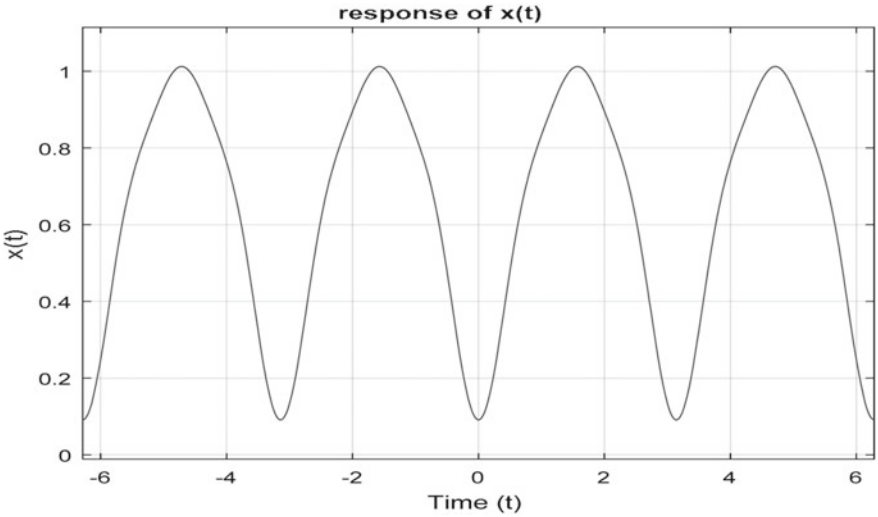


Fig. 7.8 Fourier series response $x(t) n = 3$ of Example 7.17

```
bi = [ 0, 0, 0]

ft = 2/pi - (4*cos(4*t))/(15*pi) - (4*cos(6*t))/(35*pi) - (4*cos(2*t))/(3*pi) + . . .
```

The $x(t)$ response is shown in Fig. 7.8 for $n = 3$.

Example 7.18 Write a MATLAB program to determine the trigonometric Fourier series of Example 2.5.

```
clc;
clear all;
syms t n B A pi
n = [1:3];
B=A*sin(t);
T0=2*pi;
wo=1;
up_limit=pi;
low_limit=0;
half_a0 = (1/T0)*(int(B,t,low_limit, up_limit))
ai = (2/T0)*(int(B*cos(n*wo*t),t,low_limit, up_limit))
bi = (2/T0)*(int(B*sin(n*wo*t),t,low_limit, up_limit))
ft = half_a0;
for k=1:length(n)
ft = ft + ai(k)*cos(k*wo*t) + bi(k)*sin(k*wo*t);
end;
```

```
ezplot(ft),grid
xlabel('Time (t) ');
ylabel('x(t) ');
title('Output response of x(t)');
ft
```

Output:

```
half_a0 = A/pi

ai = [ 0, -(2*A)/(3*pi), 0] + . . .

bi = [ A/2, 0, 0]

ft = A/pi + (A*sin(t))/2 - (2*A*cos(2*t))/(3*pi) + . . .
```

Example 7.19 Write a MATLAB program to determine the trigonometric Fourier series of Example 2.6.

```
clc;
clear all;
syms t n B A pi
n = [1:20];
B=t^2;
T0=2;
wo=pi;
up_limit=1;
low_limit=-1;
half_a0 = (1/T0)*(int(B,t,low_limit, up_limit))
ai = (2/T0)*(int(B*cos(n*wo*t),t,low_limit, up_limit))
bi = (2/T0)*(int(B*sin(n*wo*t),t,low_limit, up_limit))
ft = half_a0;
for k=1:length(n)
ft = ft + ai(k)*cos(k*wo*t) + bi(k)*sin(k*wo*t);
end;
ezplot(ft),grid
xlabel('Time (t) ');
ylabel('x(t) ');
title('Output response of x(t)');
ft
```

Output:

```
half_a0 = 1/3

ai = [ -4/pi^2, 1/pi^2, -4/(9*pi^2)] + . . .
```

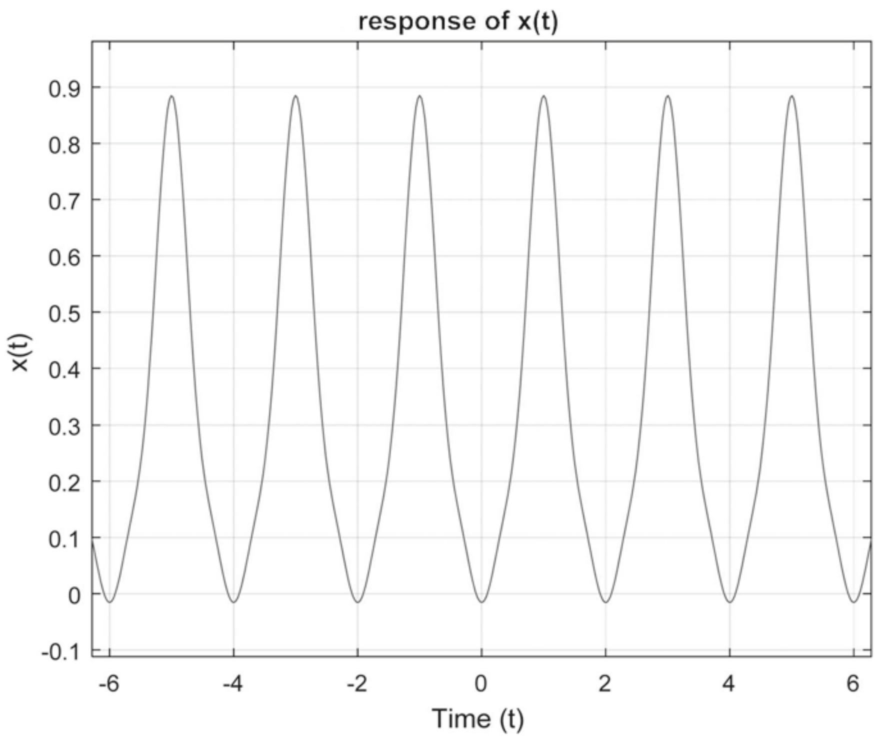


Fig. 7.9 Fourier series response $x(t)$ of Example 7.19, $n = 3$

```
bi = [ 0, 0, 0]

ft = cos(2*pi*t)/pi^2 - (4*cos(pi*t))/pi^2 - (4*cos(3*pi*t))/(9*pi^2) + 1/3+. . .
```

See (Fig. 7.9).

Example 7.20 Write a MATLAB program to determine the exponential Fourier series of Example 2.9.

```
clc;
clear all;
syms t n B A pi
n = [1:2];
B=cos(t);
T0=10;
wo=0.2*pi;
up_limit=pi/2;
low_limit=-pi/2;
D = (1/T0)*(int(B*exp(-j*wo*n*t),t,low_limit, up_limit))
Dn=vpa(D,4)
```

```
xt=0;
for k=1:length(n)
xt = xt + Dn(k)*exp(j*0.2*pi*k*t);
end;
vpa(xt,4)
```

Output:

```
D = [−(5*cos(pi^2/10))/(pi^2 − 25), −(5*cos(pi^2/5))/(4*pi^2 − 25)]

Dn = [0.1822, 0.1355]

x(t) = 0.1822*exp(t*0.6283i) + 0.1355*exp(t*1.257i)
```

Example 7.21 Write a MATLAB program to find the Fourier transform of the signal $x(t) = \delta(t-2)$ (Refer Example 3.14 (1))

```
syms t x
x = dirac(t-2);
F= fourier(x)
```

Output:

```
F =exp(-w*2i)
```

Example 7.22 Write a MATLAB program to find the Fourier transform of the signal $x(t) = \delta(t-1) - (t+1)$ (Refer Example 3.14 (2)).

```
syms t x
x = dirac(t-1)-dirac(t+1);
F= fourier(x);
simplify(F)
```

Output:

```
ans = -sin(w)*2i
```

Example 7.23 Write a MATLAB program to find the Fourier transform of the signal $x(t) = \delta(t+2) + \delta(t-2)$ (Refer Example 3.14 (3)).

```
syms t x
x = dirac(t+2)+dirac(t-2);
F= fourier(x)
simplify(F)
```

Output:

```
ans =2*cos(2*w)
```

Example 7.24 Write a MATLAB program to find the Fourier transform of the signal $x(t) = u(t-1) - u(t+1)$ (Refer Example 3.14 (4))

```
syms t x
x = heaviside(t+2)- heaviside(t-2);
F= fourier(x)
simplify(F)
```

Output:

```
ans = (2*sin(2*w))/w
```

Example 7.25 Write a MATLAB program to find the Fourier transform of the signal

$$x(t) = \frac{d}{dt}(u(-t-3) + u(t-3)$$

(Refer Example 3.14 (5))

```
syms t x
x = heaviside(-t-32)+ heaviside(t-3);
F= fourier(diff(x))
```

Output:

```
F = exp(-w*3i) - exp(w*32i)
```

Example 7.26 Write a MATLAB program to find the Fourier transform of the signal $x(t) = e{-}3tu(t-1)$ (Refer Example 3.14 (6))

```
syms t x(t)
x(t) = exp(-3*t) * heaviside(t-1);
F= fourier(x(t))
```

Output:

```
F =exp(- w*1i - 3)/(3 + w*1i)
```

Example 7.27 Write a MATLAB program to find the Fourier transform of the signal $x(t) = te^{-0.5}u(t)$ (Refer Example 3.14 (7))

```
syms t w
x=t*exp(-0.5*t) *heaviside(t);
F=fourier(x,w)
```

Output:

```
F = 1/(1/2 + w*1i)^2
```

Example 7.28 Write a MATLAB program to find the Fourier transform of the signal $x(t) = e^{-a(t-2)}u(t-2)$ (Refer Example 3.14 (8))

```
syms t w a
a=3; % Assume a=3
x=exp(-a*(t-2)) *heaviside(t-2);
F=fourier(x,w)
```

Output:

```
F = exp(-w*2i)/(a + w*1i)
```

Example 7.29 Write a MATLAB program to find the Fourier transform of the signal

$$x(t) = \frac{d}{dt}\left(5\text{rect}\left(\frac{t}{8}\right)\right)$$

(Refer Example 3.14 (18))

```
syms t x
x =5*rectangularPulse(-1, 1, (t/8))
F= fourier(diff(x))
simplify(F)
```

Output:

```
ans =sin(8*w)*10i
```

Example 7.30 Write a MATLAB program to find the Fourier transform of the signal

$$x(t) = \delta(t+2) + 5\delta(t-2)$$

(Refer Example 3.14 (19))

```
syms t x
x = dirac(t+2)+5*dirac(t+1)+dirac(t-1)+5*dirac(t-2);
F= fourier(x)
```

Output:

```
F = exp(-w*1i) + 5*exp(w*1i) + 5*exp(-w*2i) + exp(w*2i)
```

Example 7.31 Write a MATLAB program to find the Inverse Fourier transform of the function $X(j\omega) = \delta(w - w_0)$ (Example 3.18a)

```
syms t w wo
ifourier(dirac(w-wo), w, t)
```

Output:

```
ans = exp(t*wo*1i)/(2*pi)
```

Example 7.32 Write a MATLAB program to find the Inverse Fourier transform of the function

$$X(jw) = \begin{Bmatrix} 1 & w < 2 \\ 0 & \text{elsewhere} \end{Bmatrix}$$

Example 3.18(3).

```
syms t
f = rectangularPulse(-2,2,t); %
F = ifourier(f,'t');
simplify(F)
```

Output:

ans =sin(2*t)/(t*pi)

Example 7.33 Write a MATLAB program to find the Inverse Fourier transform of the function

$$X(jw) = \frac{6}{w^2 + 9}$$

Example 3.18(4).

```
clc;
syms t w wo
X=6/(9+w^2)
ifourier(X,t)
```

Output:

$x(t) = e^{-3t}u(t) + e^{3t}u(-t)$

Example 7.34 Write a MATLAB program to find the Laplace transform of ramp function (Refer Example 4.9)

```
syms t s
f=t;
laplace(f)
laplace(f,s)
```

Output:

ans =1/s^2

Example 7.35 Write a MATLAB program to find the Laplace transform of acceleration function (Refer Example 4.10)

```
syms a t s
f=(1/2)*a*t^2;
laplace(f)
laplace(f,s)
```

Output:

```
ans =a/s^3
```

Example 7.36 Write a MATLAB program to find the Laplace transform of exponential decay function (Refer Example 4.11)

```
syms a t s
f=(1/2)*a*t^2;
laplace(f)
laplace(f,s)
```

Output:

```
ans = 1/(a + s)
```

Example 7.37 Write a MATLAB program to find the Laplace transform of sine function $x(t) = \sin atu(t)$ (Refer Example 4.12)

```
syms a t s
f=sin (a*t);
laplace(f)
laplace(f,s)
```

Output:

```
ans = a/(a^2 + s^2)
```

Example 7.38 Write a MATLAB program to find the Laplace transform of cosine function $x(t) = \cos atu(t)$ (Refer Example 4.13)

```
syms a t s
f=cos (a*t);
laplace(f)
laplace(f,s)
```

Output:

```
ans = s/(a^2 + s^2)
```

Example 7.39 Write a MATLAB program to find the Laplace transform of hyperbolic sine function (Refer Example 4.14)

```
syms a t s
f=sinh(a*t);
laplace(f)
laplace(f,s)
```

Output:

```
ans = -a/(a^2 - s^2)
```

Example 7.40 Write a MATLAB program to find the Laplace transform of hyperbolic cosine function (Refer Example 4.15)

```
syms a t s
f=cosh(a*t);
laplace(f)
laplace(f,s)
```

Output:

```
ans = -s/(a^2 - s^2)
```

Example 7.41 Write a MATLAB program to find the Laplace transform of $x(t) = t^n u(t)$ function (Refer Example 4.16)

```
syms n t s
f=t^n;
laplace(f)
laplace(f,s)
```

Output:

```
ans = piecewise([-1 < real(n), gamma(n + 1)/s^(n + 1)])
```

Example 7.42 Write a MATLAB program to find the Laplace transform of $x(t) = e^{-at} \sin w_0 t$ function (Refer Example 4.17)

```
syms a t s wo
f=exp(-a*t)*sin(wo*t);
laplace(f)
laplace(f,s)
```

Output:

```
ans = wo/((a + s)^2 + wo^2)
```

Example 7.43 Write a MATLAB program to find the Laplace transform of $x(t) = t \sin w_0 t$ function (Refer Example 4.18)

```
syms t s wo
f=t*sin(wo*t);
laplace(f)
laplace(f,s)
```

Output:

```
ans=(2*s*wo)/(s^2 + wo^2)^2
```

Example 7.44 Write a MATLAB program to find the Laplace transform of $x(t) = \cos at \sin bt$ function (Refer Example 4.19)

```
syms a t s b
f=cos(a*t)*sin(b*t);
laplace(f)
laplace(f,s)
```

Output:

```
ans= (- a^2*b + b^3 + b*s^2)/(a^4 - 2*a^2*b^2 + 2*a^2*s^2 + b^4 + 2*b^2*s^2
+ s^4)
```

Example 7.45 Write a MATLAB program to find the Laplace transform of $x(t) = \sin(at + \theta)$ function (Refer Example 4.21)

```
syms a t s b
f=sin(a*t+b);
laplace(f)
laplace(f,s)
```

Output:

```
ans= (a*cos(b) + s*sin(b))/(a^2 + s^2)*s*wo)/(s^2 + wo^2)^2
```

Example 7.46 Write a MATLAB program to find the Inverse Laplace transform of

$$X(s) = \frac{(s+1)(s+3)}{(s+2)(s+4)}$$

(Refer Example 4.38(1))

```
syms t s
figure
X2=((s+1)*(s+3))/((s+2)*(s+4));
x2=ilaplace(X2,t)
ezplot(x2,[0 10])
title('x(t) Inverse form')
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

x2 = dirac(t) - (3*exp(-4*t))/2 - exp(-2*t)/2

Example 7.47 Write a MATLAB program to find the Inverse Laplace transform of

$$X(s) = \frac{10(s+4)}{s^2(s+2)}$$

(Refer Example 4.39)

```
syms t s w
figure
X2=(10*(s+4))/((s^2)*(s+2));
x2=ilaplace(X2,t)
ezplot(x2,[0 10])
title('x(t) Inverse form')
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

x2 = 20*t + 5*exp(-2*t) - 5

Example 7.48 Write a MATLAB program to find the Inverse Laplace transform of

$$X(s) = \frac{(3s^2+8s+23)}{(s+3)(s^2+2s+10)}$$

(Refer Example 4.41)

```
syms t s w
figure
X2=(3*s^2+8*s+23)/((s+3)*(s^2+2*s+10));
x2=ilaplace(X2,t)
ezplot(x2,[0 10])
title('x(t) Inverse form')
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

x2 = 2*exp(-3*t) + cos(3*t)*exp(-t)

Example 7.49 Write a MATLAB program to find the Inverse Laplace transform of

$$X(s) = \frac{(3s^2+8s+6)}{(s+2)(s^2+2s+1)}$$

(Refer Example 4.42)

```
syms t s w
figure
X2=(3*s^2+8*s+6)/((s+2)*(s^2+2*s+1));
x2=ilaplace(X2,t)
ezplot(x2,[0 10])
title('x(t) Inverse form')
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

```
x2 = exp(-t) + 2*exp(-2*t) + t*exp(-t)
```

Example 7.50 Write a MATLAB program to find the Inverse Laplace transform of

$$X(s) = \frac{10s^2}{(s+2)(s^2+4s+5)}$$

(Refer Example 4.43)

```
syms t s w
figure
X2=(10*s^2)/((s+2)*(s^2+4*s+5));
x2=ilaplace(X2,t)
ezplot(x2,[0 10])
title('x(t) Inverse form')
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

```
x2 = 40*exp(-2*t) - 30*exp(-2*t)*(cos(t) + (4*sin(t))/3)
```

Example 7.51 Write a MATLAB program to find the Inverse Laplace transform of following differential equation

$$\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 12y(t) = x(t)$$

$x(t) = u(t)$ and assume $y(0) = -2$ and $dy/dt(0) = 0$ (Refer Example 4.48)

```
syms t s Y
x=heaviside(t);
X=laplace(x,s);
y0=-2;
yd0=0;
Y1=s*Y-y0;
```

```
Y2=s*Y1-yd0;
G=Y2+7*Y1+12*Y-X;
Y=solve(G,Y);
y=ilaplace(Y,t)
ezplot(y,[0 10]);
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

y(t) =(25*exp(-4*t))/4 - (25*exp(-3*t))/3 + 1/12

Example 7.52 Write a MATLAB program to find the Inverse Laplace transform of following differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt} + x(t)$$

$x(t) = e^{-3t}u(t)$ and assume $y(0) = 9/4$, and $dy/dt(0) = 5$ (Refer Example 4.49)

```
syms t s Y
y0=9/4;
yd0=5;
x=exp(-3*t)*heaviside(t);
X=laplace(x,s);
dx=diff(exp(-3*t));
dxi=laplace(dx,s);
Y1=s*Y-y0;
Y2=s*Y1-yd0;
G=Y2+4*Y1+4*Y-X-dxi;
Y=solve(G,Y);
y=ilaplace(Y,t)
ezplot(y)
legend('Output response of y(t)')
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

y(t) =((17*exp(-2*t))/4 - 2*exp(-3*t) + (15*t*exp(-2*t))/2

Example 7.53 Write a MATLAB program to find the convolution property of Laplace transform of following signal $x_1(t) = e^{-2t}u(t)$ and $x_2(t) = e^{-3t}u(t)$ (Refer Example 4.52)

```
syms t s
x1=exp(-2*t)*heaviside(t);
```

```
x2=exp(-3*t)*heaviside(t);
X1=laplace(x1,s);
X2=laplace(x2,s);
R=ilaplace(X1*X2,t)
ezplot(R,[0 20]);
legend('Output response of y(t)')
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

```
R = exp(-2*t) - exp(-3*t)
```

Example 7.54 Write a MATLAB program to find the convolution property of Laplace transform of following signals.

$$x_1(t) = e^{-2t}u(t)$$

and

$$x_2(t) = (1 + e^{-3t})u(t)$$

(Refer Example 4.53)

```
syms t s
x1=exp(-2*t)*heaviside(t);
x2=(1+exp(-3*t))*heaviside(t);
X1=laplace(x1,s);
X2=laplace(x2,s);
R=ilaplace(X1*X2,t)
ezplot(R,[0 20]);
legend('Output response of y(t)')
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

```
R = exp(-2*t)/2 - exp(-3*t) + 1/2
```

Example 7.55 Write a MATLAB program to find the z-transform of the signal $x[n] = \{2, -1, 0, 3, 4\}$ (Refer Example 5.2 (1))

```
syms z
x=[2 -1 0 3 4];
n=[0 1 2 3 4];
X=sum(x.*(z.^-n))
```

Output:

```
X = 3/z^3 - 1/z + 4/z^4 + 2
```

Example 7.56 Write a MATLAB program to find the z-transform of the signal $x[n] = \{1, -2, 3, -2, 2\}$ (Refer Example 5.2 (2))

```
syms z
x=[1 -2 3 -1 2];
n=[-4 -3 -2 -1 0 ];
X=sum(x.*(z.^-n))
```

Output:

```
z^4 - 2*z^3 + 3*z^2 - z + 2
```

Example 7.57 Write a MATLAB program to find the z-transform of unit impulse function (Refer Example 5.2 (4))

```
syms n z a w
f=dirac(n);
ztrans(f,z)
```

Output:

```
ztrans(dirac(n), n, z)
```

$$X[z] = 1$$

Example 7.58 Write a MATLAB program to find the z-transform of unit step function (Refer Example 5.2 (5))

```
syms n z a w
f=heaviside(1)
ztrans(f,z)
```

Output:

```
ans = z/(z - 1)
```

Example 7.59 Write a MATLAB program to find the z-transform of $x[n] = e^{jwn}$ $u[n]$ (Refer Example 5.2 (11))

```
syms z,w,n
f=exp(w*n);
ztrans(f,z)
```

Output:

```
ans = z/(z - exp(w))
```

Example 7.60 Write a MATLAB program to find the z-transform of $x[n] = \cos$ $w_0 n u[n]$ (Refer Example 5.2 (12))

```
syms z,w,n
f=cos(w*n);
ztrans(f,z)
```

Output:

```
ans = (z*(z - cos(w)))/(z^2 - 2*cos(w)*z + 1)
```

Example 7.61 Write a MATLAB program to find the z-transform of $x[n] = \sin wnu[n]$ (Refer Example 5.2 (13))

```
syms z,w,n
f=cos(w*n);
ztrans(f,z)
```

Output:

```
ans = (z*sin(w))/(z^2 - 2*cos(w)*z + 1)
```

Example 7.62 Write a MATLAB program to find the z-transform of $x[n] = nu[n]$ (Refer Example 5.3 (7))

```
syms n z a w
x=n*heaviside(n);
Left=ztrans(x,z);
simplify(Left)
```

Output:

```
ans = z/(z - 1)^2
```

Example 7.63 Write a MATLAB program to find the z-transform of $x[n] = nu[n - 1]$ (Refer Example 5.3 (19))

```
syms n z
x=n*heaviside(n);
Left=ztrans(x,z);
simplify(Left)
```

Output:

```
ans = z/(z - 1)^2
```

Example 7.64 Write a MATLAB program to solve difference equation

$$y[n+2] + 1.1y[n+1] + 0.3y[n] = x[n+1] + x[n]$$

$x(n) = (-4)^{-n}u(n)$ and assume initial conditions are zero (Refer Example 5.29)

```
syms n z Y
x=-4^-n;
X=ztrans(x,z);
X1=z^(1)*X;
Y1=z^(1)*Y;
Y2=z^2*Y
G=0.3*Y+1.1*Y1+Y2-X-X1;
SOL=solve(G,Y);
y=iztrans(SOL,n)

n_s=0:30;
y_s=subs(y,n,n_s);
stem(n_s,y_s);
legend('Output response of y[n]')
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

(20*(-1/2)^n)/3 - (100*(1/4)^n)/51 - (80*(-3/5)^n)/17

Example 7.65 Write a MATLAB program to solve difference equation

$$y[n] + 2y[n-1] + 2y[n-2] = x[n]$$

$X(n) = u(n)$ and assume $y[-1] = 0$ and $y[-2] = 2$ (Refer Example 5.35)

```
clc;
clear all;
syms n z Y
x=0.2^n;
X=ztrans(x,z);
y_1=0;
y_2=1;
Y1=z^(-1)*Y+y_1;
Y2=z^(-2)*Y+z^-1*y_1+y_2;
G=Y-0.75*Y1+.125*Y2-X;
SOL=solve(G,Y);
y=iztrans(SOL,n)
n1=0:50;
y_n=subs(y,n,n1);
stem(n1,y_n)
legend('Output response of y[n]')
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

(37*(1/2)^n)/12 - (39*(1/4)^n)/8 + (8*(1/5)^n)/3

Example 7.66 Write a MATLAB program to solve difference equation

$$y[n] + 6y[n-1] + 8y[n-2] = 5x[n-1] + x[n-2]$$

$X(n) = u(n)$ and assume $y[-1] = 1$ and $y[-2] = 2$ (Refer Example 5.37), $x[n] = u[n]$.

```
clc;
clear all;
syms n z Y
x=heaviside(n+1);
X=ztrans(x,z);
X1=z^(-1)*X;
X2=z^(-2)*X;
y_1=1;
y_2=2;
Y1=z^(-1)*Y+y_1;
Y2=z^(-2)*Y+z^-1*y_1+y_2;
G=Y+6*Y1+8*Y2-5*X1-X2;
SOL=solve(G,Y);
y=iztrans(SOL,n)
n1=0:50;
y_n=subs(y,n,n1);
stem(n1,y_n)
title('y[n] in z-Transform ');
legend('Output response of y[n]')
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

(39*(-2)^n)/2 - (419*(-4)^n)/10 + 2/5

Example 7.67 Write a MATLAB program to solve difference equation

$$y[n+2] + y[n+1] + 0.24y[n] = x[n+1] + 2x[n]$$

$x(n) = (1/2)^n u(n)$ and assume initial conditions are zero (Refer Example 5.38)

```
syms n z Y
x=0.5^n;
X=ztrans(x,z);
X1=z^(1)*X;
```

```
y_1=0;
y_2=0;
Y1=z^(1)*Y+y_1;
Y2=z^(2)*Y+y_2+(z^1)*y_1;

G=Y2+Y1+0.24*Y-X1-2*X;
SOL=solve(G,Y);
y=iztrans(SOL,n)

n1=0:50;
y_n=subs(y,n,n1);
stem(n1,y_n)
legend('Output response of y[n]')
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

y[n]=(250*(1/2)^n)/99 - (80*(-2/5)^n)/9 + (70*(-3/5)^n)/11

Example 7.68 Write a MATLAB program to solve difference equation

$$y[n+2] - 9y[n+1] + 20y[n] = 4x[n+1] + 2x[n]$$

$X(n) = (1/2)^n u(n)$ and assume $y[-1] = 2$ and $y[-2] = 1$ (Refer Example 5.40)

```
clc;
clear all;
syms n z Y
x=0.5^n;
X=ztrans(x,z);
X1=z^(-1)*X;
X2=z^(-2)*X;
y_1=1;
y_2=2;
Y1=z^(-1)*Y+y_1;
Y2=z^(-2)*Y+z^-1*y_1+y_2;
G=Y-9*Y1+20*Y2-4*X1-2*X;
SOL=solve(G,Y);
y=iztrans(SOL,n)
n1=0:50;
y_n=subs(y,n,n1);
stem(n1,y_n)
legend('Output response of y[n]')
xlabel('Time (t)');
ylabel('Magnitude ');
```

Output:

$y[n] = [0.254(0.5)^n + 42.86(4)^n - 45.1(5)^n]u[n]$

Example 7.69 Write a MATLAB program to find the state equation for the transfer function (Refer Example 6.13)

$$H(S) = \frac{7s^3 + 11s^2 + 14s + 10}{s^3 + 8s^2 + 5s + 4}$$

```
clc;
clear all;
num=[7 11 14 10];
den=[1 8 5 4];
z=tf(num,den)
[A ,B,C,D]=tf2ss(num ,den)
```

Output:

```
z = 7 s^3 + 11 s^2 + 14 s + 10
    ---------------------------
    s^3 + 8 s^2 + 5 s + 4
```

Continuous-time transfer function.

$$A = \begin{bmatrix} -8 & -5 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} -45 & -21 & -18 \end{bmatrix}$$
$$D = 7$$

Example 7.70 Write a MATLAB program to determine the system function (Refer Example 6.14)

$$q = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} q(t)$$
$$y = [0 \quad 1]q(t)$$

```
clc;
clear all;
A=[-3 1;-2 0];
B=[1 ;0];
```

```
C=[0 1];
D=0;
[n , d]=ss2tf(A,B,C,D)
transferfn=tf(n,d)
```

Output:

```
                 -2
transferfn = -------------
             s^2 + 3 s + 2
```

Example 7.71 Write a MATLAB program to find the state equation for the transfer function (Refer Example 6.18)

$$H(S) = \frac{5z^4 + 7z^3 + 8z^2 + 2z + 10}{z^4 + 6z^3 + 7z^2 + 4z + 9}$$

```
clc;
clear all;
num=[5 7 8 2 10];
den=[1 6 7 4 9];
z=tf(num,den)
[A ,B,C,D]=tf2ss(num ,den)
```

Output:

$$A = \begin{bmatrix} -6 & -7 & -4 & -9 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -23 & -27 & -18 & -35 \end{bmatrix}$$

$$D = 5$$

Example 7.72 Write a MATLAB program to find the state equation (Refer Example 6.19)

$$4y[n-3] + 6y[n-2] - 5y[n-1] + y[n] = 5x[n-1] + x[n]$$

```
b0=1;
b1=5;
b2=0;
b3=0;
a1=-5;
a2=6;
a3=4;
A=[0 1 0 ; 0 0 1; -a3 -a2 -a1]
B=[ 0 0 1];
bb3=b3-b0*a3;
bb2=b2-b0*a2;
bb1=b1-b0*a1;
C= [bb3 bb2 bb1]
D=b0
```

Output:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -6 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -4 & -6 & -10 \end{bmatrix}$$

$$D = 1$$

Example 7.73 Write a MATLAB program to find the state equation canonical form-I for the transfer function (Refer Example 6.20)

$$H(S) = \frac{5z^4 + 7z^3 + 8z^2 + 2z + 10}{z^4 + 6z^3 + 7z^2 + 4z + 9}$$

```
clc;
clear all;
num=[5 7 8 2 10];
den=[1 6 7 4 9];
z=tf(num,den)
[A ,B,C,D]= tf2ss(num ,den);
A=A'
B=B'
C=C'
D=D
```

Output:

```
z =

5 s^4 + 7 s^3 + 8 s^2 + 2 s + 10
---------------------------------
s^4 + 6 s^3 + 7 s^2 + 4 s + 9

Continuous-time transfer function.
```

$$A = \begin{bmatrix} -6 & 1 & 0 & 0 \\ -7 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \\ -9 & 0 & 0 & 0 \end{bmatrix}$$

$$B = [1 \quad 0 \quad 0 \quad 0]$$

$$C = \begin{bmatrix} -23 & -27 & -18 & -35 \end{bmatrix}$$

$$D = 5$$

Example 7.74 Write a MATLAB program to find the state variables (Refer Example 6.22)

$$y[n] - 3y[n-1] - 2[y-2] = x[n] + 5x[n-1] + 6x[n-2]$$

```
clc;
clear all;
b0=1;
b1=5;
b2=6;
a1=-3;
a2=-2;
A=[0 1 ; -a2 -a1]
B=[ 0 1];
bb2=b2-b0*a2;
bb1=b1-b0*a1;
C= [ bb2 bb1]
D=b0
```

Output:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = [8 \quad 8]$$

$$D = 1$$

Example 7.75 Write a MATLAB program to determine the transfer function (Refer Example 6.23)

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad C = [3 \quad 1]; \quad D = [2]$$

```
clc;
clear all;
A=[2 -1;1 0];
B=[1 ;0];
C=[3 1];
D=2;
[n , d]=ss2tf(A,B,C,D)
transferfn=tf(n,d)
```

Output:

```
transferfn =
2 s∧2 – s + 3
-------------
s∧2 – 2 s + 1
```

Example 7.76 Determine the transfer function and the Eigen values of the system represented in state space using MATLAB.

$$\frac{dx(t)}{dt} = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} u(t)$$

$$y(t) = [2 \quad -6 \quad 5]x(t)$$

clc;
$A = [41 - 2; 102; 1 - 13];$
$B = [1; 2; 3];$
$C = [2 - 65];$
$D = 0;$
[num,den]=ss2tf(A,B,C,D)
sys=tf(num,den)
EigenValues=roots(den)

Output:

num = 0 5.0000 – 37.0000 60.0000
den = 1.0000 – 7.0000 15.0000 – 9.0000
Transfer function:
$5\,s^2 - 37\,s + 60/s^3 - 7\,s^2 + 15\,s - 9$
Eigen Values = 3.0000
3.0000
1.0000

Example 7.77 Write a Program to obtain the transfer function of the system defined by the following state space equations

$$\frac{dx(t)}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -10 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [4 \quad 5 \quad 1]x(t)$$

clc;
$A = [010; 001; -5 - 10 - 3];$
$B = [0; 0; 1];$
$C = [451];$
$D = 0;$
[num,den]=ss2tf(A,B,C,D) sys=tf(num,den)

Output:

num = 0   1.0000   5.0000   4.0000
den = 1.0000   3.0000   10.0000   5.0000
Transfer function:
$s^2 + 5\,s + 4/s^3 + 3\,s^2 + 10\,s + 5$

Example 7.78 Write a Program to obtain the state space equations for the transfer function given below

$$T(s) = \frac{5\,s + 2}{s^3 + 7\,s^2 + 3\,s + 5}$$

clc;
num=[0 0 5 2];
den=[1 7 3 5];
[A,B,C,D]=tf2ss(num,den)

Output:

$$A = \begin{bmatrix} -7 & -3 & -5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [0 \quad 5 \quad 2]$$

$$D = [0]$$

Example 7.79 Write a Program to find the state transition matrix for the following A matrix.

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$$

clc;
$A = [-31; -20]$;
T=sym('t')
STM=expm(A*t)

Output:

STM =[2/exp(2*t) – 1/exp(t), 1/exp(t) – 1/exp(2*t)]
[2/exp(2*t) – 2/exp(t), 2/exp(t) – 1/exp(2*t)]

Example 7.80 A certain control system is described by the following vector matrix differential equation

$$\frac{dx}{dt} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & -3 \\ 1 & 2 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} u(t)$$
$$y(t) = [1 \quad 1 \quad 2] x(t)$$

Determine whether the above system is completely state controllable, completely output controllable and observable.

clc;
$A = [121; -1 - 4 - 3; -123]$;
$B = [1; 4; 6]$;
$C = [112]$;
D=0;
disp('Rank of the Matrix')
Rankc=rank([B A*B A^2*B]) % To check the controllability
Ranko= rank([C' A'*C' A'^{2}*C']) % To check the observability
Rankoc= rank([C*B C*A*B C*A^2*B]) % To check the output Controllability

Output:

Rankc =3
Ranko = 3
Rankoc = 1

From the above the system is completely state controllable and observable. The system is not output controllable since the rank of the matrix is not three.

Example 7.81 Write a program to obtain the response for the following system for unit step input and $u(t) = e^{-t}$.

$$\frac{dx(t)}{dt} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = [1 \quad 0] x(t) + [0] u(t)$$

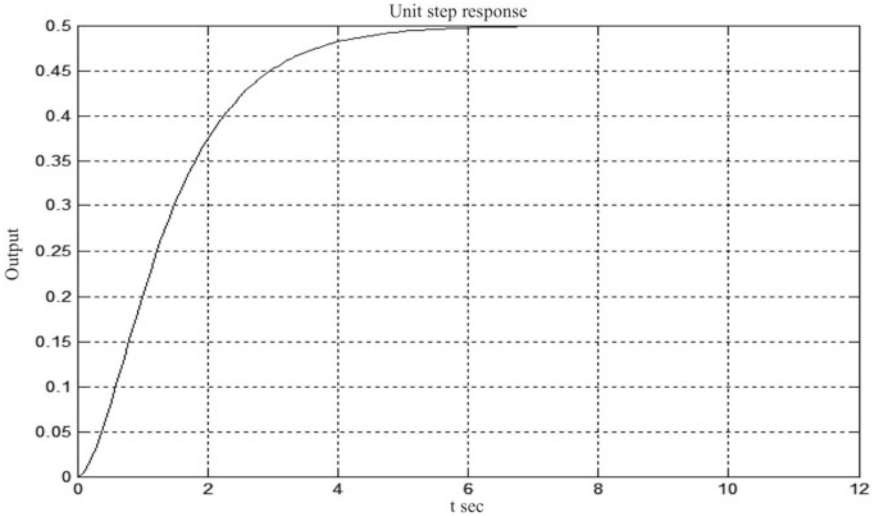


Fig. 7.10 Unit step response for Example 7.81

```
clc;
t=0:0.1:12;
A = [−31; −20];
B=[0;1];
C=[1 0];
D=[0];
y=step(A,B,C,D,1,t);
    figure(1)
    plot(t,y)
    grid
    title('Unit step Response')
    xlabel('tsec')
    ylabel('output')
    u = exp(−t)
    z=lsim(A,B,C,D,u,t)
    figure(2)
    plot(t,u,' − ',t,z,'o')
    grid
    title('Response to exponential Input u=exp(-t)')
    xlabel('tSec')
    ylabel('Exponential input')
    text(6.4,0.38,'output')
```

The output response is shown in Fig. 7.10 for unit step input.

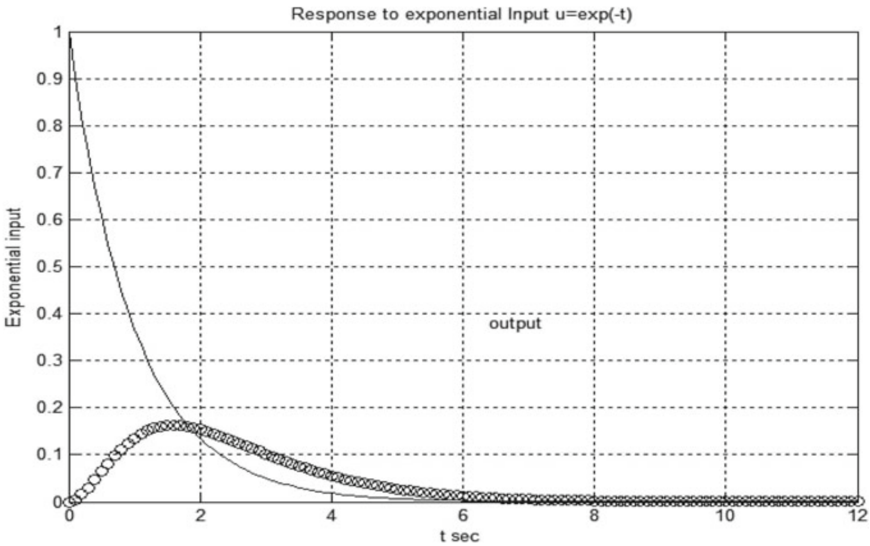


Fig. 7.11 Response for exponential Input for Example 7.81 for $x(t) = e^{-t}$

Output:

The output response of the system in Example 7.81 for $x(t) = e^{-t}$ is shown in Fig. 7.11.

Example 7.82 Write a program to obtain the response to initial conditions for the given system

$$\frac{dx(t)}{dt} = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = [1 \quad 0]$$
$$D = [0]$$

```
clc;
t=0:0.05:3
A = [−31; −20];
B=[1;0];
C=[0 1];
D=[0];
[y x]= initial(A,B,C,D,[1;-1],t);
figure(1)
plot(t,y)
grid
x1=[1 0]*x';
x2=[0 1]*x';
figure(2)
plot(t,x1,'x',t,x2,' − ')
grid
title('Response to Initial Condition')
xlabel('tSec')
ylabel('State variables x1 and x2')
gtext('x1')
gtext('x2')
```

Output:

The state variable response of Example 7.82 is shown in Fig. 7.12.

Example 7.83 Write a MATLAB program to perform unit impulse function.

Unit impulse function:

```
clc;
clear all;
close all;
x=ones(1,1);
subplot(2,3,1);
n=0;
stem(n,x);
xlabel('n');
ylabel('x');
title('unit impulse function');
```

The unit impulse function is shown in Fig. 7.13.

Example 7.84 Write a MATLAB program to perform unit step sequence.

```
clc;
clear all;
N=8;
```

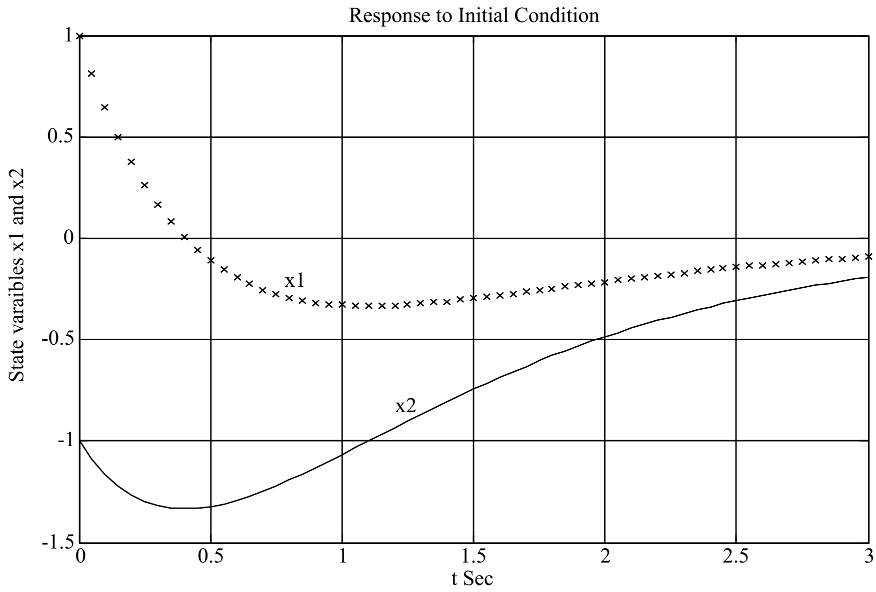


Fig. 7.12 Response for exponential input for Example 7.82

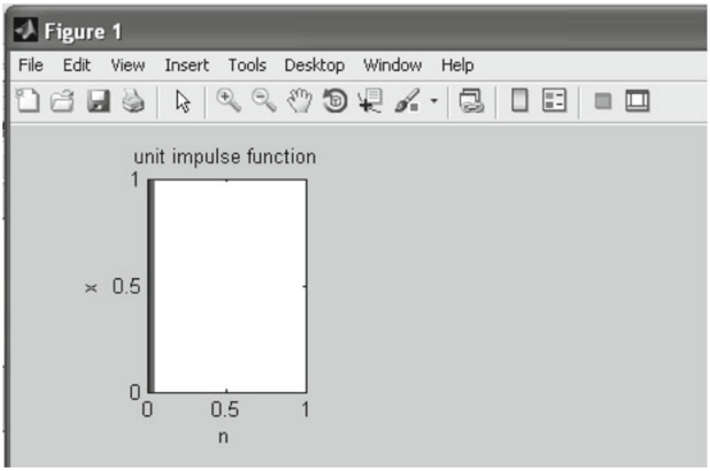


Fig. 7.13 Representation of unit impulse function

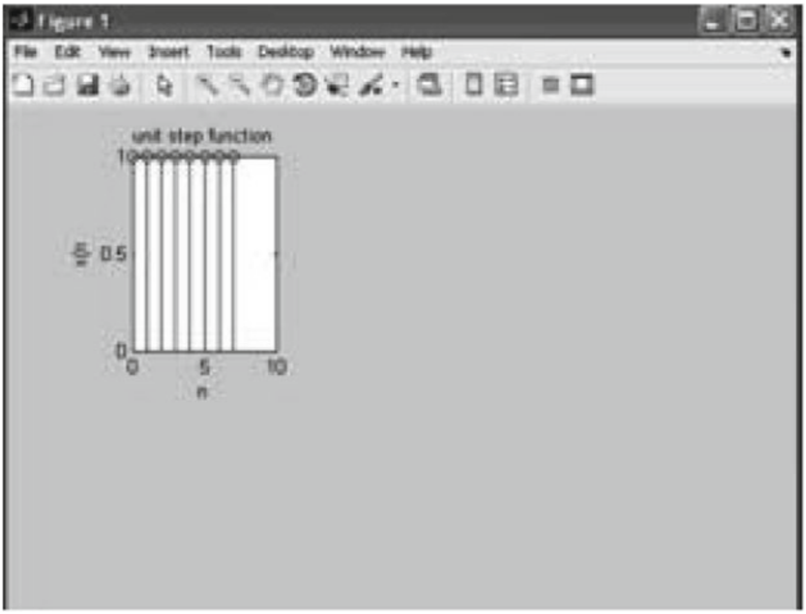


Fig. 7.14 Representation of unit step sequence

```
x=ones(1,N);
n=0:1:N-1;
subplot(2,3,1);
stem(n,x);
xlabel('n');
ylabel('x(n');
title('unit step function')
```

The unit step sequence is shown in Fig. 7.14.

Example 7.85 Write a MATLAB program to perform unit ramp sequence.

Unit ramp:

```
clc;
clear all;
N=8;
x=0:N-1;
n=0:N-1;
subplot(2,3,1);
stem(n,x);
xlabel('n');
ylabel('x(n');
title('unit ramp functin');
```

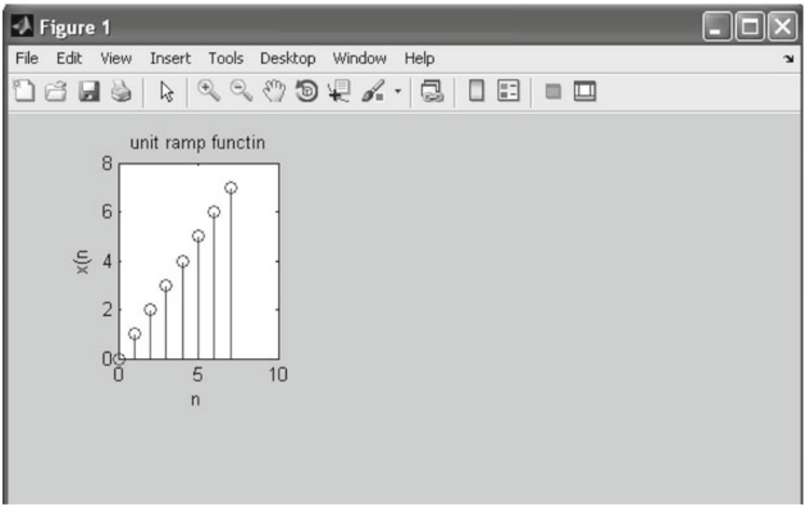


Fig. 7.15 Representation of unit ramp sequence

The unit ramp sequence is shown in Fig. 7.15.

Example 7.86 Write a MATLAB program to perform linear convolution.

$$x[n] = [1\ 2\ 3\ 4]$$
$$h[n] = [2\ 3\ 4\ 1]$$

Linear convolution:

Program:

```
clc;
clear all;
close all;
x=input('Enter the first input sequence x(n)');
h=input('Enter the second input sequence h(n)');
n1=length(x);
n2=length(h);
n=n1+n2-1;
y=conv(x,h);
disp('Linear Convolution Output is:');
disp(y);
t1=0:n1-1;
subplot(2,2,1);
stem(t1,x);
xlabel('n');
ylabel('Amplitude');
```

```
title('First input sequence:');
t2=0:n2-1;
subplot(2,2,2);
stem(t2,h);
xlabel('n');
ylabel('Amplitude');
title('Second input sequence:');
t=0:1:n-1;
subplot(2,2,3);
stem(t,y);
xlabel('n');
ylabel('Amplitude');
title('Output sequence:');
```

Output:

First sequence [1 2 3 4]
Second sequence[2 3 4 1]

o/p sequence:
[2 7 16 26 26 19 4]

Example 7.87 Write a MATLAB program to perform circular convolution for the Example 7.86.

```
clc;
clear all;
close all;
x1=input('enter');
x2=input('enter');
n1=length(x1);
n2=length(x2);
if(n1<n2)
    x1=[zeros x1(1,n2-n1)];
    elseif(n2<n1)
        x2=[zeros x2(1,n1-n2)];
else
    x1=x1;
    x2=x2;
end;
n1=length(x1);
n2=length(x2);
A=fft(x1,n1);
B=fft(x2,n2);
Y=A.*B;
y=ifft(Y);
```

```
n=length(y);
disp('Circular convolution output is:');
disp(y);
t1=0:n1-1;
subplot(2,2,1);
stem(t1,x1);
xlabel('n');
ylabel('Amplitude');
title('First input sequence:');
t2=0:n2-1;
subplot(2,2,2);
stem(t2,x2);
xlabel('n');
ylabel('Amplitude');
title('second input sequence:');
t=0:n-1;
subplot(2,2,3);
stem(t,y);
xlabel('n');
ylabel('Amplitude');
title('Output sequence:');

enter[1 2 3 4]
enter[2 3 4 5]
```

Output:

36 38 36 30

The input and output sequences are shown Fig. 7.16.

Example 7.88 Write a MATLAB program to find n point DFT of a given sequence.

$$x[n] = [1\ 1\ 1\ 1\ 0\ 0\ 0\ 0]$$

Discrete fourier transform

```
Clear all
Xn=input('enter a sequence');
L=length(xn); length of the sequence
N=input('enter the length of the DFT');
Xk=dft(xn,N)
Subplot(2,1,2),stem(abs(xk))
Xlable('\itk')
Ylabel('x(k))
subplot(2,1,2).stem(angle(xk))
```

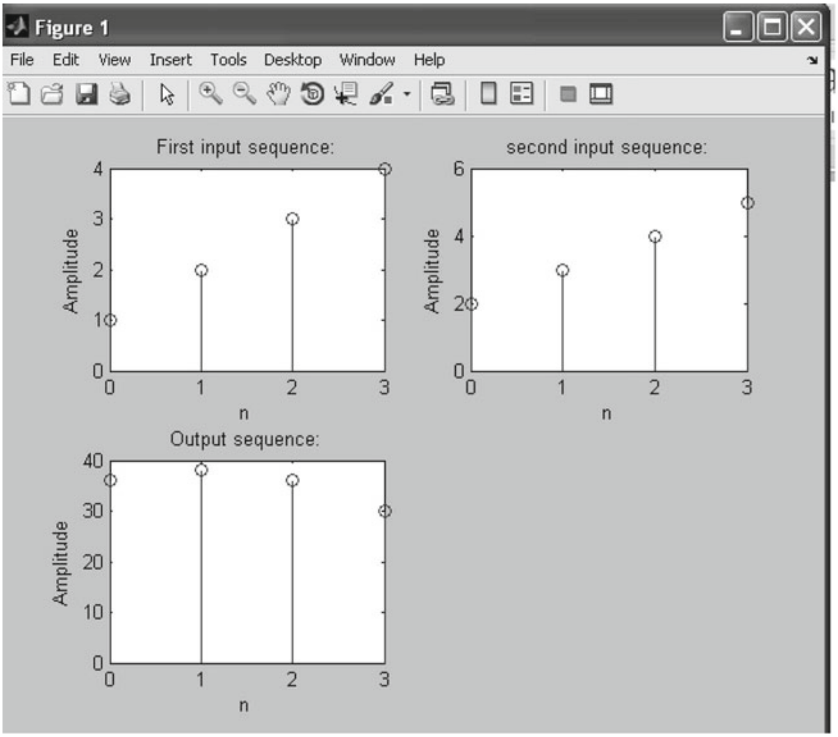


Fig. 7.16 Input/output of circular convolution

Xlabel('\itk')
Ylabel('arg(x(k))')

Output:

enter sequence [1 1 1 1 0 0 0 0]
enter the length of the DFT 8

xk=

Columns 1 through 6

4.0000 1.0000 – 2.4142*i* – 0.0000 – 0.0000*i* 1.0000 – 0.4142*i* 0 – 0.0000*i*
1.0000 + 0.4142*i*

Columns 7 through 8

0.0000 – 0.0000*i* 1.0000 + 2.4142*i*
See (Fig. 7.17).

Example 7.89 Write a MATLAB program to perform upsampling.

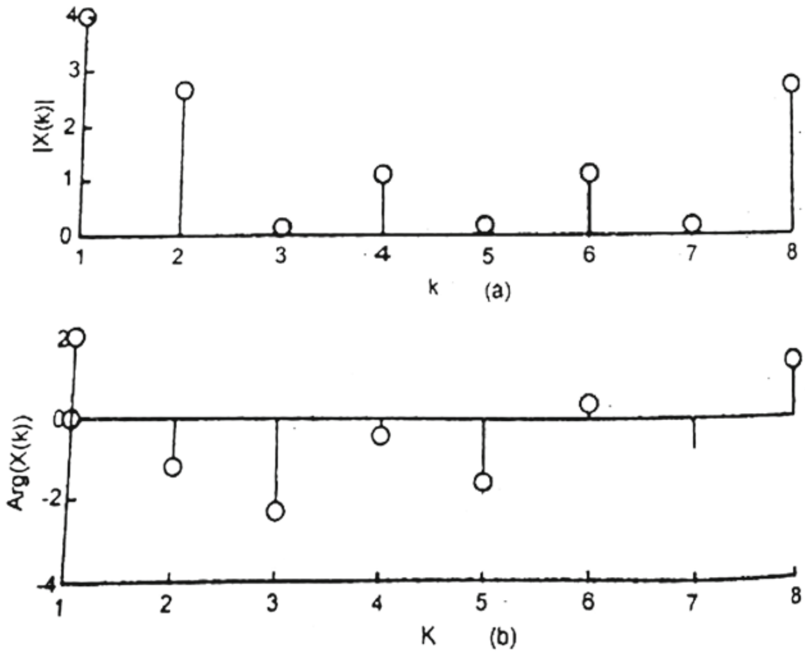


Fig. 7.17 N points DFT of $x[n] = [11110000]$

Illustration of upsampling:

```
Clear all
N=10% sequence length
N=0:1:N-1;
X=sin(2*pi*n/10)+sin(2*pi*n/5)
L=3% upsampling factor
X1=[zeros(1,L*N)];
N1=1:1:L*N j=1:L:L*N;
X1(j)=x;
Subplot(2,1,1); stem(n1,x1)
Xlabel('n'),ylabel('x')
Title('input sequence')
Subplot(2,1,2), stem(n1,x1)
Xlabel('n'),ylabel('x1')
Title('upsampled sequence');
```

The input sequence and the up sampled sequences are shown in Fig. 7.18. The input sequence is shown in Fig. 7.18a and output sequence in Fig. 7.18b.

Example 7.90 Write a MATLAB program to find Z transform.

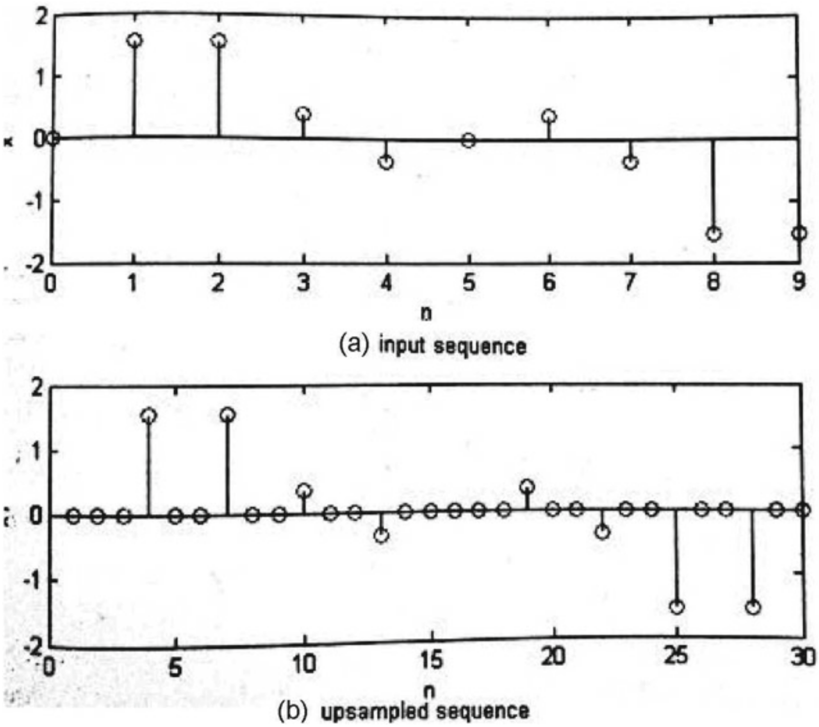


Fig. 7.18 Representation of input and upsampled sequences

Z transform:

To find the partial fraction of $H(Z)$:

```
Clear all
Clc
Num=[2];%numerator coefficients
Den=[1 -3 2];% denominator coefficients
[r,p,k]=residuez(num,den]
```

Output:

```
r = 4    -2
p= 2     1
k= [     ]
```

Example 7.91 Write a MATLAB program to estimate the power spectrum using periodogram.

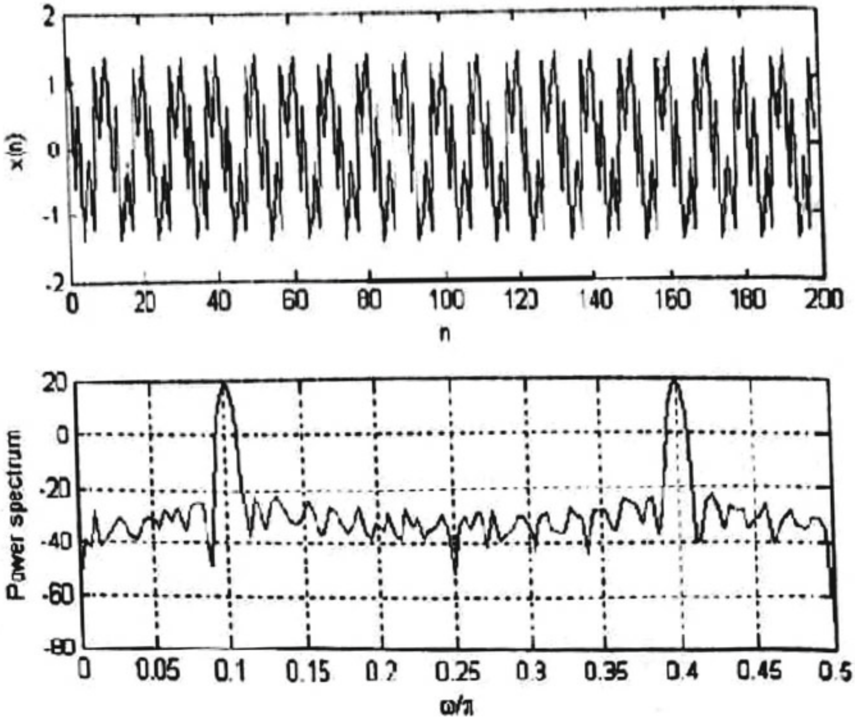


Fig. 7.19 Power spectrum is represented in Example 7.91

% power spectrum estimate using periodogram

```
Clear all
N=input ('enter the length of the sequence')
Window=hamming(n);
Nfft=input('length of the FFT");
Fs=input('sampling frequency');
N=0:1:n-1;
%signal sum of two sinusoids and random noise
X=cos(2*1*pi*f/fs)+sin(2*4*pi*n/fs)+0.01*randnsize(n);
Subplot(2,1,1),plot(n,x)
Xlabel('n'),ylabel('x(n)')
[pxx,f]=periodogram(x,window,nfft,fs)
Subplot(2,1,2)
Plot(f/fs.10*log10(pxx)); grid
Xlabel('omega/\pi'),ylabel('power spectrum')
```

The power spectrum is represented in Fig. 7.19.

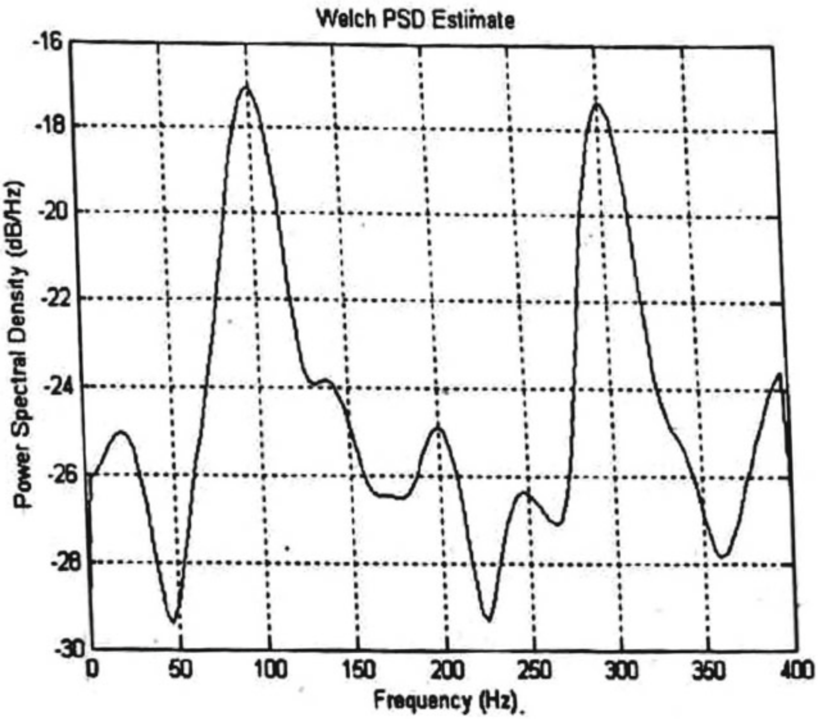


Fig. 7.20 Power spectrum estimation using Welch method in Example 7.92

Example 7.92 Write a MATLAB program to estimate power spectrum using Welch method.

% power spectrum estimate using Welch method

```
Clear all
Fs=800;
T=0.1/Fs:4;
X=cos(2*pi*t*100)+sin(2*pi*t*300))+randn(size(t));
Pwelch(x,[],0[],Fs)% uses default window overlap
```

Output:

The power spectrum estimation is shown in Fig. 7.20.

Example 7.93 Write a MATLAB program for echo cancellation.

Echo Cancelation

```
load mtlb
% To hear, type soundsc(mtlb,Fs)
```

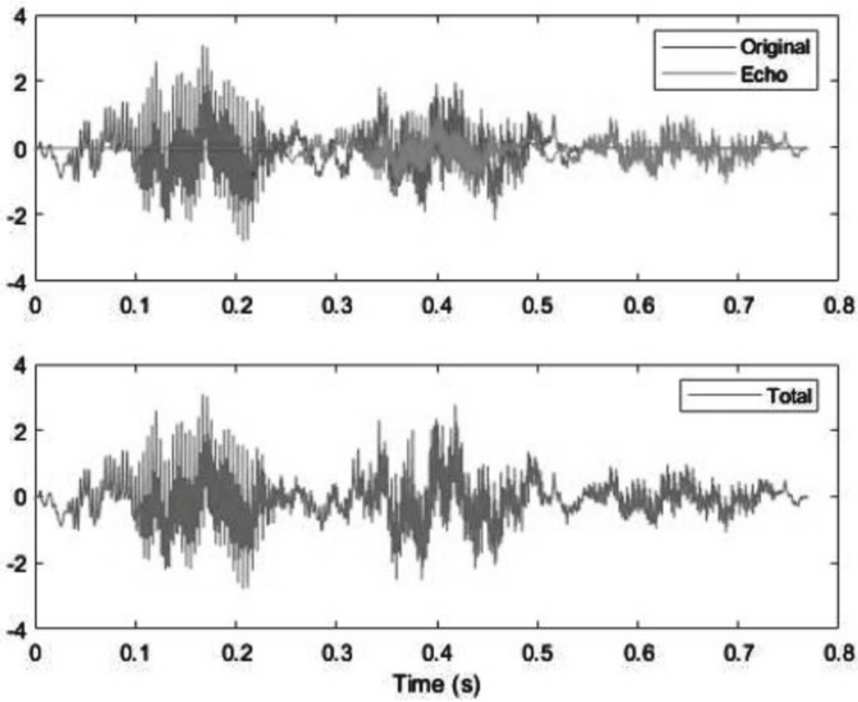


Fig. 7.21 Representation of echo cancellation in Example 7.93

```
timelag = 0.23;
delta = round(Fs*timelag);
alpha = 0.5;
orig = [mtlb;zeros(delta,1)];
echo = [zeros(delta,1);mtlb]*alpha;
mtEcho = orig + echo;
t = (0:length(mtEcho)-1)/Fs;
subplot(2,1,1)
plot(t,[orig echo])
legend('Original','Echo')
subplot(2,1,2)
plot(t,mtEcho)
legend('Total')
xlabel('Time (s)')
```

The echo cancellation is shown in Fig. 7.21.

Example 7.94 Write a MATLAB program for Speech Signal Testing.

```
clc; clear;
close all;
```

```
addpath 'func'
addpath 'func\func_pregross
\' addpath 'Speech_Processing_Toolbox'
Num_Gauss=64;
[Speech_Test0,Fs,nbits]=wavread('Test_Samples\test5\yes_no\yes.wav');
Index_use = func_cut(Speech_Test0,Fs,nbits);
Speech_Test = Speech_Test0(Index_use(1):Index_use(2));
figure;
plot(Speech_Test0);
hold on;
Len = [–1.05:0.01:1.05];
plot(Index_use(1)*ones(length(Len),1),Len,'r','linewidth',2);
hold on;
plot(Index_use(2)*ones(length(Len),1),Len,'k','linewidth',2);
hold off axis([1,length(Speech_Test0),-1.05,1.05]);
title('The simulation result of EndPoint checking');
figure; Linlin Pan Research and simulation on speech recognition by MATLAB A3
plot(Speech_Test+1.5,'b');
Speech_Test = filter([1, -0.95], 1, Speech_Test);
hold on plot(Speech_Test,'r');
legend('original','Pre emphasis');
global Show_Wind;
Show_Wind = 1;
global Show_FFT;
Show_FFT = 1;
Test_features= melcepst(Speech_Test,Fs);
figure;
surf(Test_features);
load GMM_MFCC.mat A=[0,0];
for i = 1:2 [lYM,lY]=func_multi_gauss(Test_features',
        mu_traini,sigma_traini,c_traini); A(i)=mean(lY);
end [V,I] = max(A); if I == 1 disp('The speech is: YES');
else disp('The speech is: NO');
```

The speech signal testing is shown in Fig. 7.22.

Example 7.95 Perform Live recording of 1 D speech signal using headset and plot the output waveform.

```
%%%audtest%%%%
USING audiorecorder function:

Fs=8000;
nBits=8;
nChannels=1;
recObj = audiorecorder(Fs,nBits,nChannels);
```

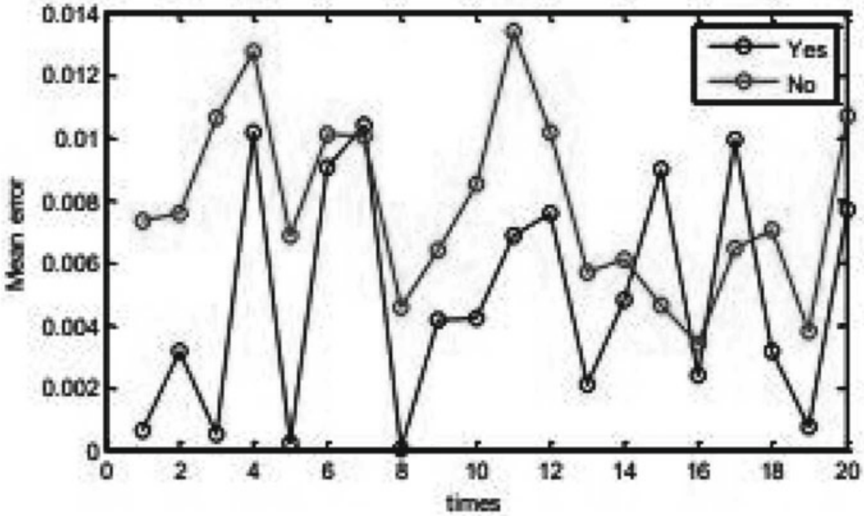


Fig. 7.22 Representation of speech signal testing for Example 7.94

```
disp('Start speaking.')
recordblocking(recObj, 5);
disp('End of Recording.');
% Play back the recording.
play(recObj);
myRecording = getaudiodata(recObj);
plot(myRecording);
% Plot the waveform.
```

Output waveform obtained from the above Program is plotted below in Fig. 7.23.

Output waveform:

Example 7.96 Write a MATLAB program for the acquisition of 2D (image) signal.

```
%%%Read gray scale image%%imgrd
clc;
clear all;
a=imread('cameraman.tif'); %Read the gray scale image
[M N]=size(a);
Figure;
subplot(3,2,1);
imshow(a);
%%%%Apply 2D DCT to the image
b=dct2(a);
subplot(3,2,2);
```

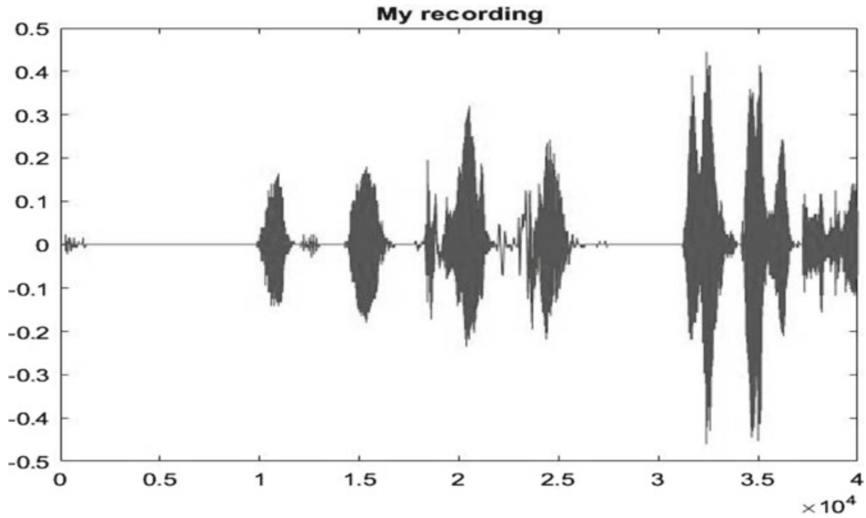


Fig. 7.23 Plot of 1 D speech signal of Example 7.95

```
imshow(abs(b),[]);
subplot(3,2,3);
e=idct2(b);
subplot(3,2,3);
imshow(e,[]);
```

Output obtained from the above Program is shown below in Fig. 7.24.

Output:

Example 7.97 Perform time scaling operations on 1D signal and analyze the process in time and frequency domains.

```
clc;
clear all;
Fs = 1000;      % Sampling frequency
T = 1/Fs;       % Sampling period
L = 1500;       % Length of signal
t = (0:L-1)*T; % Time vector
f=50;
X = 0.7*sin(2*pi*f*t);
Y= 2*sin(2*pi*(2*f)*t);
plot(1000*t(1:50),X(1:50))
title('Signal-1')
xlabel('t (milliseconds)')
ylabel('X(t)')
```

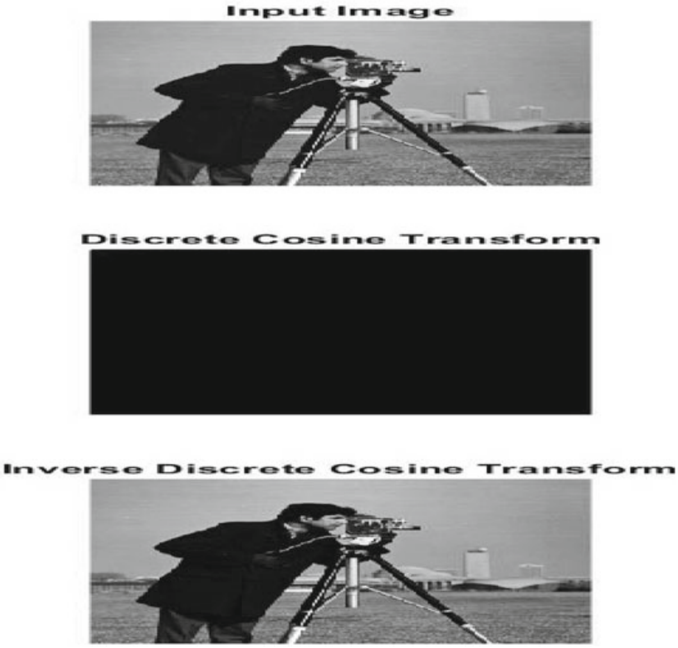


Fig. 7.24 Output image of DCT and IDCT of Example 7.96

```
plot(1000*t(1:50),Y(1:50))

title('Signal-2')
xlabel('t (milliseconds)')
ylabel('Y(t)')
z1 = fft(X);
P2 = abs(z1/L);
P1 = P2(1:L/2+1);
P1(2:end-1) = 2*P1(2:end-1);
f = Fs*(0:(L/2))/L;
plot(f,P1)
title('Single-Sided Amplitude Spectrum of X(t)')
xlabel('f (Hz)')
ylabel('|P1(f)|')
z2=fft(Y);
P3 = abs(z2/L);
P4 = P3(1:L/2+1);
P4(2:end-1) = 2*P4(2:end-1);
f = Fs*(0:(L/2))/L;
plot(f,P4)
title('Single-Sided Amplitude Spectrum of Y(t)')
```

```
xlabel('f (Hz)')
ylabel('|P1(f)|')
```

Output waveform obtained from the above Program is plotted in Fig. 7.25.

Output waveform:

Example 7.98 Perform Convolution operation on two speech signals and analyze the process in time and frequency domains.

(a) Time Domain

```
[sig1, fs] = audioread('example1.wav');
% import the song
t = [1:length(sig1)]/fs;
% soundsc(sig1,fs);
subplot(3,1,1)
plot(t, sig1) % plot the song
xlabel('t (second)')
ylabel('Relative signal strength')
title('Song')

[sig2, fs] = audioread('SpeechDFT-16-8-mono-5secs.wav');
% soundsc(sig2,fs);% import the song
x=sig2;
x(length(sig1))=0; % zero-pad if lenth(sig2) < sig1
x=x(1:length(sig1));
t1 = [1:length(sig2)]/fs;
subplot(3, 1, 2)
plot(t1, sig2) % plot the song
xlabel('t1 (second)')
ylabel('Relative signal strength')
title('Speech signal')
w =conv2(sig1,x,'same');
soundsc(w,fs);
% t2 = 0:1:10;
t2=[1:length(sig1)]/fs;
subplot(3,1,3);
plot(t2,w);
xlabel('t2 (second)')
ylabel('Relative signal strength')
title('Convolved Signal')
```

Output waveform obtained from the above Program is plotted and shown in Fig. 7.26.

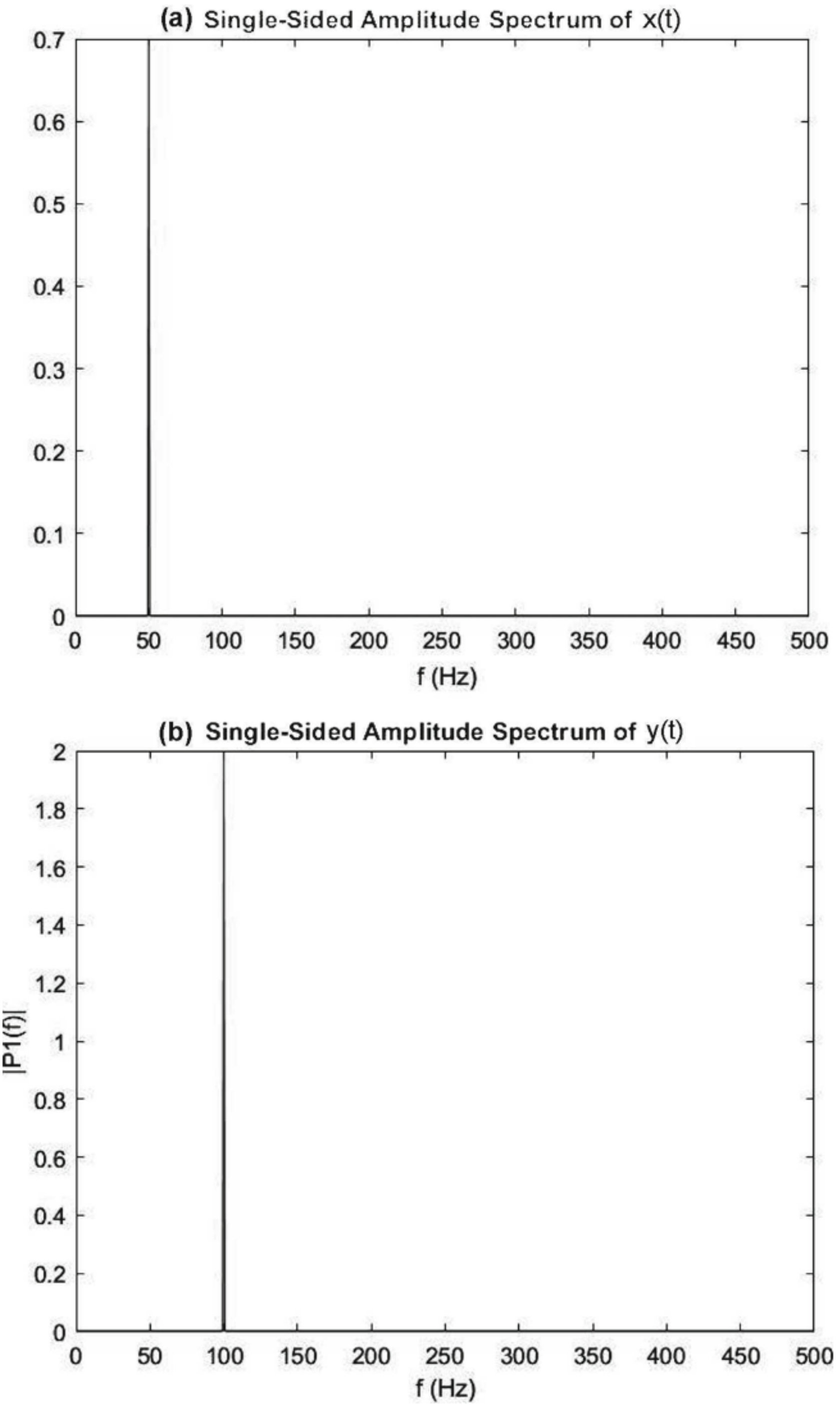


Fig. 7.25 Output of 1D single sided amplitude spectrum. **a** $x(t)$ and **b** $y(t)$ of Example 7.97

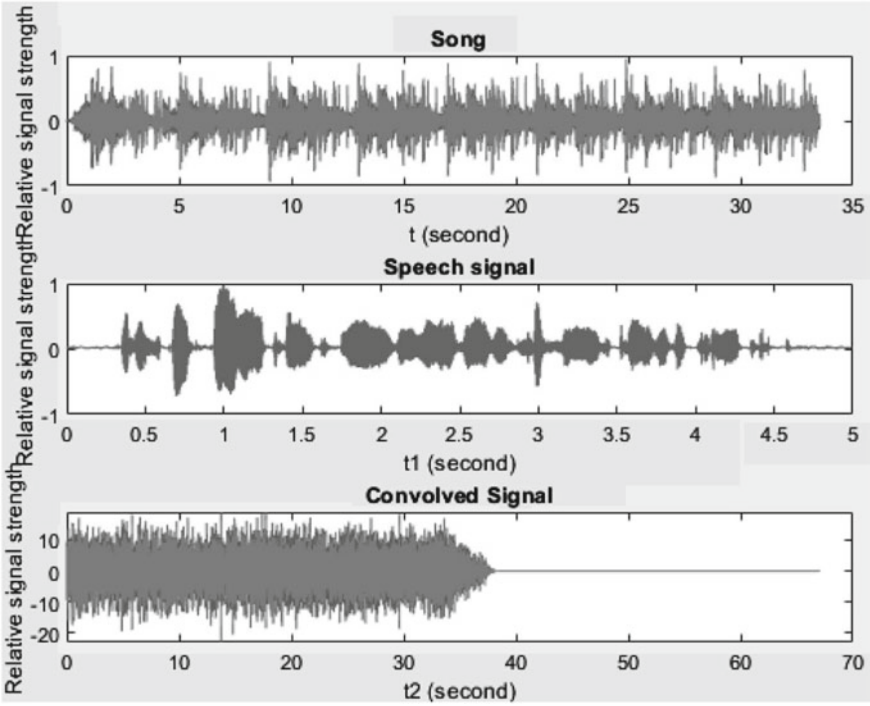


Fig. 7.26 Output of convolution of two speech signals in time domain analysis of Example 7.98a

Output Waveform:

(b) Frequency Domain

```
[sig1,fs] = audioread('example1.wav');
% import the song
f = [1:length(sig1)];
soundsc(sig1,fs);
% plot(t, sig1) % plot the song
X = fft(sig1);
X = fftshift(X);%rearranges a Fourier transform X by
% shifting the zero-frequency component to the center of the array.
Xmag = abs(X);

subplot(3,1,1)
plot(f,Xmag);
xlabel('Frequency')
ylabel('Relative signal strength')
title('Song')
[sig2, fs] = audioread('SpeechDFT-16-8-mono-5secs.wav');
```

```
soundsc(sig2,fs);% import the song
x=sig2;
x(length(sig1))=0; % zero-pad if lenth(sig2) < sig1
x=x(1:length(sig1));
f1 = [1:length(x)];
% plot(t1, sig2) % plot the song
X1 = fft(x);
X1 = fftshift(X1);%rearranges a Fourier transform X by
% shifting the zero-frequency component to the center of the array.
Xmag1 =abs(X1);
% delta_f = fs./(N.*1000);

% nf = -N./2:1:N/2-1;
% f = nf .* delta_f;
subplot(3, 1, 2)
plot(f1,Xmag1);
xlabel('Frequency')
ylabel('Relative signal strength')
title('Speech Signal')
w =conv2(sig1,x,'same');
soundsc(w,fs);
% t2 = 0:1:10;
% t2=[1:length(sig1)]/fs;
X2 = fft(w);
X2 = fftshift(X2);%rearranges a Fourier transform X by
% shifting the zero-frequency component to the center of the array.
Xmag2 = abs(X2);
f2=[1:length(sig1)];
subplot(3,1,3);
plot(f2,Xmag2);
% plot(t2,w);
xlabel('Frequency')
ylabel('Relative signal strength')
title('Convolved Signal')
```

Output waveform obtained from the above Program is plotted and shown in Fig. 7.27.

Output Waveform:

Example 7.99 Perform Correlation operations on two speech signals and analyze the process in time and frequency domains.

(a) Time Domain

```
load mtlb

% soundsc(mtlb,Fs)
```

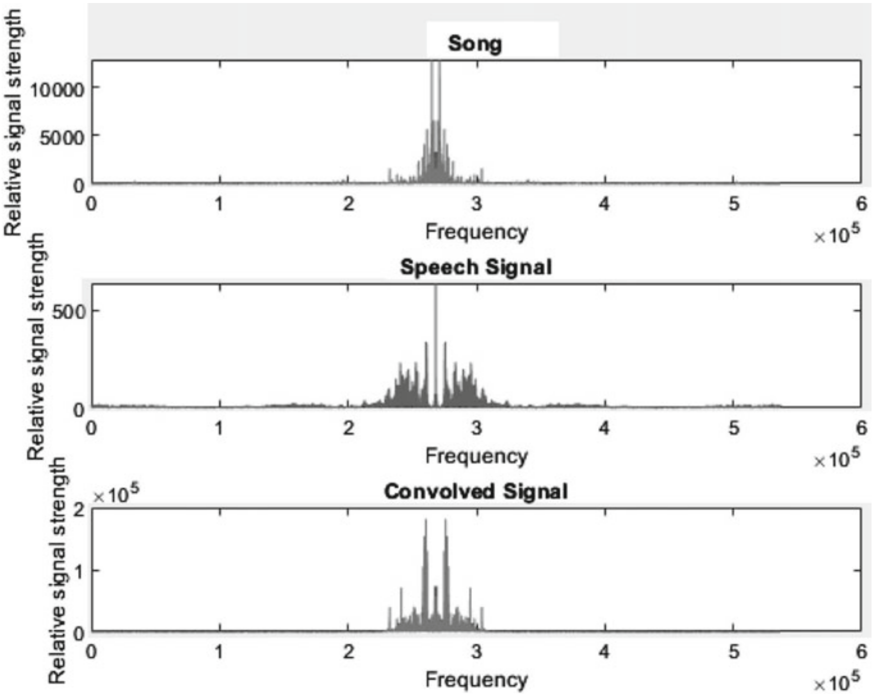


Fig. 7.27 Output of convolution of two speech signals in frequency domain analysis of Example 7.98b

```
timelag = 0.23;
delta = round(Fs*timelag);
alpha = 0.5;

orig = [mtlb;zeros(delta,1)];
echo = [zeros(delta,1);mtlb]*alpha;

mtEcho = orig + echo;
t = (0:length(mtEcho)-1)/Fs;

subplot(2,1,1)
plot(t,[orig echo])
legend('Original','Echo')

subplot(2,1,2)
plot(t,mtEcho)
legend('Total')
xlabel('Time (s)')
% soundsc(mtEcho,Fs)
```

```
[Rmm,lags] = xcorr(mtEcho,'unbiased');

Rmm = Rmm(lags>0);
lags = lags(lags>0);

figure
plot(lags/Fs,Rmm)
xlabel('Lag (s)')
[ ,dl] = findpeaks(Rmm,lags,'MinPeakHeight',0.22);

tNew = filter(1,[1 zeros(1,dl-1) alpha],mtEcho);
soundsc(mtNew,Fs)
subplot(2,1,1)
plot(t,orig)
legend('Original')

subplot(2,1,2)
plot(t,mtNew)
legend('Filtered')
xlabel('Time (s)')
```

Output waveform obtained from the above Program is plotted and is shown in Fig. 7.28.

Output Waveform:

(b) Frequency Domain

```
[sig1,fs] = audioread('example1.wav');
% import the song
f = [1:length(sig1)];
soundsc(sig1,fs);
% plot(t, sig1) % plot the song
X = fft(sig1);
X = fftshift(X);%rearranges a Fourier transform X by
% shifting the zero-frequency component to the center of the array.
Xmag = abs(X);

subplot(3,1,1)
plot(f,Xmag);
xlabel('Frequency')
ylabel('Relative signal strength')
title('Song')
[sig2, fs] = audioread('SpeechDFT-16-8-mono-5secs.wav');
soundsc(sig2,fs);% import the song
x=sig2;
x(length(sig1))=0; % zero-pad if lenth(sig2) < sig1
```

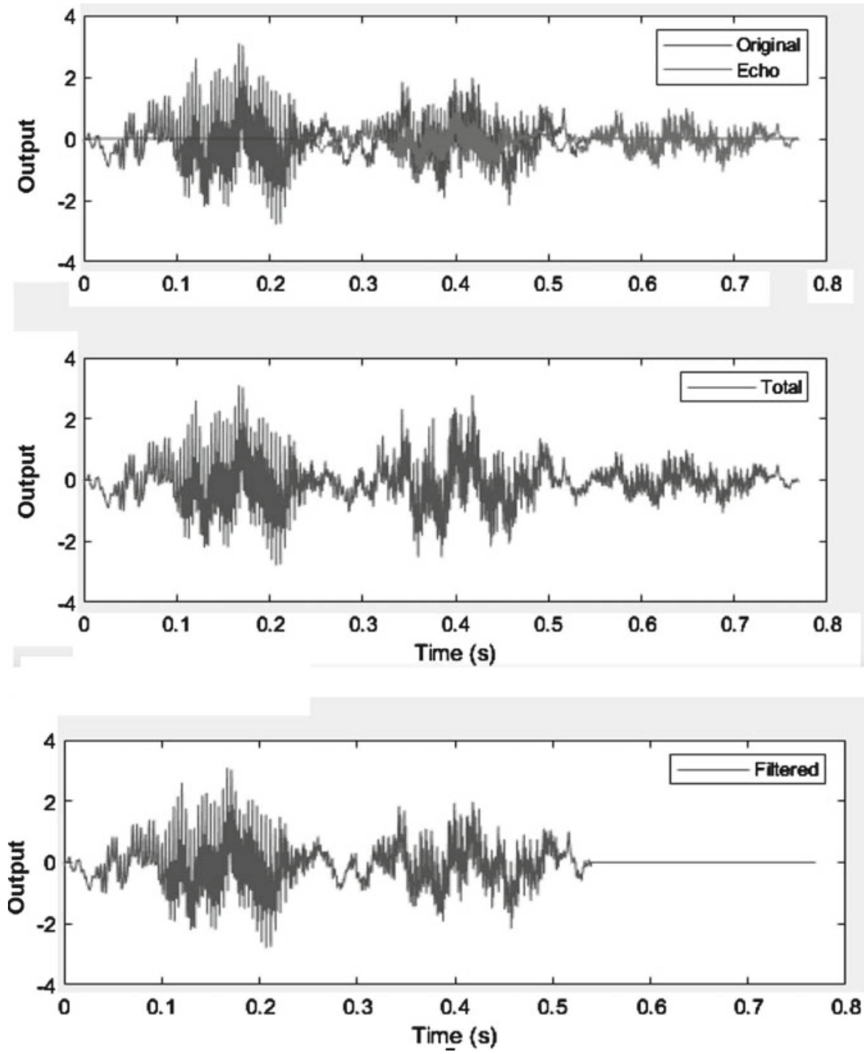


Fig. 7.28 Output of correlation of two speech signals in time domain analysis of Example 7.99a

```
x=x(1:length(sig1));
f1 = [1:length(x)];
% plot(t1, sig2) % plot the song
X1 = fft(x);
X1 = fftshift(X1);%rearranges a Fourier transform X by
% shifting the zero-frequency component to the center of the array.
Xmag1 =abs(X1);
% delta_f = fs./(N.*1000);
```

```
% nf = -N./2:1:N/2-1;
% f = nf .* delta_f;
subplot(3, 1, 2)
plot(f1,Xmag1);
xlabel('Frequency')
ylabel('Relative signal strength')
title('Speech Signal')
w =conv2(sig1,x,'same');
soundsc(w,fs);
% t2 = 0:1:10;
% t2=[1:length(sig1)]/fs;
X2 = fft(w);
X2 = fftshift(X2);%rearranges a Fourier transform X by
% shifting the zero-frequency component to the center of the array.
Xmag2 = abs(X2);
f2=[1:length(sig1)];
subplot(3,1,3);
plot(f2,Xmag2);
% plot(t2,w);
xlabel('Frequency')
ylabel('Relative signal strength')
title('Convolved Signal')
```

Output waveform obtained from the above Program is plotted and is shown in Fig. 7.29.

Output Waveform:

Example 7.100 Downsample and upsample the speech signal by an integer factor 2 and 4. Analyze the process in frequency and time domains.

```
% A speech signal is downsampled and upsampled, the spectra
% are plotted, and the signals are run through the sound card.
% %————————————————————————————

clc;
clear all;
close all;
%%
%————————————————————————————
% Parameters
D =4; % downsampling/upsampling factor
%%
%————————————————————————————
%Read and play back data sampled at 8192 HzHz
Fs = 8192;
```

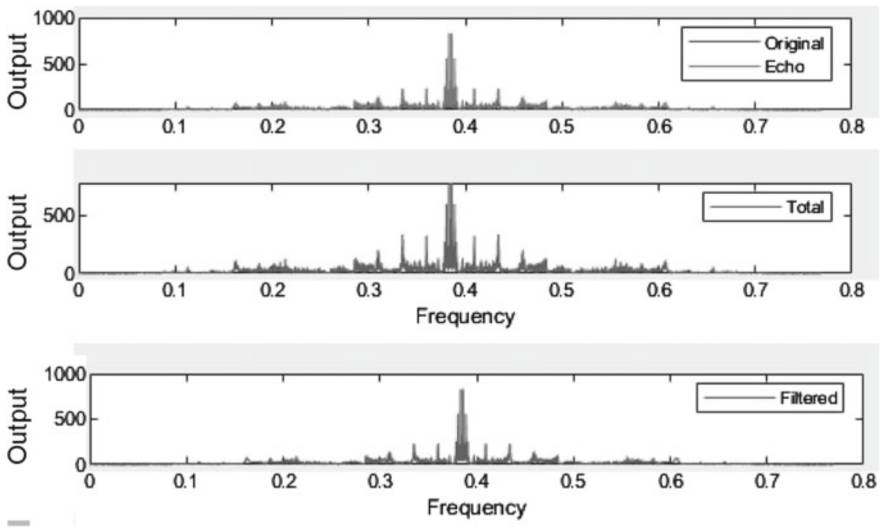


Fig. 7.29 Output of correlation of two speech signals in frequency domain analysis of Example 7.99b

```
data=load('data_1.txt','ascii');
x=data;
x(8192)=0; % zero-pad if lenth(data) < 8192
x=x(1:8192);
N = length(x);
%%
%————————————————————————————
```

a. % Downsampling by D

```
n = 1:1:N;
t = (n-1)./Fs;
z = zeros(1,N);
z(1:ceil(N/D)) = x(1:D:N); % ceil(x) will round of the elements to the nearest integer
z(ceil(N/D)+1:N) = zeros(1,N-ceil(N/D));
figure;
subplot(2,1,1), plot(t,x);% ploting
xlabel('t sec');
title('Original utterance'); % utterence spoken word
subplot(2,1,2), plot(t,z);xlabel('t sec');
title('Utterance downsampled by D');
X = fft(x);
X = fftshift(X);%rearranges a Fourier transform X by
% shifting the zero-frequency component to the center of the array.
Xmag = abs(X);
```

```
Z = fft(z);
Z = fftshift(Z);
Zmag = abs(Z);
delta_f = Fs./(N.*1000);
nf = -N./2:1:N/2-1;
f = nf .* delta_f;
figure;
subplot(2,1,1), plot(f,Xmag);
xlabel('f kHz');
title('Original utterance');
subplot(2,1,2), plot(f,Zmag);
xlabel('f kHz');
title('Utterance downsampled by D');
input('Original utterance')
soundsc(x,Fs);
input('Utterance downsampled by D')
soundsc(z,Fs);
input('Utterance downsampled by D, played at Fs/D')
soundsc(z,Fs./D);
%%

%———————————————————————————————————————
```

b. % Upsampling by D

```
z = zeros(1,D.*N);
z(1:D:D.*N) = x(1:N);
x_extend = x;
x_extend(N+1:D.*N) = zeros(1,(D-1).*N);
n_extend = 1:1:D.*N;
t_extend = n_extend./Fs;
figure;
subplot(2,1,1), plot(t_extend,x_extend);
xlabel('t sec');
title('Original utterance');
subplot(2,1,2), plot(t_extend,z);
xlabel('t sec');
title('Utterance upsampled by D');
X_extend = fft(x_extend);
X_extend = fftshift(X_extend);
Xmag_extend = abs(X_extend);
Z = fft(z);
Z = fftshift(Z);
Zmag = abs(Z);
delta_f = Fs./(D.*N.*1000);
nf = -D.*N./2:1:D.*N/2-1;
```

```
f = nf .* delta_f;
figure;
subplot(2,1,1), plot(f,Xmag_extend);
xlabel('f kHz');
title('Original utterance');
subplot(2,1,2), plot(f,Zmag);
xlabel('f kHz');
title('Utterance upsampled by D');
input('Original utterance')
soundsc(x,Fs);
input('Utterance upsampled by D')
soundsc(z,Fs);
input('Utterance upsampled by D, played at Fs*D')
soundsc(z,Fs*D);
```

Output waveform obtained from the above Program is plotted and is shown in Fig. 7.30.

Output Waveform: The down sampling for a factor of 4 is shown in Fig. 7.30a. Down sampling for a factor of 2 is shown in Fig. 7.30b. The up sampling for a factor of 2 is shown in Fig. 7.30c. The up sampling for a factor of 4 is shown in Fig. 7.30d.

7.2 Application of Python Program to Solve Engineering Problems

Example 7.101 Write a Python program for daily recording covid cases. Get the input from the user.

Program:

```
#covid case wave
import matplotlib.pyplot as plt
x=[]
y=[]
n=int(input("Enter the number of days to be recorded»> "))
for i in range(n):
    y.append(int(input(f"Enter the case recorded on day i+1» ")))
for i in range(n):
    i+=1
    x.append(i)
print(x)
print(y)
plt.plot(x,y)
plt.xlabel('Days')
plt.ylabel('Number of cases')
```

(a)

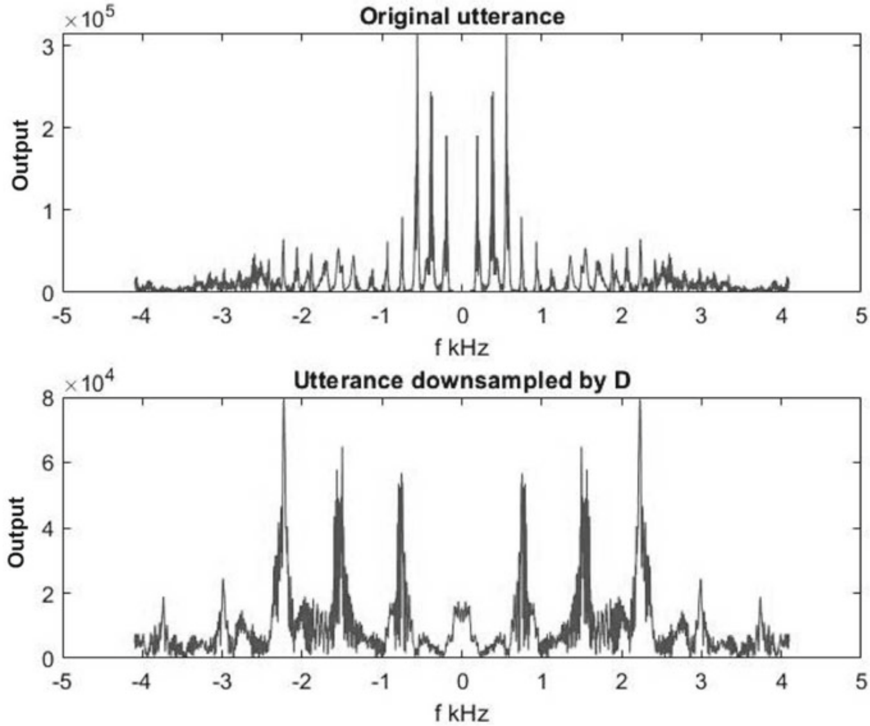


Fig. 7.30 **a** Output of downsampling the speech signal by a factor of 4 of Example 7.100. **b** Output of downsampling the speech signal by a factor of 2 of Example 7.100. **c** Output of upsampling the speech signal by a factor of 2 of Example 7.100. **d** Output of upsampling the speech signal by a factor of 4 of Example 7.100

```
plt.title('Covid case graph')
plt.show()
```

The covid cases graph is represented in Fig. 7.31.

Output:

The graph of daily covid cases is shown in Fig. 7.31.

Example 7.102 Write a Python program to perform deposit and withdrawing cash account.

Program:

```
class Bank_Account:
def_init_(self):
    self.balance=0
```

(b)

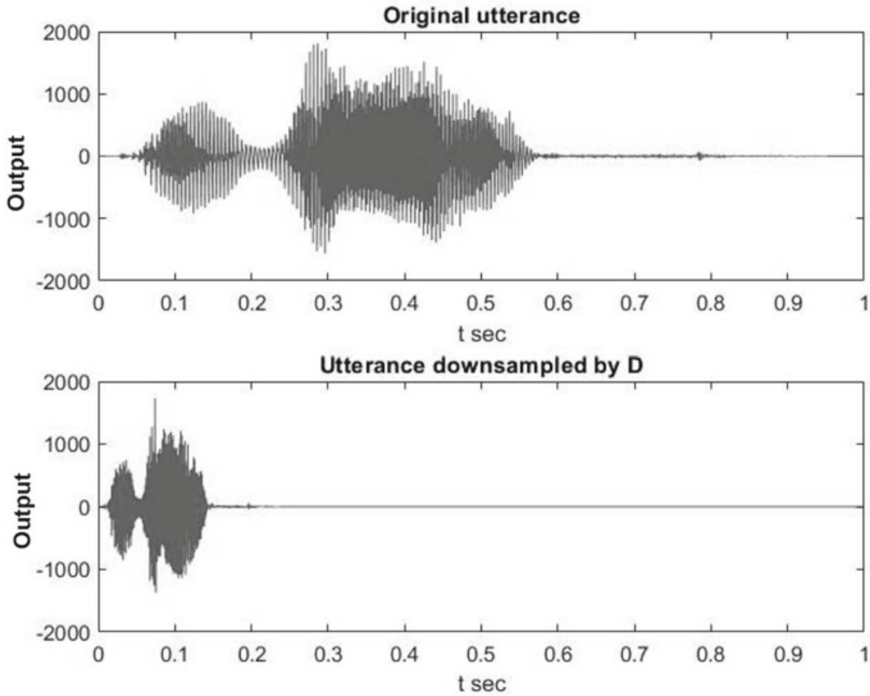


Fig. 7.30 (continued)

```
    print("Hello!!! Welcome to the Deposit & Withdrawal Machine")
def deposit(self):
    amount=float(input("Enter amount to be Deposited: "))
    self.balance += amount
    print("\n Amount Deposited:",amount)
def withdraw(self):
    amount = float(input("Enter amount to be Withdrawn: "))
    if self.balance>=amount:
        self.balance-=amount
        print("\n You Withdrew:", amount)
        else:
    print("\n Insufficient balance ")
def display(self):
    print("\n Net Available Balance=",self.balance)
s = Bank_Account()
s.deposit()
s.withdraw()
s.display()
```

(c)

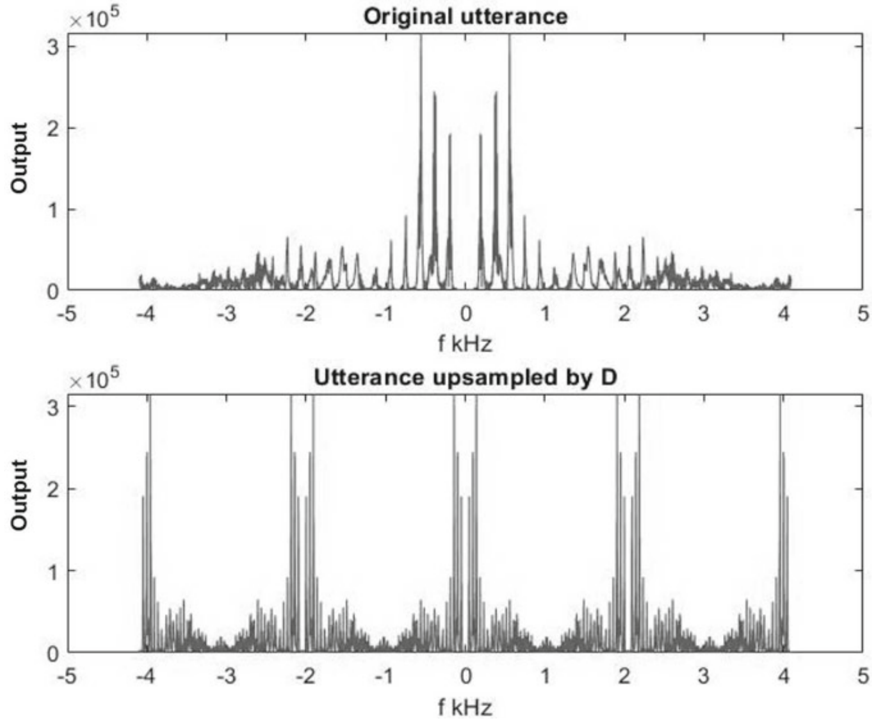
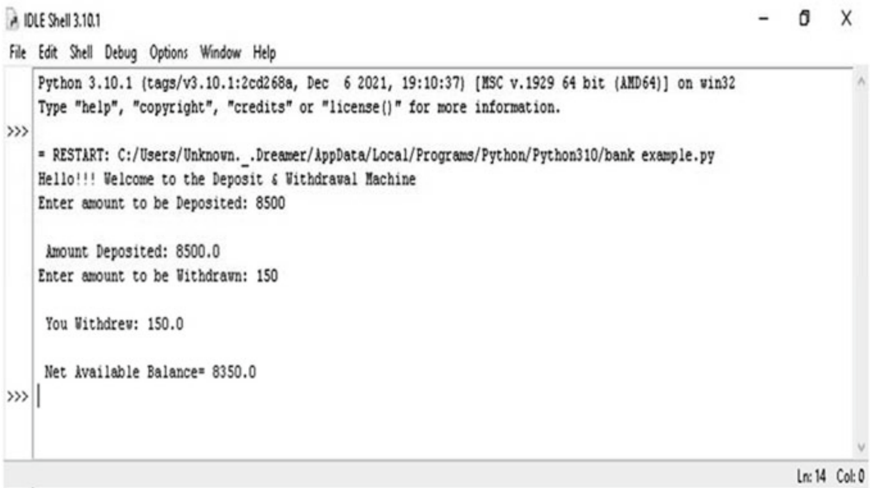


Fig. 7.30 (continued)

Output:



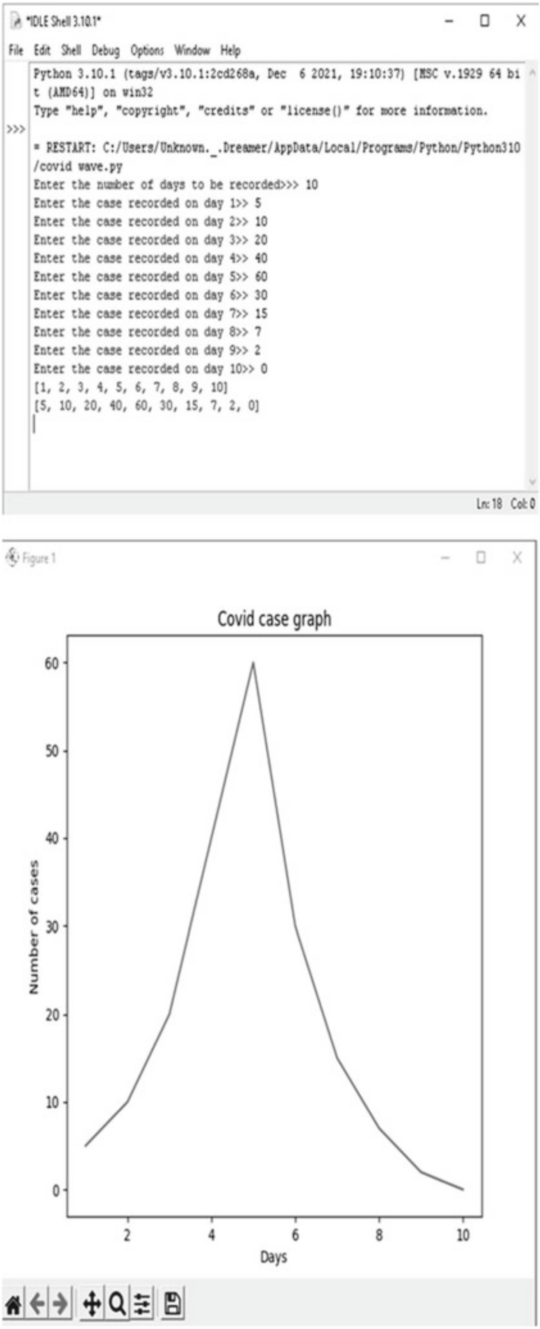


Fig. 7.31 Graph of daily covid cases of Example 7.101

Example 7.103 Write a Python program to calculate Electricity consumption bill.

The problem statement is given below.

Problem statement:

- Get the input from the user.
- The first 100 units are free.
- The next 100 units it costs 1.5rs per unit.
- For the next 300 units it costs 3rs per unit.
- For more than 500 units, the first 100 units are free, the next 100 units costs 3.50rs per unit, the next 300 units costs 4.60rs per unit and the rest of the units used costs 6.60rs per unit.

Program

```
unit=int(input("Enter the amount of units used» "))
cal=unit
cost=0
if unit>0 and unit<=100:
    print("the first 100 unit is free")
elif unit>100 and unit<=200:
    unit=unit-100
    cost=unit*1.5
elif unit>200 and unit<=500:
    unit=unit-200
    cost=(unit*3)+(100*2)
elif unit>500:
    unit=unit-500
    cost=(unit*6.60)+(100*3.50)+(300*4.60)
else:
    print("Invalid input")
print(f"The total electricity bill amount you have to pay for cal units is Rs.cost")
```

Output 1:

Enter the amount of units used» 248

The total electricity bill amount you have to pay for 248 units is Rs.344

Output 2:

Enter the amount of units used» 576

The total electricity bill amount you have to pay for 576 units is Rs.2231.6

Example 7.104 Write a Python program to implement a line graph for the given points (10,10), (20,78), (30,40), (40,45), (50,20), (60,60), (70,30), (80,20), (90,90), (100,10).

Program

```
#open cmd
#pip install matplotlib
import matplotlib.pyplot as plt
x=[10,20,30,40,50,60,70,80,90,100]
y=[10,78,30,45,20,60,30,20,90,10]
plt.plot(x, y)
plt.show()
```

Output:

The graph of distance between points is shown in Fig. 7.32.

Example 7.105 Write a Python program to manage retail shop billing system with orders of technical items.

Program:

```
product_name=[]
product_quantity=[]
product_price=[]
company_name='Retail Store'
company_address='Malik street,maathur'
company_city='Trichy'
message='Thanks for shopping with us today!'
length=int(input("Enter the number of product purchased»> "))
for i in range(length):
    product_name.append(input("Enter the product name: "))
    product_quantity.append(int(input("Enter the product quantity: ")))
    product_price.append(int(input("Enter the product price: ")))
print("\n\t\t#### BILL ####")
print("*"*50)
print("\t\t{}".format(company_name.title()))
print("\t\t{}".format(company_address.title()))
print("\t\t{}".format(company_city.title()))
print("*"*50)
print("\tProduct Name\t Qantity \tPrice")
print("-"*50)
i=0
for i in range(length):
    print("\t",product_name[i],"\t",product_quantity[i],"\t\t",product_price[i])
print("="*50)
print("\t\t\t\tTotal")
total=0
i=0
for i in range(length):
```

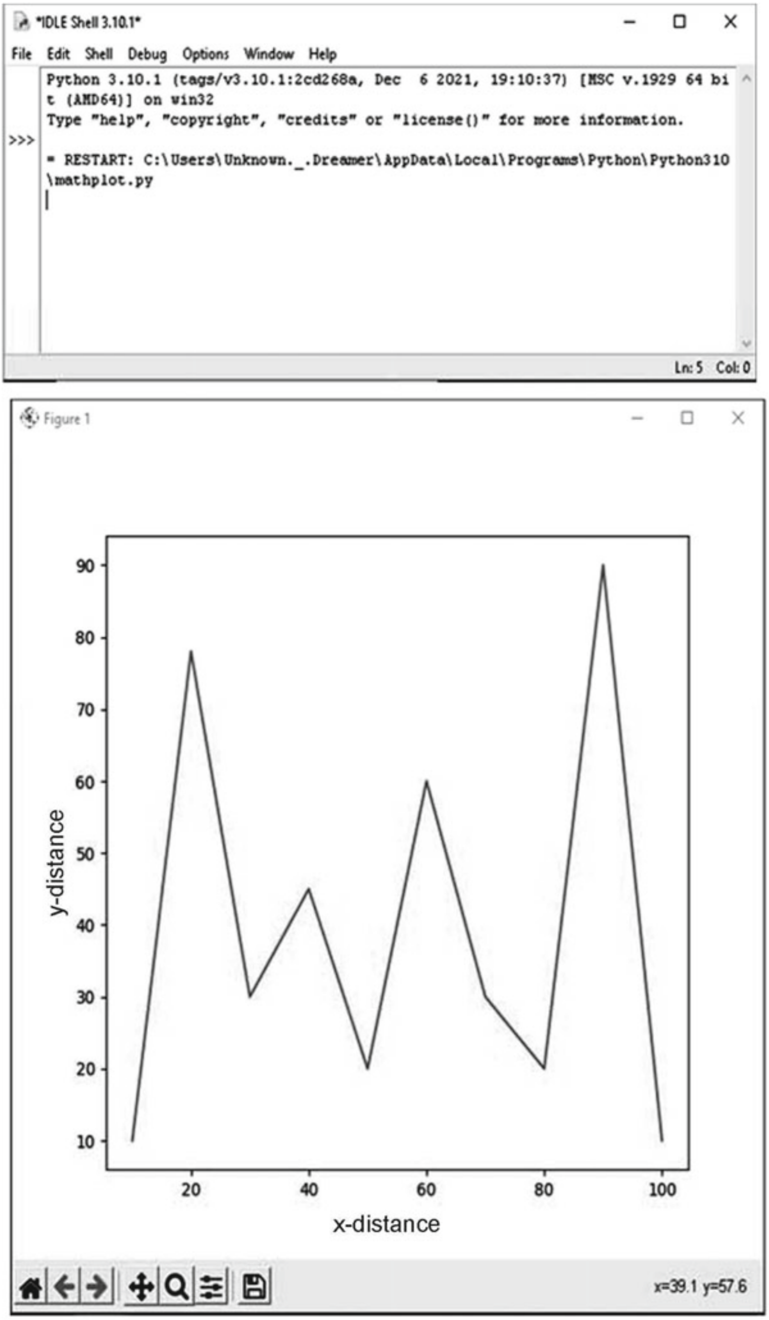
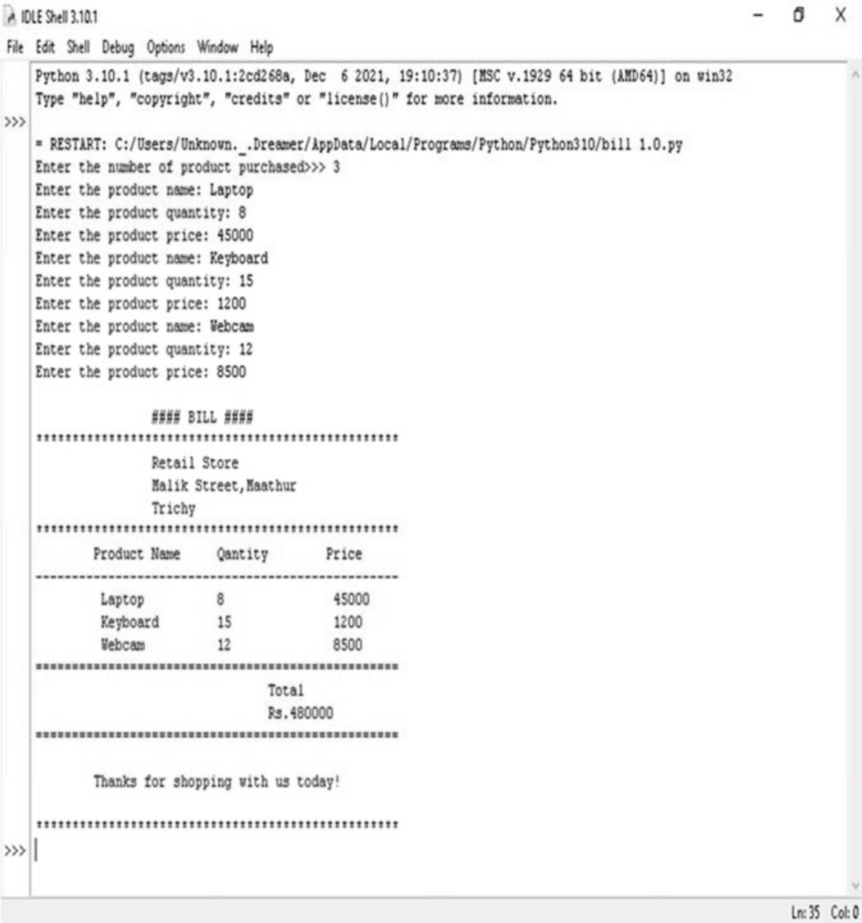


Fig. 7.32 Graph of distance between two points of Example 7.104

```
    total+=(product_price[i]*product_quantity[i])
print("\t\t\t\tRs.{}".format(total))
print("="*50)
print("\n\t{}\n".format(message))
print("*"*50)
```

Output:



```
IDLE Shell 3.10.1
File Edit Shell Debug Options Window Help
Python 3.10.1 (tags/v3.10.1:2cd268a, Dec  6 2021, 19:10:37) [MSC v.1929 64 bit (AMD64)] on win32
Type "help", "copyright", "credits" or "license()" for more information.
>>>
= RESTART: C:/Users/Unknown._.Dreamer/AppData/Local/Programs/Python/Python310/bill 1.0.py
Enter the number of product purchased>>> 3
Enter the product name: Laptop
Enter the product quantity: 8
Enter the product price: 45000
Enter the product name: Keyboard
Enter the product quantity: 15
Enter the product price: 1200
Enter the product name: Webcam
Enter the product quantity: 12
Enter the product price: 8500

                #### BILL ####
**************************************************
                Retail Store
                Malik Street,Maathur
                Trichy
**************************************************
        Product Name     Qantity        Price
--------------------------------------------------
         Laptop          8              45000
         Keyboard        15             1200
         Webcam          12             8500
==================================================
                                Total
                                Rs.480000
==================================================

        Thanks for shopping with us today!

**************************************************
>>> |
Ln: 35  Col: 0
```

Example 7.106 Write a Python program for Signal Processing on Graphs.

Program:

```
#pip install pygsp
#pip install numpy
from pygsp import graphs, filters
G = graphs.Logo()
```

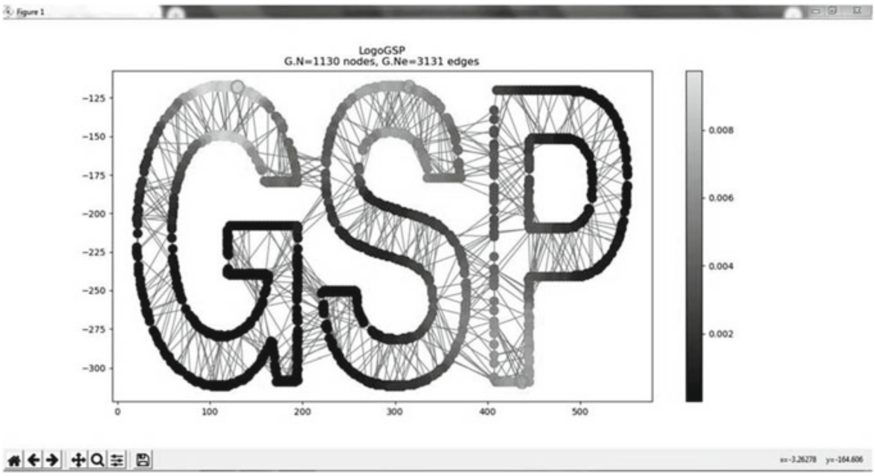


Fig. 7.33 Output response of Example 7.106

```
G.estimate_lmax()
g = filters.Heat(G, tau=100)
import numpy as np
DELTAS = [20, 30, 1090]
s = np.zeros(G.N)
s[DELTAS] = 1
s = g.filter(s)
G.plot_signal(s, highlight=DELTAS, backend='matplotlib')
```

Output:

The signal processing plot is shown in Fig. 7.33.

Example 7.107 Develop a Python program for logical gates

Program:

```
def AND (a, b):
    if a == 1 and b == 1:
        return True
    else:
        return False

def NAND (a, b):
    if a == 1 and b == 1:
        return False
    else:
        return True
```

```
def OR(a, b):
    if a == 1 or b ==1:
        return True
    else:
        return False

def XOR (a, b):
    if a != b:
        return True
    else:
        return False

def NOT(a):
        return not a

def NOR(a, b):
    if(a == 0) and (b == 0):
        return True
    elif(a == 0) and (b == 1):
        return False
    elif(a == 1) and (b == 0):
        return False
    elif(a == 1) and (b == 1):
        return False

def XNOR(a,b):
    if(a == b):
        return True
    else:
        return False
while True:
    print(’’’
1.AND Gate
2.OR Gate
3.NAND Gate
4.NOR Gate
5.XOR Gate
6.XNOR Gate
7.NOT Gate
8,Exit
’’’)
    a=int(input("Enter the Choice» "))
    if a==1:
        print("Enter ’1’ for Ture and ’0’ for false")
        a=int(input("Enter the first condition: "))
```

```
        b=int(input("Enter the second condition: "))
        print(AND(a,b))
    elif a==2:
        print("Enter '1' for Ture and '0' for false")
        a=int(input("Enter the first condition: "))
        b=int(input("Enter the second condition: "))
        print(OR(a,b))
    elif a==3:
        print("Enter '1' for Ture and '0' for false")
        a=int(input("Enter the first condition: "))
        b=int(input("Enter the second condition: "))
        print(NAND(a,b))
    elif a==4:
        print("Enter '1' for Ture and '0' for false")
        a=int(input("Enter the first condition: "))
        b=int(input("Enter the second condition: "))
        print(NOR(a,b))
    elif a==5:
        print("Enter '1' for Ture and '0' for false")
        a=int(input("Enter the first condition: "))
        b=int(input("Enter the second condition: "))
        print(XOR(a,b))
    elif a==6:
        print("Enter '1' for Ture and '0' for false")
        a=int(input("Enter the first condition: "))
        b=int(input("Enter the second condition: "))
        print(XNOR(a,b))
    elif a==7:
        print("Enter '1' for Ture and '0' for false")
        a=int(input("Enter the first condition: "))
        b=int(input("Enter the second condition: "))
        print(NOT(a,b))
    elif a==8:
        print("-"*20)
        break
    else:
        print("Invalid Input")
print()
```

Output:

```
1.AND Gate
2.OR Gate
3.NAND Gate
4.NOR Gate
5.XOR Gate
6.XNOR Gate
7.NOT Gate
8,Exit

Enter the Choice>> 1
Enter '1' for Ture and '0' for false
Enter the first condition: 1
Enter the second condition: 1
True

1.AND Gate
2.OR Gate
3.NAND Gate
4.NOR Gate
5.XOR Gate
6.XNOR Gate
7.NOT Gate
8,Exit

Enter the Choice>> 2
Enter '1' for Ture and '0' for false
Enter the first condition: 1
Enter the second condition: 0
True
```

Example 7.108 Write a Python program to calculate the employee salary.

Program:

```
e_name=input("Enter the name of Employee \n")
c_name=input("Enter the company name \n")
salary=float(input("Enter the salary of Employee \n"))
if(salary>50000):
    tax=0.15*salary
    netsalary=salary-tax
    print("The net salary of "+e_name+" worked in " +c_name+ " is",netsalary)
else:
    netsalary=salary
    print("No taxalbe Amount")
    print("The net salary of "+e_name+" worked in " +c_name+ " is",salary)
```

Output:

```
Enter the name of Employee
John
Enter the company name
JO SOFT
Enter the salary of Employee
80000
The net salary of John worked in JO SOFT is 68000.0
>>>
```

Index

S. Palani, *Basic System Analysis*,
https://doi.org/10.1007/978-3-031-28280-5